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Aspects of F-Theory GUTs

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The basic tools for model building in F-theory are reviewed and applied to the construction of $SU(5)$ models. The flux mechanism for gauge symmetry breaking and doublet triplet splitting is analysed. A short account for the gauge coupling unification and the role of flux and Kaluza-Klein thresholds is also given.

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1. Introduction

During the last four decades there has been significant progress in our understanding of the world of elementary particles. The predictions of Standard Model (SM) of Electroweak and Strong interactions developed in early 70's are now confirmed by an enormous amount of experimental data. Nowadays, highly sophisticated experiments like those performed at the Large Hadron Collider (LHC) at CERN intend to find the missing ingredient -the Higgs field- to complete the anticipated picture of the SM. However, from the theoretical point of view, it was soon realised that the SM falls rather short for a complete and final theory of elementary particles and their fundamental forces. Among other shortcomings, the SM involves a large number of arbitrary parameters, while the gauge symmetry of the model is a product of gauge groups rather than a simple unified one. Furthermore, gravity is not included, therefore the SM cannot be considered a truly unified theory of all fundamental forces. However, from the extrapolation of the three SM gauge couplings there are hints indicating that they probably merge at a common mass scale at high energies and therefore it is expected that indeed, there is a larger symmetry, i.e., a Grand Unified Group (for reviews see [1]) with the SM gauge group incorporated in it. Nevertheless, the SM alone cannot account for this since it is plagued by quadratic divergences at large energies where the Unification of couplings is expected. The large difference between the weak energy scale where the strong, and electroweak interactions are manifest (of the order of 100 GeV) and the expected energy where they unify (around 10^{16} GeV), is known as the hierarchy problem. There is belief that these difficulties might be evaded if supersymmetry is introduced. There are significant theoretical reasons indicating that if supersymmetry (for reviews see [2]) is indeed the solution to the SM drawbacks, then it should be relevant at low scales accessible to present day experiments. The existence of superpartners will be checked at LHC in the forthcoming years.

Essential role in the theoretical developments and in particular the way supersymmetry is contributing to the solutions of the various theoretical issues, has been played by String Theory according to which our world is “immersed" in a “hyperspace" consisting of ten space-time dimensions where six of them are compactified and extremely small to be observed. String Theory reconciles in a nice way supersymmetry, Grand Unified symmetries and unification of gauge couplings at a high (string) scale. Besides, quantization of gravity occurs naturally in the context of String Theory since the ultraviolent infinities can be avoided. Thus, one of the most important tasks is to embed the successful Standard Model of electroweak and strong interactions in a unified String derived model. The unification of all interactions can be realised in a quantum gravity theory free of anomalies. At present, the only candidate theory for this role is String Theory.

Some two decades ago a major effort had been devoted to develop unified models in the context of Heterotic String Theory and there was a perception that it was the only one that includes the Standard Model. The great progress made in recent years has shown that other theories such as Type I Strings can also reproduce the Standard Model. One of the interesting features of Type I string theory is that the scale where the theory leaves its trace could in principle be very low [3], even at the order of a few TeV, and therefore gives us the opportunity to solve the problem of hierarchy without requiring the existence of supersymmetry. Additionally the low unification scenario allows the possibility to seek experimental evidence in appropriately designed experiments. In this scenario an important role is played by extensive solitonic-type objects that appear in the Type I theory and are known as Dp-branes [4]. Our world could be localised on such a brane immersed in a higher dimensional space. In this scenario, known as Brane-World scenario, the interactions of the Standard Model are confined on the brane while the gravitational interactions are spread throughout the whole 10d space and this explains the fact that the gravitational interactions in four-dimensional world are weaker compared to other fundamental interactions.

The last decade, considerable efforts were concentrated in model building and the fermion masses from intersecting D-brane configurations (for related reviews see [5]) embedded in a ten dimensional space. In effective field theory models emerging from intersecting D-branes, the matter fields are represented by strings attached on pairs of different D-brane stacks and they are localised at the intersections. The gauge symmetry of these constructions consists of gauge group factors $U(n_1)\times\cdots\times U(n_k)$, with matter accommodated in the various available bifundamental representations. Hence, in this context, the Standard Model gauge group could naturally emerge from some appropriate D-brane configuration. Since the various D-brane stacks span different dimensions of the ten dimensional space, the corresponding gauge couplings $g_{1,\dots,k}$, depend on different world volumes and therefore they generally have different values. Therefore, although there are many interesting features and success in the above approach, these models do not incorporate the anticipated gauge coupling unification in a natural way since there is no underlying symmetry that would force these couplings to be equal at the unification (string) scale. We note that it is possible to assume a D-brane set up with $U(5)$ gauge group[1], which contains all SM group factors in a single gauge symmetry and leads automatically to gauge coupling unification at the string scale. The main shortcoming of this possibility however, -in the context of intersecting D branes- is the absence of the tree level perturbative Yukawa coupling $10_M\cdot 10_M\cdot 5_H$ to provide fermion masses. We will see how these issues are resolved in the context of F-theory models.

2. The Framework

Work done during the last few years provides convincing evidence that the above drawbacks can be evaded when the desired grand unified theory symmetries (GUTs) are realised in F-theory[6][2] compactified on Calabi-Yau fourfolds. Recent progress in F-theory model building [10]-[30] has shown that old successful GUTs including the SU(5), SO(10) models etc, are naturally realised on the world-volume of non-perturbative seven branes wrapping appropriate compact surfaces. The rather interesting fact in F-theory constructions is that because they are defined on a compact elliptically fibered Calabi-Yau complex four dimensional Manifold the exceptional groups $\mathscr{E}_{6,7,8}$, can be naturally incorporated into the theory too [10, 11, 13, 30]. Although exceptional gauge symmetries suffer from several drawbacks when realised in the context of four-dimensional grand unified theories, in the case of F-theory models they are more promising as new possibilities arise for the symmetry breaking mechanisms and the derivation of the desired massless spectrum.

Present studies have led to remarkable progress on model building in F-theory [20]-[74] with a considerable amount of them focusing on three generation $SU(5)$-GUT models. The vital issues of proton decay, the Higgs mixing term and the fermion mass structure require the computation of

[1] Notice that in intersecting D-brane constructions of this type, the available gauge symmetries are of $U(N)$ and $SO(N)$ type, whilst exceptional groups are absent.

[2] For reviews see [7, 8, 9].

Yukawa couplings [20, 23, 60, 34, 35, 36, 37, 49, 65, 52, 53]. F-model building gave rise to some interesting mechanisms to generate Yukawa hierarchy either with the use of fluxes [20, 49] and the notion of T-branes [64] or with the implementation of the Froggatt-Nielsen mechanism [34, 35, 36, 37, 65]. In [49] (and further in [53, 54]) it is shown that when three-form fluxes are turned on in F-theory compactifications, rank-one fermion mass matrices receive corrections, leading to masses for lighter generations and CKM mixing. Flipped $SU(5)$ [20, 62, 35, 57, 59, 58], as well as some examples of $SO(10)$ F-theory models [20, 39, 40] were also considered. Many of these models predict exotic states below the unification scale, and the renormalization group (RG) analysis of gauge coupling unification including the effect of such states and flux effects has been discussed in a series of papers [43]-[48, 69]. Other phenomenological issues such as neutrinos from KK-modes[51], proton decay [42] and the origin of CP violation [63] have also been discussed. A systematic classification of semi-local F-theory GUTs arising from a single $\mathscr{E}_8$ point of local enhancement, leading to simple GUT groups based on $\mathscr{E}_6$, $SO(10)$ and $SU(5)$ on the del Pezzo surface has been presented in [71]. Here I focus on some phenomenological aspects of effective F-theory models mainly with $SU(5)$ symmetry. To make this presentation self-contained in the next section I review in brief the basics of F-theory and elliptic fibration. Section 4 is devoted to the methodology of F-theory model building. In section 5 the spectral cover approach is reviewed whilst the remaining sections deal with various phenomenological issues of specific examples in the context of $SU(5)$ models.

3. Rudiments of F-theory and Elliptic fibration

We start with a short description of the salient features of F-theory and F-theory model building following mainly the works of [10] and [11, 13][3]. F-theory can be considered as a 12-dimensional theory which arises from the geometrization of the type IIB 10-dimensional string theory. The effective theory is described by the type IIB supergravity whose bosonic field content contains the metric g_{MN} the dilaton field e^{ϕ} and the p-form potentials C_p, $p=0,2,4$ which imply the corresponding field strengths $F_{p+1}=dC_p$. An important observation is that when p-form magnetic fluxes are turned on in the internal manifold, new string vacua may appear and a tree-level moduli potential will be generated. Further, there are two scalars contained in the aforementioned bosonic spectrum, namely C_0 and e^{ϕ} which can be combined into a complex modulus

$$\tau = C_0 + \imath e^{-\phi} \equiv C_0 + \frac{\imath}{g_s} \tag{3.1}$$

In addition, it is convenient to define the field combinations

$$G_3 = F_3 - \tau H_3 \tag{3.2}$$
$$\tilde{F}_5 = F_5 - \frac{1}{2} G_2 \wedge H_3 - \frac{1}{2} B_2 \wedge F_3 \tag{3.3}$$

The 5-form field defined in (3.3) has to obey the selfduality condition $*\tilde{F}_5 = \tilde{F}_5$ where $*$ stands for the Hodge star. With these ingredients one can write an action leading to the correct equations of motion [7]

$$S_{IIB} \propto \int d^{10}x\sqrt{-g}R - \frac{1}{2}\int \frac{1}{(\mathrm{Im}\tau)^2} d\tau\wedge *d\bar{\tau} + \frac{1}{\mathrm{Im}\tau} G_3 \wedge *\overline{G}_3 + \frac{1}{2}\tilde{F}_5\wedge *\tilde{F}_5 + C_4\wedge H_3 \wedge F_3$$

[3] see also [9, 18]

The action is invariant under the following $SL(2,Z)$ duality transformations

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d} \tag{3.4}$$

$$\begin{pmatrix} H \\ F \end{pmatrix} \rightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} \tag{3.5}$$

together with $\tilde{F}_5 \rightarrow \tilde{F}_5$ and $g_{MN} \rightarrow g_{MN}$. This action looks like it has been obtained from a compactified 12-dimensional theory on a torus with modulus τ defined in (3.1). The F_3, H_3 fields appear in S_{IIB} as if they have been obtained from a 12-d field strength $\widehat{F}_4$ reduced along the two radii of the torus. In F-theory τ is interpreted as the complex structure modulus of an elliptic curve generating a complex fourfold which constitutes the elliptic fibration over the CY threefold. Since the fibration relies on the $\tau = C_0 + \imath/g_s$, this means that the gauge coupling is not a constant and the resulting compactification is not perturbative. Hence, according to the above picture, F-theory[6] is defined on a background $R^{3,1} \times X$ with $R^{3,1}$ our usual space-time and X an elliptically fibered Calabi-Yau (CY) complex fourfold with a section over a complex three-fold base B_3.

In [75] a specific example was presented where there is an equivalence between F-theory compactifications on a K3 surface and the Heterotic theory compactification on T^2. A K3 surface is a complex smooth regular manifold with trivial canonical bundle. The general elliptically fibered K3 is described by the Weierstrass equation

$$y^2 \, = x^3 + f(z,w)\, x u^4 + g(z,w)\, u^6 \tag{3.6}$$

where z, w, x, y, u are parameters of the fibration and f, g homogeneous polynomials of degree 8 and 12 respectively. The equation is invariant under the following two rescalings

$$\{z,w,x,y,u\} \rightarrow \{\lambda z, \lambda w, \lambda^4 x, \lambda^6 y, u\} \; ; \; \{z,w,x,y,u\} \rightarrow \{z, w, \mu^2 x, \mu^3 y, \mu u\}$$

Indeed, for the first rescaling the left hand side becomes $y^2 \rightarrow \lambda^{12} y^2$ and the same weight emerges for the right hand side of (3.6). Similarly, one finds that for the second rescaling from both terms of the equation a weight μ^6 is factored out. There are five coordinates compared to two rescalings and one equation, thus the equation describes a two complex dimensional surface. For the first rescaling we observe that the sum of the weights is $1+1+4+6+0 = 12$, i.e. equal to the weight 12, and the second is $0+0+2+3+1 = 6$ is equal to the weight of the second equivalent equation. Therefore, this is a CY manifold.

Fixing $u = 1, w = 1$ the above equation becomes

$$y^2 \, = x^3 + f(z)x + g(z) \tag{3.7}$$

We now observe that f, g transform as sections $f \in K_{B_3}^{-4}, g \in K_{B_3}^{-6}$. This can be understood if we assign the scalings $x \rightarrow \lambda^2 x$ and $y \rightarrow \lambda^3 y$ so that (3.7) becomes $\lambda^6 y^2 = \lambda^6 x^3 + \tilde{f} \lambda^2 x + \tilde{g}$ implying $\tilde{f} \rightarrow \lambda^4 f$ and $\tilde{g} \rightarrow \lambda^6 g$.

The functions $f(z), g(z)$ now are considered 8 and 12 degree polynomials in z. For each point of the base, the equation describes a torus labeled by the coordinate z. (To get an intuition, note that fixing f, g to be real numbers, (3.6) reduces to elliptic curves, see fig.1).

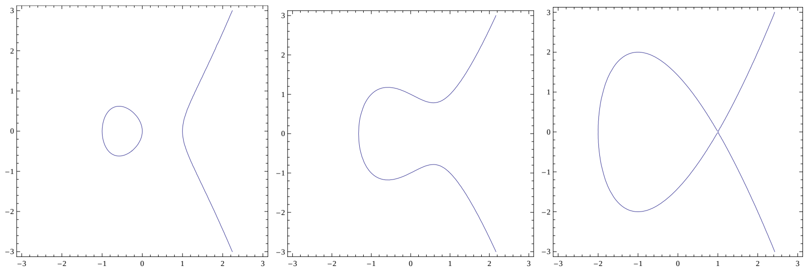

Figure 1: Fixing the values of the polynomials (f,g) to certain real numbers in the Weierstraß equation, elliptic fibrations reduce to elliptic curves. The three cases correspond to the three possible cases of the discriminant, bigger, smaller or equal to zero respectively.

The modular parameter of the torus is related to the functions f,g through the $SL(2,Z)$ modular invariant function $j(\tau)$

$$j(\tau) = \frac{4(24f)^3}{4f^3+27g^2} \tag{3.8}$$

where

$$j(\tau) = e^{-2\pi i\tau} + 744 + \mathscr{O}(e^{2\pi i\tau}) \tag{3.9}$$

The curve described by (3.6) is non-singular provided that the discriminant

$$\Delta = 4\,f^3 + 27\,g^2 \tag{3.10}$$

in non-zero. At the zero loci of the discriminant Δ, (i.e., at $\Delta = 0$) the elliptic curve becomes singular with one cycle shrinking to zero size and the fiber degenerates[4]. There are 24 zeros z_i of the discriminant which are in general distinct and different from the zeros of f,g. This corresponds to 24 7-branes located at z_i, $i = 1,2,\ldots,24$. In the vicinity of such a point using (3.8) and (3.10) we have

$$j(\tau(z)) \sim \frac{1}{z-z_i} \to \tau(z) \approx \frac{1}{2\pi i}\log(z-z_i) \tag{3.11}$$

up to $SL(2,Z)$ transformations. In the limit $z \to z_i$, we observe that $\tau \to i\infty$ and since $\tau = C_0 + i/g$ this means that we are in the weak coupling regime since $g \to 0$. Further, since $\ln(z-z_i) = \ln|z-z_i| + i\,\theta$, performing a complete rotation around z_i, τ undergoes a monodromy $\tau \to \tau+1$, or equivalently

$$C_0 \to C_0+1, \to \oint_{z_i} F_1 = \oint_{z_i} dC_0 = 1$$

This implies the existence of a 7-brane at z_i, while totally there are 24 such branes in the compact transverse space. However, since the space is compact the sum $\oint_{z_i} F_1$ must vanish. Further considerations along these lines lead to the conclusion that F-theory is strongly coupled. There are

[4]The Discriminant locus may have several irreducible components, so that $\Delta = \sum_i n_i S_i$ where S_i are the divisors of B_3 and n_i represent their multiplicities. The singularities of the CY 4-fold are developed along the divisors with $n_i > 1$.

limiting cases however where F-theory has a perturbative expansion. Indeed, suppose that f^3/g^2 is constant which can be satisfied by assuming that [75]

$$g = \phi^3,\ f = a\phi^2,\ \phi = \prod_{i=1}^{4}(z - z_i)$$

Substituting one finds

$$j(\tau) = \frac{4(27a)^3}{27 + 4a^3}$$

which gives a weak coupling regime everywhere on the base for $27 + 4a^3 \approx 0$ i.e., $a \sim -\frac{3}{4^{1/3}}$.

We discuss now how this geometric picture is associated to the gauge group structure and the spectrum of an effective low energy theory model. Recall first that in intersecting D-brane constructions non-abelian gauge symmetries emerge when more than one D-branes coincide. While a single D-brane is associated to a $U(1)$ symmetry, when n of $D6$ branes coincide the gauge group becomes $SU(n)$. In F-theory when D7 branes coincide at certain point, then at this point there is a singularity of the elliptic fibration. The *singularities* of the manifold are classified with respect to the vanishing order of the polynomials f, g and the zeros of the discriminant Δ. They determine the *gauge group* and *the matter content* of the *F-theory* compactification. By adjusting the coefficients of the polynomials f, g we can obtain $\mathscr{A}, \mathscr{D}, \mathscr{E}$ types of gauge groups.

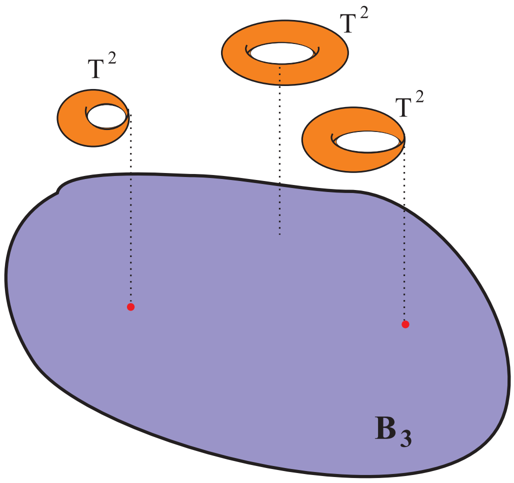

Figure 2: CY four-fold constituting an elliptic fibration over a three-fold base B_3 (only two dimensions are shown). Every point of B_3 is represented by a torus with modulus $\tau = C_0 + \imath/g_s$. Red points represent 7-branes, orthogonal to B_3. The torus degenerates at these 'points' (vanishing cycle). Going around the non-trivial cycle, the vanishing cycle undergoes monodromy.

3.1 Tate's Algorithm

According to the interpretation above, in F-theory the gauge symmetry is associated to the singularities of the internal compact manifold. A systematic analysis of these singularities has started with the work of Kodaira [76]. Given the form of the Weierstrass equation

$$y^2 = x^3 + f(z)x + g(z)$$

ord(f)	ord(g)	ord(Δ)	fiber type	Singularity
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	$\mathscr{E}_6$
3	≥ 5	9	III^*	$\mathscr{E}_7$
≥ 4	5	10	II^*	$\mathscr{E}_8$

Table 1: Kodaira's classification of Elliptic Singularities

the Kodaira classification relies on the vanishing order of the polynomials f,g and the discriminant Δ. This is summarised in Table 3.1. A useful tool for the analysis of the gauge properties of an F-theory GUT is Tate's algorithm [77] [5]. Tate's Algorithm gives an algorithmic procedure to describe the singularities of the elliptic fiber and determine the local properties of the associated gauge group.

We follow [78] to sketch how this analysis works for a few simple cases. We assume a small set U of the base and that the restriction $S|_U$ will have a defining equation $\{z=0\}$. In this patch we expand the coefficients of the Weierstrass equation in powers of z

$$f(z)=\sum_n f_n z^n,\ g(z)=\sum_m g_m z^m \tag{3.12}$$

Plugging into the discriminant the above expansions we get

$$\begin{aligned}\Delta &= 4\left[f_0+f_1z+O(z^2)\right]^3+27\left[g_0+g_1z+O(z^2)\right]^2\\ &= 4f_0^3+27g_0^2+(12f_1f_0^2+54g_0g_1)z+(12f_2f_0^2+12f_1^2f_0+27g_1^2+54g_0g_2)z^2+O(z^3)\end{aligned}$$

We can demand z/Δ which requires that the zeroth order coefficient is zero, i.e. $4f_0^3+27g_0^2=0$. Assuming that f_0,g_0 are simple functions of a new variable t, $f_0=at^2, g_0=bt^3$, the coefficients a,b must obey

$$4a^3+27b^2=0$$

which is satisfied for $a=-1/3, b=2/27$, thus

$$f_0=-\frac{1}{3}t^2,\ \ g_0=\frac{2}{27}t^3 \tag{3.13}$$

The discriminant now becomes

$$\Delta=\frac{4}{3}t^3\left(f_1t+3g_1\right)z+\left(\frac{4f_2t^4}{3}+4g_2t^3-4f_1^2t^2+27g_1^2\right)z^2+O(z^3) \tag{3.14}$$

[5] For recent advances on Tate's classification see [78, 79, 80].

We turn now to the Weierstrass equation. To put it in the Tate form we make the substitution

$$x \to x + \frac{1}{3}t$$

Substituting and reorganising in powers of x, we get

$$y^2 = x^3 + t\,x^2 + (f_2z^2 + f_1z)x + \frac{1}{3}\left[(f_1t + 3g_1)z + (f_2t + 3g_2)z^2 + (f_3t + 3g_3)z^2 + \cdots\right]$$

By redefining $g_i \to \tilde{g} = g_i + f_it/3$ to absorb the terms $\sim t$, we write

$$y^2 = x^3 + t\,x^2 + (f_1z + f_2z^2 + \cdots)x + \tilde{g}_1z + \tilde{g}_2z^2 + \cdots$$

This is the Tate form I_1. The discriminant takes also the simpler form

$$\Delta = 4t^3\tilde{g}_1\,z + \left(4\tilde{g}_2t^3 - f_1\,(18\tilde{g}_1 + t\,f_1)\,t + 27\tilde{g}_1^2\right)z^2 + O(z^3) \tag{3.15}$$

Let us now examine the conditions to obtain z^2/Δ. From the form (3.15) obtained for Δ we see that the coefficient of z is zero if

$$\tilde{g}_1 \equiv \frac{f_1}{3}t + g_1 = 0$$

This condition eliminates also several other terms, the result being

$$\Delta = (4\tilde{g}_2t - f_1^2)t^2\,z^2 + O(z^3)$$

In addition, the Weierstrass equation becomes

$$y^2 = x^3 + t\,x^2 + (f_1z + f_2z^2 + \cdots)x + \tilde{g}_2z^2 + \cdots$$

which is the Tate form for I_2 in Table 2. For global obstructions with regard to the general validity of the Tate forms see [78].

The procedure can be continued for the next order and so on. Partial results are summarised in the Table 2. (for complete results see Table of refs[77, 16, 78]).

We then write the general Tate form of the Weierstrass equation as follows

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6 \tag{3.16}$$

with a_n being polynomial functions on the base. The indices of the coefficients a_n have been chosen so to indicate the section they belong to, i.e. $a_n \in K_{B_3}^n$. Thus each term is a section $K_{B_3}^{-6}$.

To make connection with the previous standard form (3.7) of the Weierstrass equation we complete the square and the cube as follows. We form the square on the left hand side

$$\left(y + \frac{a_1x + a_3}{2}\right)^2 \;=\; x^3 + a_2x^2 + a_4x + a_6 + \left(\frac{a_1x + a_3}{2}\right)^2$$

while we equate the RHS with

$$(x+\lambda)^3 + f\,(x+\lambda) + g$$

Comparing, we get

$$f = \frac{1}{48}\left(24a_1a_3-\left(a_1^2+4a_2\right)^2\right)+a_4$$
$$g = \frac{1}{864}\left(a_1^6+12a_1^4a_2-36a_1^3a_3+48a_1^2a_2^2\right.$$
$$\left.-72a_4\left(a_1^2+4a_2\right)-144a_1a_2a_3+64a_2^3+216a_3^2\right)+a_6 \quad (3.17)$$

Using the definitions

$$\beta_2 = a_1^2+4a_2,\ \beta_4 = a_1a_3+2a_4,\ \beta_6 = a_3^2+4a_6$$

the functions f,g can be rewritten in a simpler form

$$f = -\frac{1}{48}\left(\beta_2^2-24\beta_4\right)$$
$$g = -\frac{1}{864}\left(-\beta_2^3+36\beta_2\beta_4-216\beta_6\right) \quad (3.18)$$

If we further define

$$\beta_8 = \beta_2a_6-a_1a_3a_4+a_2a_3^2-a_4^2$$

we can write the discriminant

$$\Delta = -\beta_2^2\beta_8-8\beta_4^3-27\beta_6^2+9\beta_2\beta_4\beta_6$$

f,g are assumed to be functions of a complex coordinate z on the base B_3.

In summary, we have the following picture: assuming a hypersurface $S\in B_3$ singularity of $\mathscr{ADE}$ type at $z=0$ at a certain point of the base we generate a fibration of this base parametrised by the coordinate z. As $z\neq 0$ the original symmetry breaks leading to a subgroup. Going around this z-point where the fiber degenerates we return to the same singularity up to a monodromy action. In general the effect of the monodromy action cannot be absorbed by some gauge transformation and as a result the gauge symmetry is not fully restored. Thus, one ends up with a reduced gauge symmetry. The order of vanishing of $a_i = b_iz^{n_i}$ characterises the type of singularity, i.e, the **gauge group** supported by the divisor S. For example, the choice

$$a_1 = -b_5, a_2 = b_4z, a_3 = -b_3z^2, a_4 = b_2z^3, a_6 = z^5b_0$$

where b_i are independent of z, lead to the equation

$$y^2 = x^3+b_0z^5+b_2xz^3+b_3yz^2+b_4x^2z+b_5xy \quad (3.19)$$

which as can be seen from table 2 implies an $\mathbf{SU(5)}$ Singularity. The coefficients b_i are in general non-vanishing and can be seen as sections of line-bundles on S. We denote with c_1 the 1^{st} Chern class of the **Tangent** Bundle to S_{GUT} and $-t$ the 1^{st} Chern class of the **Normal** Bundle to S_{GUT}. It is also customary to define the quantity

$$\eta = 6c_1-t$$

Type	**Group**	a_1	a_2	a_3	a_4	a_6	Δ
I_0	0	0	0	0	0	0	0
I_1	–	0	0	1	1	1	1
I_2	–	0	0	1	1	2	2
I_{2n}^s	$SU(2n)$	0	1	n	n	$2n$	$2n$
I_{2n+1}^s	$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
I_1^{*s}	$SO(10)$	1	1	2	3	5	7
IV^{*s}	$\mathscr{E}_6$	1	2	3	3	5	8
III^{*s}	$\mathscr{E}_7$	1	2	3	3	5	9
II^s	$\mathscr{E}_8$	1	2	3	4	5	10

Table 2: Particular cases of Tate's algorithm. (For the complete results see [77, 16].) The order of vanishing of the coefficients $a_i \sim z^{n_i}$, the discriminant Δ and the corresponding gauge group. The highest singularity allowed in the elliptic fibration is $\mathscr{E}_8$.

while $c_1(B_3)|_S = c_1(S) - t$. Returning to (3.19) defining the $SU(5)$ singularity, the various coefficients b_k and parameters x, y, z are sections of line bundles as they appear in the following Table

$$\begin{array}{ll} \textit{section} & c_1(\textit{bundle}) \\ x: & 2(c_1 - t) \\ y: & 3(c_1 - t) \\ z: & -t \\ b_k: & \eta - k c_1 = (6-k)c_1 - t \end{array} \tag{3.20}$$

With these definitions, each term of the equation (3.19) is a section of the same class $6(c_1 - t)$, in accordance with (3.6). Indeed, for example

$$b_2 x z^3 \;:\; \{(6-2)c_1 - t\} + \{2c_1 - 2t\} - 3t \;=\; 6(c_1 - t) \tag{3.21}$$

Substituting $a_i = b_i z^{n_i}$, the β_k take the form

$$\begin{aligned} \beta_2 &= b_5^2 + 4b_4 z \\ \beta_4 &= b_3 b_5 z^2 + 2b_2 z^3 \\ \beta_6 &= b_3^2 z^4 + 4b_0 z^5 \\ \beta_8 = \frac{\beta_2\beta_6 - \beta_4^2}{4} &= z^5(\mathscr{R} + z(4b_0 b_4 - b_2^2)) \\ \mathscr{R} &= b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2 \end{aligned}$$

We can check how the symmetry is enhanced for certain choices. For example, choosing $b_5 = 0$ we see that $\beta_2 \propto z b_4$, $\beta_4 \propto z^3 b_2$ etc while the discriminant becomes $\Delta \propto z^7$. Comparing with Tate's results in Table 2, we see that this corresponds to an $SO(10)$ singularity. Thus, a matter curve is defined along the intersection with another brane where we expect to find the **10** of $SU(5)$ in the adjoint decomposition of $SO(10)$, therefore we write

$$\Sigma_{10} = \{b_5 = 0\}$$

Demanding $\mathscr{R}=0$, we see that $\Delta \sim z^6$ and this corresponds to an $SU(6)$ singularity. The $SU(6)$ adjoint induces the **5** of $SU(5)$, therefore we define the matter curve

$$\Sigma_5 = \{\mathscr{R}=0\}$$

Further enhancements are obtained setting additional coefficients equal to zero. They result to triple intersections of branes which define points in the internal geometry where the Yukawa couplings are formed. Choosing $b_4 = b_5 = 0$, we can proceed as above and see that we obtain an $\mathscr{E}_6$ enhancement. This involves the top Yukawa coupling. Similarly, $b_3 = b_5 = 0$ implies an $SO(12)$ enhancement which is the origin of the bottom mass term:

$$\{b_5 = b_4 = 0\} \to \lambda_t,\ \{b_5 = b_3 = 0\} \to \lambda_b$$

Using the homology classes derived previously for b_i's we can also deduce those of the matter curves. In particular we find

$$[\Sigma_{10}] = c_1 - t,\ [\Sigma_5] = 8c_1 - 3t,$$

or

$$[\Sigma_5] - 3[\Sigma_{10}] - 5c_1 = 0$$

The last one is equivalent to the anomaly cancellation condition [27].

4. Model building

In the previous section we have analysed in some detail the geometric singularities and their interpretation as gauge symmetries. In the present section we describe the basic steps for model building following closely the analysis of [11, 13].

The ultimate goal is to associate this geometrical conception to a GUT model and make a choice of a compact 'surface' S of suitable topological type to build an effective field theory with the desired massless spectrum. Hence, the required set up consists of a 7-brane stack wrapping a compact Kähler surface S of two complex dimensions while the gauge theory of a particular model is associated with the geometric singularity of the internal space [16, 10, 11, 13].

To make a more specific choice of S we must require some further phenomenological constraints. For example, we have pointed out in the introduction that the MSSM spectrum drives the three SM gauge couplings to a common value at a high scale which is nevertheless at least two orders smaller than the Planck scale. It has been argued [11] that in order to achieve a decoupling limit of gravity the spacetime filling sevenbrane associated to the gauge symmetry G_S must wrap a del Pezzo surface. The simplest ones are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. There are eight more del Pezzo surfaces dP_n constructed from an operation known as 'blow up' of $\mathbb{P}^2$ at generic points. (for a detailed discussion see [11, 13]). We may further specify this choice to the del Pezzo dP_8 surface since all other del Pezzo surfaces can be obtained from this one by blowing down various two cycles of the latter. In correspondence with the del Pezzo surfaces, it is now possible to assume singularities associated to exceptional gauge symmetries $\mathscr{E}_8$ and its subgroups, which incorporate the known successful GUT symmetries such as $SU(5)$ and $SO(10)$.

We discuss now the breaking mechanism of the gauge group down to SM. In general, in F-theory there are two mechanisms available. Higgs mechanism and fluxes (we mention also discrete

Wilson lines used in the heterotic string, but this mechanism will not be implemented here). We aim to build a unified theory, the minimal one being $SU(5)$, thus a Higgs breaking mechanism requires the adjoint representation. But if S is a del Pezzo surface, zero mode adjoint Higgs fields are not at our disposal. Even if for some other choice of S the Higgs adjoint is available, this usually leads to a conventional GUT model with resulting spectrum involving undesired matter fields. For example, the $SU(5)$ GUT breaking by the 24-Higgs adjoint allows in the spectrum the dangerous triplet fields which mediate proton decay. The alternative possibility is to turn on $U(1)$ fluxes on the worldvolume of the 7-brane. We will see that the breaking of the GUT group with this mechanism gives the opportunity to eliminate unwanted fields from the light spectrum. We note in passing that in heterotic string theory the $U(1)$ flux mechanism cannot be implemented for the $SU(5)$ breaking. This would require a flux along $U(1)_Y$ and the corresponding gauge boson would develop a string scale mass via the Green-Schwarz mechanism. On the contrary, in F-theory we can arrange so that although the cohomology class of the flux on the seven-brane can be non-trivial, it can represent a trivial class in the base of the compactification. Thus, we can break $SU(5)$ turning on a $U(1)_Y$ flux for example, while keeping the corresponding gauge boson massless.

Next we come to the matter and Higgs fields. Matter can be found in the bulk from the decomposition of the adjoint representation as well as on Riemann surfaces which are located at the intersection between the GUT model seven-brane and additional seven-branes. In several cases, the bulk matter can be of exotic type and has to be eliminated by some suitable condition. We will see that this is possible when the GUT symmetry is $SU(5)$ but it is not true for $SO(10)$ and possible higher groups. It is possible however to turn on singlet vevs with appropriate $U(1)$-'charges' to make some of these states massive or to associate some of them to ordinary matter.

Suppose then that we start with a legible gauge symmetry group G_S associated to the singularity of S. To determine the massless spectrum of a given GUT model, we first turn on a non-trivial background field configuration on S along some subgroup H_S of G_S. Then the effective field theory gauge group is given by the commutant subgroup of H_S in G_S, i.e,

$$G_S \supset \Gamma_S \times H_S$$

Let us start with the matter in the bulk. The spectrum is found in representations which arise from the decomposition of the adjoint of G_S under $\Gamma_S \times H_S$ [6]

$$\mathrm{ad}(G_S) = \bigoplus \tau_j \otimes T_j \tag{4.1}$$

In general, the net number of chiral minus anti-chiral states is given in terms of a topological index formula [11], $n_\tau - n_{\tau^*} = \chi(S, \mathscr{T}_j^*) - \chi(S, \mathscr{T}_j)$ where τ^* is the dual representation of τ, $\mathscr{T}$ is the bundle transforming in the representation T and χ is the Euler character [7]. If $h^i = \dim_C H^i$, i.e. the dimension of the Dolbeault cohomology groups, then $\chi = h^0 - h^1 + h^2$. Moreover, if S is a del Pezzo (or Hirzebruch) surface then $H^2_{\bar\partial}(S, T_j) = 0$ while when the holomorphic bundle T_j is irreducible and non-trivial we also have $H^0_{\bar\partial}(S, T_j) = 0$.

[6]If H_S contains semi-simple $U(1)$ factors, Γ_S corresponds to a proper subgroup of the four-dimensional subgroup. This is the case of $G_S = \mathscr{E}_6$ with H_S, where the commutant is $SO(10) \times U(1)$.

[7]For $H_S = SU(n)$, the spectrum on the bulk is always non-chiral since the corresponding instanton solutions have vanishing first Chern class.

As a more specific example, let us assume that the bulk gauge group is $\mathscr{E}_6$. Under the decomposition $\mathscr{E}_6 \supset SO(10)\times U(1)$, we get

$$78 \;\to\; 45_0 + 1_0 + 16_{-3} + \overline{16}_3 \tag{4.2}$$

Thus, in addition to the adjoint of $SO(10)$ we also get the zero modes 16_{-3} and $\overline{16}_3$ characterised by the line bundles $\mathscr{L}^{-3}$ and $\mathscr{L}^{+3}$ respectively. If we assign the number of states by n_{16} and $n_{\overline{16}}$ respectively in order to obtain chirality we need to have $n_{16}-n_{\overline{16}}\neq 0$. We recall that the number of states is minus the Euler character $n_{16}=-\chi(S,L)$. For S a del Pezzo and $\mathscr{L}$ a line bundle over S, the Riemann-Roch theorem states

$$\begin{aligned}
\chi(S,\mathscr{L}) &= 1+\frac{1}{2}c_1(\mathscr{L})\cdot c_1(\mathscr{L})+\frac{1}{2}c_1(\mathscr{L})\cdot c_1(S)\\
\chi(S,\mathscr{L}^*) &= 1+\frac{1}{2}c_1(\mathscr{L}^*)\cdot c_1(\mathscr{L}^*)+\frac{1}{2}c_1(\mathscr{L}^*)\cdot c_1(S)\\
&= 1+\frac{1}{2}c_1(\mathscr{L})\cdot c_1(\mathscr{L})-\frac{1}{2}c_1(\mathscr{L})\cdot c_1(S)
\end{aligned} \tag{4.3}$$

Therefore, the difference

$$\chi(S,\mathscr{L}^*)-\chi(S,\mathscr{L}) = -c_1(\mathscr{L})\cdot c_1(S)$$

counts the number of chiral states $n_{16}-n_{\overline{16}}$.

Now in this set up one may assume other 7-branes spanning different directions of the internal space. In particular, when other seven branes $S_1,S_2,\cdots$ intersect with the GUT brane wrapping the surface S, they form Riemann surfaces [8] denoted subsequently with Σ_i. Chiral matter and Higgs fields reside on these Riemann surfaces thus we call them matter curves. Along these matter curves gauge symmetry is enhanced. These chiral states appear in bifundamental representations in close analogy to the case of intersecting D-brane models. Along the intersections the rank of the singularity increases. Designating G_S the gauge group on the surface S and G_{S_i} that associated with S_i, the gauge group on Σ_i is enhanced to $G_{\Sigma_i}\supset G_S\times G_{S_i}$ whose the adjoint in general decomposes as

$$\mathrm{ad}(G_{\Sigma_i}) = \mathrm{ad}(G_S)\oplus \mathrm{ad}(G_{S_i})\oplus(\oplus_j U_j\otimes (U_i)_j) \tag{4.4}$$

with $U_j,(U_i)_j$ being the irreducible representations of G_S,G_{S_i}. In the simple case of $G_S=SU(n)$, $G_{S_i}=SU(m)$, and $G_{\Sigma_i}=SU(n+m)$ for example, the chiral $\mathscr{N}=1$ multiplet is the bifundamental $(n,\overline{m})$.

We assume that a non-trivial background gauge field configuration acquires a value in a subgroup $H_S\subset G_S$ and similarly in $H_{S_i}\subset G_{S_i}$. If $G_S\supset \Gamma_S\times H_S$ and $G_{S_i}\supset\Gamma_{S_1}\times H_{S_i}$, with Γ_{S,S_i} being the corresponding maximal G_{S,S_i} subgroups, the $G_S\times G_{S_i}$ symmetry breaks to the commutant group $\Gamma=\Gamma_S\times\Gamma_{S_i}$. Denoting also $H=H_S\times H_{S_i}$, the decomposition of $U\otimes U_i$ into irreducible representations of $\Gamma\times H$ give

$$U\otimes U_i = \oplus_j (r_j,R_j) \tag{4.5}$$

[8] A Riemann Surface (RS) is a connected Hausdorff topological space together with a complex structure; according to the Riemann famous mapping theorem, a simply connected RS is isomorphic to: the Riemann sphere, or to $\mathscr{C}$, or to the open unit disc $|z|<1, z\in\mathscr{C}$.

Let us see how this works for $G_S = \mathscr{E}_6$ [11, 71]. Choosing a $H_S = U(1)$ flux, $\mathscr{E}_6$ breaks to $SO(10) \times U(1)$. If we set $G_{\Sigma_1} = SU(3)$ and recall the breaking pattern

$$\mathscr{E}_8 \supset \mathscr{E}_6 \times SU(3) \supset SO(10) \times U(1) \times SU(3)$$

we have the decomposition of the $\mathscr{E}_8$ adjoint

$$248 \;\rightarrow\; (78,1) + (27,3) + (\overline{27},\overline{3}) + (1,8) \tag{4.6}$$

Matter and Higgs fields are found on matter curves and in the bulk, in the following representations

$$(27,3) \;\rightarrow\; (1,3)_4 + (10,3)_{-2} + (16,3)_1 \tag{4.7}$$

$$(\overline{27},\overline{3}) \;\rightarrow\; (1,\overline{3})_{-4} + (10,\overline{3})_2 + (\overline{16},\overline{3})_{-1} \tag{4.8}$$

$$(78,1) \;\rightarrow\; (45,1)_0 + (1,1)_0 + (16,1)_{-3} + (\overline{16},1)_3 \tag{4.9}$$

The net number of chiral fermions in a specific representation is given as before by $n_{r_j} - n_{r_j^*}$. In the case of an algebraic curve Σ_i the Euler character is written as a function of the genus g of the Riemann surface Σ_i and the first Chern class [11]:

$$n_{r_j} - n_{r_j^*} \;=\; (1-g)\,\mathrm{rk}\,(\Sigma_i, K_{\Sigma_i}^{1/2} \otimes \mathscr{R}_j) + \int_{\Sigma_i} c_1(\Sigma_i, K_{\Sigma_i}^{1/2} \otimes \mathscr{R}_j) \tag{4.10}$$

with $K_{\Sigma_i}^{1/2}$ being the spin bundle over Σ_i and $\mathscr{R}_j$ the corresponding bundle which transforms as a representation R_j. A recent analysis on $\mathscr{E}_6$ can be found in [71].

4.1 Two or... three things we should know about Del Pezzo surfaces

Since the role of the compact surface S is pivotal for the properties of the model, let us review a few things about them.

• We are mainly interested to del Pezzo surfaces. The simplest ones are $\mathbb{P}^1 \times \mathbb{P}^1$ ($= \mathbb{F}_0$ e.g. a Hirzebrough surface [9]) and $dP_0 = \mathbb{P}^2$. The are eight more del Pezzo surfaces dP_n constructed from an operation known as 'blow up' of $\mathbb{P}^2$ at generic points. To blow-up a surface (manifold) at a marked point, we remove the point and replace it with a line gluing it in such a way so that we still get a manifold. The points of this line correspond to different directions from the marked point on the plane. Del Pezzo surfaces are obtained by applying the 'blow-up' operation up to eight points on the plane. A dP_n is generated by the hyperplane divisor H from $\mathbb{P}^2$ and the exceptional divisors $E_{1,\ldots,8}$ with intersection numbers

$$H \cdot H = 1,\; H \cdot E_i = 0,\; E_i \cdot E_j = -\delta_{ij} \tag{4.11}$$

The canonical divisor (and the first Chern class $c_1(dP_n)$) is given by

$$K_S \;=\; -c_1(dP_n) = -3H + \sum_{i=1}^{n} E_i \tag{4.12}$$

The dP_n generators C_i are given in Table 3.

Surface	Generators C_i	Indices	# of Gen.
dP_1	$E_1, H-E_1$	1	2
dP_2	$E_i, H-E_1-E_2$	$i=1,2$	3
dP_3	$E_i, H-E_i-E_j$	$i,j=1,2,3$	6
dP_4	$E_i, H-E_i-E_j$	$i,j=1,2,3,4$	10
dP_5	$E_i, H-E_i-E_j, 2H-E_i-E_j-E_k-E_l-E_m$	$i,j,k,l,m=1,2,3,4,5$	16
$\dots$	$\dots$	$\dots$	$\dots$
dP_8	$E_k,\ H-E_k-E_l,\ 2H-\sum_{j=1}^{5}E_{n_j}$ $3H-2E_k-\sum_{j=1}^{6}E_{n_j},\ 4H-2(E_k+E_l+E_m)-\sum_{j=1}^{5}E_{n_j}$	$k,l,...=1,2,\dots,8$	240

Table 3: The generators of a few del Pezzo surfaces (see [30]). All effective classes can be written as linear combinations of C_i with coefficients non-negative integers.

• The effective class C of a curve can be written as a sum of the generators C_i, $C=\sum_i n_iC_i$ for $n_i>0$. The characteristic property of a del Pezzo surface is that c_1 is ample, that is, it has positive intersection with every effective curve. This in particular implies that K must have positive self-intersection,

$$K_S\cdot K_S = 9-n$$

which gives the restriction $n\leq 8$.

A Kähler class can be defined as follows

$$\omega = AH-\sum_{i=1}^{n} a_iE_i$$

For a line bundle L on del Pezzo with $c_1(L)=\sum_{i=1}^{n} m_iE_i$, ($m_i$ integers) the condition $\omega\cdot c_1(L)=0$ implies

$$\sum_i a_im_i = 0$$

while for sufficiently large A, for any divisor D, the intersection is positive $\omega\cdot D>0$.

• To see the connection of dP_n with exceptional algebras let's define the generators (for $n\geq 3$)

$$a_1 = E_1-E_2,\ \dots,\ a_{n-1}=E_{n-1}-E_n,\ a_n = H-E_1-E_2-E_3 \tag{4.13}$$

Using the dot product for the E_i, H generators, we get

$$a_i\cdot a_j = 2\delta_{ij}-\delta_{i,j+1}-\delta_{j,i+1} = \begin{cases} 2 & i=j \\ -1 & i=j+1 \\ -1 & j=i+1 \end{cases} \tag{4.14}$$

The intersection product of a_i's is identical to minus the Cartan matrix for the dot product of the simple roots of the corresponding algebra E_n. In the particular case of dP_2 there is only one generator E_1-E_2 which is identified as a root of $SU(2)$.

[9] A Hirzebruch surface is a $\mathbb{P}^1$ fibration over a $\mathbb{P}^1$; the general type is classified by an integer index n, and denoted by $\mathbb{F}_n$. It is spanned by two generators $\mathscr{S},\mathscr{E}$ with the properties $\mathscr{S}\cdot\mathscr{S}=-n$, $\mathscr{S}\cdot\mathscr{E}=1$, $\mathscr{E}\cdot\mathscr{E}=0$. The canonical divisor (and Chern class) is given by $K_S=-c_1(S)=-2\mathscr{S}-(n+2)\mathscr{E}$ and any effective class is a combination $a\mathscr{S}+b\mathscr{E}$, with $a,b\geq 0$.

4.2 The $SU(5)$ model

The F-theory derivation of the $SU(5)$ model has attracted the interest of many authors. So, let us consider now that we have a singularity $G_S = SU(5)$. We assume that the gauge symmetry breaks to SM by turning on a $U(1)_Y$ flux. We write the decomposition of the $SU(5)$ gauge multiplet as

$$24 \to R_0 + R_{-5/6} + R_{5/6}$$

where

$$R_0 = (8,1)_0 + (1,3)_0 + (1,1)_0,\; R_{-5/6} = (3,2)_{-5/6},\; R_{5/6} = (\bar{3},2)_{5/6}\cdot \tag{4.15}$$

As we have seen above, massless fields in the bulk are given by the Euler characteristic χ. In order to avoid the massless exotics $R_{\pm 5/6}$ we impose the condition $\chi(S, L^{\pm 5/6}) = 0$. Taking the difference

$$0 = \chi(S, L^{5/6}) - \chi(S, L^{-5/6}) = c_1(L^{5/6}) \cdot c_1(dP_8) = -c_1(L^{5/6}) \cdot K_S$$

so $c_1(L^{5/6}) \cdot K_S = 0$ which means that $c_1(L^{5/6})$ is orthogonal to K_S, i.e. it is a vector in the orthogonal complement of the canonical class. Substituting to the Euler character we find

$$\chi(S, L^{5/6}) = 0 \;\Rightarrow\; 1 + \frac{1}{2} c_1(L^{5/6}) \cdot c_1(L^{5/6}) = 0 \tag{4.16}$$

The vanishing of the latter implies

$$c_1(L^{5/6}) \cdot c_1(L^{5/6}) \;=\; -2 \tag{4.17}$$

This is the condition for $c_1(L^{5/6})$ to correspond to a root of E_n (see (4.14)) while it implies a fractional line bundle $L = \mathscr{O}(E_i - E_j)^{1/5}$ (yet consistent with bulk gauge field configurations [13]).

To obtain the chiral and Higgs spectrum, we should consider the intersections with other branes. Recall that chiral matter and Higgs fields reside on the 10 and $5, \bar{5}$ representations.

In the F-theory set up, the 5 and $\bar{5}$ reside on curves where $SU(5)$ enhances to $SU(6)$. Similarly the 10's are localised on curves where $SU(5)$ enhances to $SO(10)$. When three of these matter curves meet at one point, a trilinear Yukawa coupling is generated while the gauge symmetry is further enhanced. There is a pellucid way to see these enhancements with the help of Dynkin diagrams. In figure 3 we start with an A_4 singularity which corresponds to $SU(5)$. There are two ways to extend this diagram: in the first one we observe that the symmetry is enhanced to $SU(6)$ and the 5-representation of $SU(5)$ is found in the decomposition of the $SU(6)$ adjoint, $35 \to 24_0 + 1_0 + 5_6 + \bar{5}_{-6}$. In the second case we observe from figure 3 that we can also have an $SO(10)$ enhancement while the 10 of $SU(5)$ is found in the adjoint decomposition $45 \to 24_0 + 1_0 + 10_4 + \overline{10}_{-4}$. The top Yukawa coupling $10 \cdot 10 \cdot 5$ originates from an $\mathscr{E}_6$ enhancement and the bottom $10 \cdot \bar{5} \cdot \bar{5}$ from an $SO(12)$.

For an implementation of the above, consider the particular case of $SU(5)$ toy model discussed in section 17 of ref [13]. The surface S is of the del Pezzo type dP_8 which is generated by the hyperplane divisor H from $\mathbb{P}^2$ and the exceptional divisors $E_{1,\dots,8}$ with intersection numbers and canonical divisor for dP_8 given by (4.11) and (4.12). Denoting with C and g the class and the genus of a given matter curve respectively, one has

$$C \cdot (C + K_S) \;=\; 2g - 2 \tag{4.18}$$

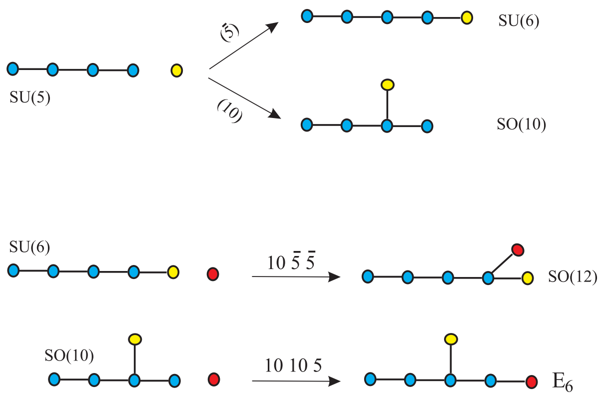

Figure 3: Enhancements of the A_4 singularity at double and triple intersections.

In this particular example the 10_M chiral matter of the three generations resides on one Σ_{10}, with $C = 2H - E_1 - E_5$ and the three $\bar{5}_M$ on a single Σ_5^1 curve with $C = H$. Higgs fields 5_H and $\bar{5}_{\bar{H}}$ reside on different $\Sigma_5^{2,3}$ matter curves with classes $C = H - E_1 - E_3$ and $H - E_2 - E_4$ respectively, giving $g = 0$ for all curves, three families and a Higgs pair. Further details of this model can be worked out using the properties of dP_8 and can be found in [13].

Next we will discuss in detail the $SU(5)$ model and other cases in the spectral cover picture.

5. Spectral cover approach

An equivalent description of the supersymmetric configurations of the 8-dimensional gauge theory can be given in terms of adjoint scalars and gauge fields, corresponding to the so called Higgs bundle picture [10]. In the spectral cover picture we concentrate in the vicinity of the chosen surface S associated to the GUT group G_S, while its neighborhood is described by a spectral surface. The intersections of the spectral cover with the surface S encode the information about the spectrum and its properties.

In local F-theory models we consider the maximum singularity of the elliptic fibrations, i.e. $\mathscr{E}_8$, thus assuming that our effective theory has a GUT group G_S the spectral cover group corresponds to its commutant with respect to $\mathscr{E}_8$. We recall that all viable gauge groups G_S embedded in $\mathscr{E}_8$, can be inferred by the embedding formula [11]

$$\frac{\mathscr{E}_n \times SU(m)}{Z_m} \subset \mathscr{E}_8, \;\; n+m=9 \tag{5.1}$$

Of particular interest are the cases where G_S is one of the phenomenologically viable GUTs $\mathscr{E}_6, SO(10)$ or $SU(5)$. The corresponding decompositions are

$$\mathscr{E}_8 \supset \mathscr{E}_6 \times SU(3) \to \mathscr{E}_6 \times U(1)^3 \to [SO(10) \times U(1)] \times U(1)^2 \tag{5.2}$$

$$\mathscr{E}_8 \supset \mathscr{E}_5\{= SO(10)\} \times SU(4) \tag{5.3}$$

$$\to [SU(5) \times U(1)] \times SU(4) \to [SU(5) \times U(1)] \times U(1)^3$$

$$\mathscr{E}_8 \supset SU(5) \times SU(5)_{\perp} \to SU(5) \times U(1)^4 \tag{5.4}$$

A complete list of all possibilities can be found in [11]. Here we will construct the $SU(5)$ and flipped $SU(5)$ models, while similar analysis for the $\mathscr{E}_6$ model can be found in [71].

5.1 The $SU(5)$ model from the spectral cover

If we take $G_S = SU(5)$ the corresponding spectral surface is $SU(5)_\perp$. Matter resides in the adjoint representation of $\mathscr{E}_8$ which in this case decomposes as

$$248 = (24,1)+(1,24)+(5,10)+(\bar{5},\overline{10})+(10,\bar{5})+(\overline{10},5)$$

The decomposition appears under $SU(5)_{GUT}\times SU(5)_\perp$ where the $SU(5)_\perp$ is the group describing the bundle in the vicinity.

We label the weights of $SU(5)_\perp$ with t_i subject to $\sum_{i=1}^5 t_i = 0$, while we assume further breaking of $SU(5)_\perp$ to

$$SU(5)_\perp \to U(1)^4$$

Thus the 10 representations of $SU(5)_{GUT}$ originate from the $(10,\bar{5})$ component they reside on matter curves $\Sigma_{10_{t_i}}$ and are characterised by the weights t_i. Similarly, the $5/\bar{5}$ representations reside on $\Sigma_{5_{t_i+t_j}}$.

The corresponding spectral cover equation is obtained by defining the homogeneous coordinates

$$z\to U,\; x\to V^2,\; y\to V^3$$

so that the Weierstrass equation becomes

$$0 = b_0U^5+b_2V^2U^3+b_3V^3U^2+b_4V^4U+b_5V^5$$

with U,V being sections of $-t$ and c_1-t respectively. We can turn this equation to a fifth degree polynomial in terms of the affine parameter $s=U/V$:

$$P_5 = \sum_{k=0}^{5} b_k s^{5-k} \;=\; b_5+b_4s+b_3s^2+b_2s^3+b_1s^4+b_0s^5$$

where we have divided by the fifth power V^5, so that each term in the last equation becomes section of c_1-t. The roots of the spectral cover equation [10, 22]

$$0 \;=\; b_5+b_4s+b_3s^2+b_2s^3+b_0s^5 \propto \prod_{i=1}^{5}(s+t_i) \tag{5.5}$$

are identified as the $SU(5)$ weights t_i.

In the above the coefficient b_1 is taken to be zero since it corresponds to the sum of the roots which for $SU(n)$ is always zero, $\sum t_i = 0$. Also, it can be seen that the coefficient b_5 is equal to the product of the roots, i.e. $b_5 = t_1t_2t_3t_4t_5$ and the Σ_{10} curves where the corresponding matter multiplets are localised are determined by the five zeros

$$\Sigma_{10_i},\quad b_5 = \prod_{=i=1}^{5} t_i = 0 \to t_i = 0,\quad i=1,2,3,4,5 \tag{5.6}$$

The model effectively appears with a symmetry $SU(5)_{GUT}\times U(1)^4$. In order to write a Yukawa term, this symmetry should be respected. Thus, writing the coupling involving the up quark masses

$$\mathscr{W} \supset 10_{t_i}\,10_{t_j}\,5_{-t_i-t_j}$$

would appear to involve two different generations. On the other hand, phenomenology requires a rank one mass matrix at tree-level to account for the heavy top mass. A similar conclusion holds for the bottom mass term. More generally, the known hierarchical fermion mass spectrum and the heaviness of the third generation however, is compatible with rank one structure of the mass matrices at tree-level. This requires a solution where at least two of the curves are identified through some (discrete) symmetry.

This idea of identification is corroborated also by the following fact. In the spectral cover approach, we have seen that the properties of the manifold are encoded into the coefficients b_i. Matter curves on the other hand are associated to the roots t_i which are polynomial solutions with factors combinations of b_i's, thus

$$b_i = b_i(t_j)$$

Generically, the inversion of these equations will lead to branchcuts. The solutions $t_j = t_j(b_i)$ are then subject to monodromy actions.

To get a feeling of the procedure we present an example (given in [30]) by considering the simplest case of the $\mathscr{Z}_2$ monodromy. Suppose that two of the roots in (5.5) do not factorize. This implies that the second degree polynomial

$$a_1 + a_2 s + a_3 s^2 = 0$$

cannot be expressed in simple polynomials of the base coordinates. The solutions can be written

$$s_1 = \frac{-a_2 + \sqrt{w}}{2a_3}, \; s_2 = \frac{-a_2 - \sqrt{w}}{2a_3}$$

with $w = a_2^2 - 4a_1 a_3$. These exhibit branchcuts and since

$$\sqrt{w} = e^{i\theta/2}\sqrt{|w|}$$

under a 2π rotation around the brane configuration $\theta \to \theta + 2\pi$ we get $\sqrt{w} \to -\sqrt{w}$ and

$$s_1 \leftrightarrow s_2$$

This means that the two branes interchange locations $s = s_1$ and $s = s_2$. This is equivalent of taking the quotient of the parent theory with a $\mathscr{Z}_2$ symmetry. If this is among $t_1 \leftrightarrow t_2$ the coupling now reads

$$\mathscr{W} \supset 10_{t_1}\, 10_{t_2}\, 5_{-t_1-t_2} \to 10_{t_1}\, 10_{t_1}\, 5_{-2t_1}$$

providing a diagonal mass term since the two curves are identified.

Since the $SU(5)$ spectral cover is described by the 5-degree polynomial shown above, the various monodromy actions are associated to the possible ways of splitting the polynomial according to

$$\begin{aligned}
\mathscr{Z}_2 :&\quad (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s)(a_6 + a_7 s)(a_8 + a_9 s)\\
\mathscr{Z}_2 \times \mathscr{Z}_2 :&\quad (a_1 + a_2 s + a_3 s^2)(a_4 + a_5 s + a_6 s^2)(a_7 + a_8 s)\\
\mathscr{Z}_3 :&\quad (a_1 + a_2 s + a_3 s^2 + a_4 s^3)(a_5 + a_6 s)(a_7 + a_8 s)\\
\mathscr{Z}_4 :&\quad (a_1 + a_2 s + a_3 s^2 + a_4 s^3 + a_5 s^4)(a_6 + a_7 s)\\
\mathscr{Z}_3 \times \mathscr{Z}_2 :&\quad (a_1 + a_2 s + a_3 s^2 + a_4 s^3)(a_5 + a_6 s + a_7 s^2)\\
\text{no split} :&\quad (a_1 + a_2 s + a_3 s^2 + a_4 s^4 + a_5 s^5)
\end{aligned}$$

6. The case of $\mathscr{Z}_2$ monodromy

Up to this point we have discussed the constraints from the gauge symmetry G_S that should be imposed on the Yukawa sector of the effective field theory. We have seen that the $U(1)$ factors are not entirely independent since they undergo a series of monodromies. In general, the theory must be the quotient by some monodromy group which leaves the roots of the gauge symmetry G_S invariant. In the following we attempt to implement the constraints obtained from the previous symmetry breaking stages into the $SU(5)_{GUT}$ model imposing a $\mathscr{Z}_2$ monodromy among t_1, t_2. Expanding, we may determine the homology class for each of the coefficients a_i by comparison with the b_k's. Thus, one gets

$$\begin{aligned}
b_0 &= a_3a_5a_7a_9 \\
b_1 &= a_3a_5a_7a_8 + a_3a_4a_9a_7 + a_2a_5a_7a_9 + a_3a_5a_6a_9 \\
b_2 &= a_3a_5a_6a_8 + a_2a_5a_8a_7 + a_2a_5a_9a_6 + a_1a_5a_9a_7 + a_3a_4a_7a_8 + a_3a_4a_6a_9 + a_2a_4a_7a_9 \\
b_3 &= a_3a_4a_8a_6 + a_2a_5a_8a_6 + a_2a_4a_8a_7 + a_1a_7a_8a_5 + a_2a_4a_6a_9 + a_1a_5a_6a_9 + a_1a_4a_7a_9 \\
b_4 &= a_2a_4a_8a_6 + a_1a_5a_8a_6 + a_1a_4a_8a_7 + a_1a_4a_6a_9 \\
b_5 &= a_1a_4a_6a_8
\end{aligned} \tag{6.1}$$

We first solve the constraint $b_1 = 0$. We make the Ansatz [36]

$$a_2 = -c(a_5a_7a_8 + a_4a_9a_7 + a_5a_6a_9),\; a_3 = ca_5a_7a_9$$

Substituting into b_n's we get

$$\begin{aligned}
b_0 &= c\,a_5^2a_7^2a_9^2 \\
b_2 &= a_1a_5a_7a_9 - \left(a_5^2a_7^2a_8^2 + a_5a_7\,(a_5a_6 + a_4a_7)\,a_9a_8 + \left(a_5^2a_6^2 + a_4a_5a_7a_6 + a_4^2a_7^2\right)a_9^2\right)c \\
b_3 &= a_1\,(a_5a_7a_8 + a_5a_6a_9 + a_4a_7a_9) - (a_5a_6 + a_4a_7)\,(a_5a_8 + a_4a_9)\,(a_7a_8 + a_6a_9)\,c \\
b_4 &= a_1\,(a_5a_6a_8 + a_4a_7a_8 + a_4a_6a_9) - a_4a_6a_8\,(a_5a_7a_8 + a_5a_6a_9 + a_4a_7a_9)\,c \\
b_5 &= a_1a_4a_6a_8
\end{aligned}$$

Next, we observe that we have to determine the homology classes $[a_i]$ of nine unknowns $a_1, \dots a_9$ in terms of the b_k-classes $[b_k]$. From (6.1) we deduce that the latter satisfy the general equation $[b_k] = [a_l] + [a_m] + [a_n] + [a_p]$ for $k+l+m+n+p = 24$. Three classes are left unspecified which we choose them to be $[a_l] = \chi_l, l = 5,7,9$. The rest are computed easily and presented in Table 4.

The Σ_{10} curves are found setting $s = 0$ in the polynomial

$$b_5 \equiv \Pi_5(0) = a_1a_4a_5a_6 = 0 \to a_1 = 0, a_4 = 0, a_5 = 0, a_6 = 0 \tag{6.2}$$

Thus, after the monodromy action, we obtain four curves (one less compared to no-monodromy case) to arrange the appropriate pieces of the three (3) families. The Σ_5 curves are treated similarly. To determine the properties of the fiveplets we need the corresponding spectral cover equation. This is a 10-degree polynomial

$$\mathscr{P}_{10}(s) = \sum_{n=1}^{10} c_n s^{10-n} = b_0 \prod_{i,j}(s - t_i - t_j),\; i<j,\; i,j = 1,\dots,5$$

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$\eta-2c_1-\chi$	$\eta-c_1-\chi$	$\eta-\chi$	$-c_1+x_5$	x_5	$-c_1+x_7$	x_7	$-c_1+\chi_9$	χ_9

Table 4: Homology classes for coefficients a_i for the $\mathscr{Z}_2$ ($SU(5)$) case

Field	$U(1)_i$	homology	$U(1)_Y$-flux	$U(1)$-flux
$10^{(1)}=10_3$	$t_{1,2}$	$\eta-2c_1-\chi$	$-N$	M_{10_1}
$10^{(2)}=10_1$	t_3	$-c_1+\chi_7$	N_7	M_{10_2}
$10^{(3)}=10_2$	t_4	$-c_1+\chi_8$	N_8	M_{10_3}
$10^{(4)}=10'_2$	t_5	$-c_1+\chi_9$	N_9	M_{10_4}
$5^{(0)}=5_{h_u}$	$-t_1-t_2$	$-c_1+\chi$	N	$M_{5_{h_u}}$
$5^{(1)}=5_2$	$-t_{1,2}-t_3$	$\eta-2c_1-\chi$	$-N$	M_{5_1}
$5^{(2)}=5_3$	$-t_{1,2}-t_4$	$\eta-2c_1-\chi$	$-N$	M_{5_2}
$5^{(3)}=5_x$	$-t_{1,2}-t_5$	$\eta-2c_1-\chi$	$-N$	M_{5_3}
$5^{(4)}=5_1$	$-t_3-t_4$	$-c_1+\chi-\chi_9$	$N-N_9$	M_{5_4}
$5^{(5)}=5_{h_d}$	$-t_3-t_5$	$-c_1+\chi-\chi_8$	$N-N_8$	$M_{5_{h_d}}$
$5^{(6)}=5_y$	$-t_4-t_5$	$-c_1+\chi-\chi_7$	$N-N_7$	M_{5_6}

Table 5: Field representation content under $SU(5)\times U(1)_{t_i}$, their homology class and flux restrictions [36] for the model [37]. Superscripts in the first column are numbering the curves, while subscripts indicate the family, the Higgs etc. For convenience, only the properties of $10,5$ are shown. $\overline{10},\overline{5}$ are characterized by opposite values of $t_i\to -t_i$ etc. Note that the fluxes satisfy $N=N_7+N_8+N_9$ and $\sum_i M_{10_i}+\sum_j M_{5_j}=0$ while $\chi=\chi_7+\chi_8+\chi_9$.

We can convert the coefficients $c_n=c_n(t_j)$ to functions of $c_n(b_j)$. In particular we are interested for the value $\mathscr{P}_{10}(0)$ given by the coefficient c_{10} which can be expressed in terms of b_k according to

$$c_{10}(b_k)=b_3^2b_4-b_2b_3b_5+b_0b_5^2=0$$

Using the equations $b_k(a_i)$ and the Ansatz, we can split this equation into seven factors which correspond to the seven distinct fiveplets left after the $\mathscr{Z}_2$ monodromy action.

$$\begin{aligned}P_5=&(a_1-ca_4(a_7a_8+a_6a_9))\times(a_1-c(a_5a_6+a_4a_7)a_8)\times(a_1-ca_6(a_5a_8+a_4a_9))\\&\times(a_4a_7a_9+a_5(a_7a_8+a_6a_9))\times(a_5a_6+a_4a_7)\times(a_5a_8+a_4a_9)(a_7a_8+a_6a_9)\end{aligned} \tag{6.3}$$

Their homologies can be specified using those of a_i. Notice that in the first line of the above the three factors correspond to three fiveplets of the same homology class $[a_1]=\eta-2c_1-\chi$. The complete spectrum is presented in Table 5. Recall now that the $SU(5)$ multiplets decompose to Standard Model multiplets according to

$$\begin{aligned}10&\to(3,2)_{\frac{1}{6}}+(\bar{3},1)_{-\frac{2}{3}}+(1,1)_1\to(Q,u^c,e^c)\\5&\to(3,1)_{-\frac{1}{3}}+(1,2)_{\frac{1}{2}}\to(d^c,\ell)\end{aligned} \tag{6.4}$$

We have pointed out that in F-theory constructions one of the possible ways to break the GUT symmetry is to turn on a flux on the worldvolume of the seven-brane supporting the unified gauge group. In the present case, the $SU(5)$ gauge symmetry can be broken by turning on a non-trivial flux along the hypercharge with $Q_Y = \text{diag}\{-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\}$. As a result, $SU(5)$ multiplets residing on certain curves where the flux restricts non-trivially, might split. This means that some SM pieces of the (6.4) decomposition could be swept away by flux. In the case of Higgs fiveplets in particular, this mechanism could be used to remove the unwanted triplets. To implement this idea in a specific scenario, we recall first the $SU(5)$ embedding to $\mathscr{E}_8$

$$\mathscr{E}_8 \to SU(5)_{GUT} \times U(1)^4$$

The $SU(5)$ chiral and Higgs matter fields descend from the adjoint representation of the $\mathscr{E}_8$ symmetry and reside on the various curves denoted with $\Sigma_{10_j}, \Sigma_{\bar{5}_i}$. Suppose that M_{10_j}, M_{5_i} are two integers representing the number of 10 and $\bar{5}$ representations in a specific construction. The $U(1)$ fluxes (those not included in $SU(5)_{GUT}$) together with the tracelessness condition $\sum_i F_{U(1)_i} = 0$ imply the following condition on the numbers of multiplets [36, 55]

$$\sum_i M_5^i + \sum_j M_{10}^j = 0 \tag{6.5}$$

Consider first the case that we have all 10-type chiral matter accommodated only on one Σ_{10} curve and all chiral states $\bar{5}$ respectively on a single $\Sigma_{\bar{5}}$ curve. Then condition (6.5) implies the relation $M_{10} = -M_5 = M$.

We denote with $N_{Y_5}, N_{Y_{10}}$ the corresponding units of Y flux which splits the $SU(5)$ multiplets according to

$$\Sigma_{\bar{5}} : \begin{cases} n_{(\mathbf{3},\mathbf{1})_{-1/3}} - n_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} = M_5 \\ n_{(\mathbf{1},\mathbf{2})_{1/2}} - n_{(\mathbf{1},\mathbf{2})_{-1/2}} = M_5 + N_{Y_5} \end{cases} \quad \Sigma_{10} : \begin{cases} n_{(\mathbf{3},\mathbf{2})_{1/6}} - n_{(\bar{\mathbf{3}},\mathbf{2})_{-1/6}} = M_{10} \\ n_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} - n_{(\mathbf{3},\mathbf{1})_{2/3}} = M_{10} - N_{Y_{10}} \\ n_{(\mathbf{1},\mathbf{1})_{1}} - n_{(\mathbf{1},\mathbf{1})_{-1}} \quad = M_{10} + N_{Y_{10}} \end{cases} \tag{6.6}$$

Notice that these formulae count the number of 5-components minus those of $\bar{5}$ and the number of 10 components minus those of $\overline{10}$. Since we know that families are accommodated on $\bar{5}$'s we require $n_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} > n_{(\mathbf{3},\mathbf{1})_{-1/3}}$ which implies $M_5 < 0$. Similarly, because the remaining pieces of fermion generations live on 10's, we wish to end up with 10-components after the symmetry breaking, hence we should have $M_{10} > 0$. For example, for exactly three generations we should demand $M_{10} = -M_5 = 3$ and $N_{Y_j} = 0$. In general various curves belong to different homology classes and flux restricts non-trivially to some of them, thus $N_{Y_j} \neq 0$ at least for some values of j.

6.1 A realistic model with Doublet-Triplet splitting

We will discuss here the model of [37] which emerges from the general class [36] presented in Table 5. The first two columns give the field content under $SU(5) \times U(1)_{t_i}$ for the case of $\mathscr{Z}_2$ monodromy. The third column presents the homology classes expressed in terms of c_1, η and the χ_i the latter being unspecified subject only to the condition $\chi = \chi_7 + \chi_8 + \chi_9$. If $\mathscr{F}_Y$ denotes the $U(1)_Y$ flux, to avoid a Green-Schwarz mass for the corresponding gauge boson we must require $\mathscr{F}_Y \cdot \eta = \mathscr{F}_Y \cdot c_1 = 0$. Then, we get $N_i = \mathscr{F}_Y \cdot \chi_i$ and consequently $N = \mathscr{F}_Y \cdot \chi = N_7 + N_8 + N_9$. Using these facts, all remaining entries of column 4 in Table 5 are easily deduced.

We now take the flux parameters to be $M_{10_{1,2,3}} = 1$, $M_{5_{1,2,4}} = -1$ and $N = 0$, while we have the freedom to choose $N_{7,8,9}$ subject only to the constraint $N = N_7 + N_8 + N_9$. This choice of M_i, N_j's ensures the existence of three 10 and three $\bar{5}$ representations which are needed to accommodate the three chiral families.

Next we use the $U(1)_Y$ flux mechanism to realise the doublet triplet splitting and make the model free from dangerous color triplets at scales below M_{GUT}. We choose $M_{5_{h_u}} = 1$, to accommodate the Higgs 5_{h_u}. In addition we choose $M_{5_{h_d}} = 0$ and $N_8 = 1$ so that we are left only with the h_d-doublet in the corresponding Higgs fiveplet

$$\Sigma_{5_{h_d}} : \begin{cases} n_{(3,1)_{-1/3}} - n_{(\bar{3},1)_{1/3}} & = M_{5_5} = 0 \\ n_{(1,2)_{1/2}} - n_{(1,2)_{-1/2}} & = M_{5_5} + N - N_8 = -1 \end{cases} \tag{6.7}$$

In order to satisfy the trace conditions we choose $M_{5_6} = -1$, $N_7 = -1$ so that $\bar{5}^{(6)}$ has only a colour triplet component:

$$\Sigma_{5^{(6)}} : \begin{cases} n_{(3,1)_{-1/3}} - n_{(\bar{3},1)_{1/3}} & = M_{5_6} = -1 \\ n_{(1,2)_{1/2}} - n_{(1,2)_{-1/2}} & = M_{5_6} + N - N_7 = 0 \end{cases} \tag{6.8}$$

We observe that in this simple example we have succeeded to disentangle the colour triplet from the Higgs curve at the price of generating however a new one in a different matter curve. Yet, this allows the possibility of realising the doublet-triplet splitting since we can generate a heavy mass M_D for the triplet by coupling it to an antitriplet via the appropriate superpotential term [37]. This way we obtain the corresponding Higgs doublets light.

However from Table 5 one may see that the matter on the $\Sigma_{10^{(2,3)}}$ curves will be affected by the $N_{7,8}$ flux. In particular the content of $10/\overline{10}$-representations on $\Sigma_{10^{(2,3)}}$ splits as follows

$$\Sigma_{10^{(2)}} : \begin{cases} n_{(3,2)_{1/6}} - n_{(\bar{3},2)_{-1/6}} & = M_{10_2} = 1 \\ n_{(\bar{3},1)_{-2/3}} - n_{(3,1)_{2/3}} & = M_{10_2} - N_7 = 2 \\ n_{(1,1)_1} - n_{(1,1)_{-1}} & = M_{10_2} + N_7 = 0 \end{cases} \tag{6.9}$$

$$\Sigma_{10^{(3)}} : \begin{cases} n_{(3,2)_{1/6}} - n_{(\bar{3},2)_{-1/6}} & = M_{10_3} = 1 \\ n_{(\bar{3},1)_{-2/3}} - n_{(3,1)_{2/3}} & = M_{10_3} - N_8 = 0 \\ n_{(1,1)_1} - n_{(1,1)_{-1}} & = M_{10_3} + N_8 = 2. \end{cases} \tag{6.10}$$

We observe that in the presence of flux one $e^c = (1,1)_1$ state is 'displaced' from $\Sigma_{10^{(2)}}$ to the $\Sigma_{10^{(3)}}$ curve. A similar dislocation occurs for one $u^c = (\bar{3},1)_{-2/3}$ of $\Sigma_{10^{(3)}}$ which 'reappears' in $\Sigma_{10^{(2)}}$. We note that this fact implies a different texture for the up, down and charged lepton mass matrices. It can be checked that the particular distribution of the chiral matter on the specific matter curves can lead to interesting results with respect to the fermion mass structure and other phenomenological properties of the model [37]. For clarity, the final distribution of the MSSM spectrum along the available matter curves is summarized in Table 6.

We close the section with a few remarks about the $SU(5)$ singlets. These are found on curves extending away from the GUT surface S. In particular, singlet fields inhabit on curves in B_3 that project down to the curves on the GUT surface [33]. However, some of their properties could in principle be captured by the corresponding defining equation. Thus, if we work in analogy with the

Chiral Matter											
	M	N	Q	u^c	e^c		M	N	d^c	L
$10^{(1)}\,(F_3)$	1	0	1	1	1	$5^{(4)}\,(\bar{f}_1)$	-1	0	-1	-1
$10^{(2)}\,(F_{2,1})$	1	-1	1	2	0	$5^{(1)}\,(\bar{f}_2)$	-1	0	-1	-1
$10^{(3)}\,(F_{1,2})$	1	1	1	0	2	$5^{(2)}\,(\bar{f}_3)$	-1	0	-1	-1
$10^{(4)}\,(-)$	0	0	0	0	0	$5^{(3)}\,(-)$	0	0	0	0

Higgs and Colour Triplets				
	M	N	T	$h_{u,d}$
$5^{(0)}\,(h_u, T)$	1	0	1	1
$5^{(5)}\,(h_d)$	0	-1	0	-1
$5^{(6)}\,(\bar{T})$	-1	1	-1	0

Table 6: The distribution of the chiral and Higgs matter content of the minimal model along the available curves, after the $U(1)_Y$ flux is turned on. The three families $F_i = 10_i, \bar{f}_j = \bar{5}_j$ are assigned on the curves as indicated. The Higgs doublets $h_{u,d}$ and $T/\bar{T}$ triplets descend from three different curves.

non-abelian representations, we could determine their homologies by examining the polynomial equation $\prod_{i\neq j}(t_i - t_j)$ in terms of b_n's. The zeroth order term of the polynomial gives [71]

$$\begin{aligned} P_0 = \; & 3125 b_5^4 b_0^5 + 256 b_4^5 b_0^4 - 3750 b_2 b_3 b_5^3 b_0^4 + 2000 b_2 b_4^2 b_5^2 b_0^4 + 2250 b_3^2 b_4 b_5^2 b_0^4 \\ & -1600 b_3 b_4^3 b_5 b_0^4 - 128 b_2^2 b_4^4 b_0^3 + 144 b_2 b_3^2 b_4^3 b_0^3 - 27 b_3^4 b_4^2 b_0^3 + 825 b_2^2 b_3^2 b_5^2 b_0^3 \\ & -900 b_2^3 b_4 b_5^2 b_0^3 + 108 b_3^5 b_5 b_0^3 + 560 b_2^2 b_3 b_4^2 b_5 b_0^3 - 630 b_2 b_3^3 b_4 b_5 b_0^3 \\ & +16 b_2^4 b_4^3 b_0^2 - 4 b_2^3 b_3^2 b_4^2 b_0^2 + 108 b_2^5 b_5^2 b_0^2 + 16 b_2^3 b_3^3 b_5 b_0^2 - 72 b_2^4 b_3 b_4 b_5 b_0^2 \end{aligned}$$

which subsequently should be written in terms of a_i's. This can be factorised [71] to give the homologies of the singlet fields θ_{ij}.

6.2 Flipped $SU(5)$

Flipped $SU(5)$ can naturally emerge in the context of F-theory [35, 57, 62]. This can be easily noticed in the spectral cover approach where the second $SU(5)$ in the chain $\mathscr{E}_8 \to SU(5) \times U(5)_\perp$ breaks to $U(1)_X \times SU(4)$

$$\mathscr{E}_8 \to SU(5) \times U(5)_\perp \to [SU(5) \times U(1)_X] \times SU(4) \to [SU(5) \times U(1)_X] \times U(1)^3$$

$U(1)_X$ can be chosen to accommodate part of the hypercharge while monodromies may be imposed among the remaining abelian factors $U(1)^3 \subset SU(4)$. The $SO(10) = \mathscr{E}_5$ embedding of $SU(5) \times U(1)_X$ can be easily detected through the following $\mathscr{E}_8$ breaking pattern

$$\mathscr{E}_8 \supset \mathscr{E}_5 \times SU(4) \to [SU(5) \times U(1)_X] \times SU(4) \to [SU(5) \times U(1)_X] \times U(1)^3$$

The adjoint representation of $\mathscr{E}_8$ then has the $SO(10)\times SU(4)$ and successively the $SU(5)\times SU(4)\times U(1)_X$ decomposition given by

$$\begin{aligned}248 &\to (45,1)+(16,4)+(\overline{16},\bar{4})+(10,6)+(1,15)\\ &\to (24,1)_0+(1,15)_0+(1,1)_0+(1,4)_{-5}+(1,\bar{4})_5+(10,4)_{-1}+(10,1)_4\\ &\quad +(10,\bar{4})_1+(10,1)_{-4}+(\bar{5},4)_3+(\bar{5},6)_{-2}+(5,\bar{4})_{-3}+(5,6)_2 \end{aligned}\tag{6.11}$$

In flipped $SU(5)$ we have the following accommodation of fields. The chiral matter fields, -as in the ordinary $SU(5)$- constitute the three components of the $16\in SO(10)$, ($16=10_{-1}+\bar{5}_3+1_{-5}$ under the $SU(5)\times U(1)_X$ decomposition). However, the definition of the hypercharge includes a component of the external $U(1)_X$ in such a way that flips the positions of u^c, d^c and e^c, ν^c, while leaves the remaining unaltered. Indeed, employing the hypercharge definition $Y=\frac{1}{5}\left(x+\frac{1}{6}y\right)$ where x is the charge under the $U(1)_X$ and y the diagonal generator in $SU(5)$, we obtain the following 'flipped' embedding of the SM representations

$$\begin{aligned}F_i &= 10_{-1} = (Q_i, d_i^c, \nu_i^c)\\ \bar{f}_i &= \bar{5}_{+3} = (u_i^c, \ell_i)\\ \ell_i^c &= 1_{-5} = e_i^c\end{aligned}\tag{6.12}$$

The Higgs fields are found in

$$H\equiv 10_{-1} = (Q_H, D_H^c, \nu_H^c)\ ,\ \bar{H}\equiv \overline{10}_{+1} = (\bar{Q}_H, \bar{D}_H^c, \bar{\nu}_H^c)\tag{6.13}$$

$$h\equiv 5_{+2} = (D_h, h_d)\ ,\ \bar{h}\equiv \bar{5}_{-2} = (\bar{D}_h, h_u)\tag{6.14}$$

There is a remarkable fact in the flipped $SU(5)$ model, which is going to be crucial for the viability in the F-theory construction: we observe that matter antifiveplets carry different $U(1)_X$ charges from the Higgs anti-fiveplets, thus they are distinguished from each other. Consequently, they do not contain exactly the same components. Several R-parity violating terms are not allowed because of this distinction.

For rank one mass textures these couplings predict $m_t = m_{\nu_\tau}$ at the GUT scale. However, in contrast to the standard $SU(5)$ model, down quarks and lepton mass matrices are not related, since at the $SU(5)\times U(1)_X$ level they originate from different Yukawa couplings. Indeed, the mass terms descend from the following $SU(5)\times SU(4)\times U(1)_X$ invariant trilinear couplings

$$\mathscr{W}_d = 10_{-1}\cdot 10_{-1}\cdot 5_2^h \to Q_i u_j h_d\tag{6.15}$$

$$\mathscr{W}_u = 10_{-1}\cdot \bar{5}_3\cdot \bar{5}_{-2}^{\bar{h}} \to Q u^c h_u + \ell \nu^c h_u\tag{6.16}$$

$$\mathscr{W}_l = 1_{-5}\cdot \bar{5}_3\cdot 5_2^h \to e^c \ell h_d\tag{6.17}$$

This gives the opportunity to obtain a correct fermion mass hierarchy at M_W [10]. Moreover, a higher order term providing Majorana masses for the right-handed neutrinos can be written

$$\mathscr{W}_{\nu^c} = \frac{1}{M_S}\overline{10}_{\bar{H}}\overline{10}_{\bar{H}}\, 10_{-1}\, 10_{-1}\tag{6.18}$$

[10] E.g., we can evade the naive M_{GUT}-mass matrix relation $m^0_{down}=m^0_{lepton}$ of the minimal $SU(5)$ GUT. We know that in order to obtain the observed lepton and down quark mass spectrum at low energies, at the GUT scale we should have the relations $m^0_\tau\approx m^0_b$, $m^0_\mu\approx 3m^0_s$ and $m^0_e\approx 1/3\, m^0_d$.

$F \in 10^j, j=1,2,3$	$\bar{f} \in \bar{5}^j, j=1,2,3$	$\ell^c \in 1^j, j=1,2,3$
$(10,4)_{-1} : \begin{cases} 10^{(1)}_{-1} : \{t_1,t_2\} \\ 10^{(2)}_{-1} : \{t_3\} \\ 10^{(3)}_{-1} : \{t_4\} \end{cases}$	$(\bar{5},4)_3 : \begin{cases} \bar{5}^{(5)}_{3} : \{t_1,t_2\} \\ \bar{5}^{(6)}_{3} : \{t_3\} \\ \bar{5}^{(7)}_{3} : \{t_4\} \end{cases}$	$(1,4)_{-5} : \begin{cases} 1^{c(1)}_{-5} : \{t_1,t_2\} \\ 1^{c(2)}_{-5} : \{t_3\} \\ 1^{c(3)}_{-5} : \{t_4\} \end{cases}$

Table 7: Matter curves (labeled by the $SU(4)$ weights t_i, where $\sum_i t_i = 0$), available to accommodate the fermion generations in the case of $\mathscr{Z}_2$ monodromy in flipped $SU(5)$.

In the present context, the above terms descend from the following $SU(5)\times SU(4)\times U(1)$ invariant trilinear couplings

$$\mathscr{W}_{down} \in (10,4)_{-1}\cdot(10,4)_{-1}\cdot(5,6)_2 \tag{6.19}$$

$$\mathscr{W}_{up} \in (10,4)_{-1}\cdot(\bar{5},4)_3\cdot(\bar{5},\bar{6})_{-2} \tag{6.20}$$

$$\mathscr{W}_{\ell} \in (1,4)_{-5}\cdot(\bar{5},4)_3\cdot(5,6)_2 \tag{6.21}$$

Further, in general the following Higgs terms can be written

$$H_iH_ih_j + \bar{H}_i\bar{H}_i\bar{h}_j \tag{6.22}$$

When $H_{,}\bar{H}_i$ acquire vevs, one obtains mass terms for the colour triplets

$$\langle H_i\rangle d^c_{H_i}D_j + \langle \bar{H}_i\rangle \bar{d}^c_{H_i}\bar{D}_j \tag{6.23}$$

As we have explained in previous sections, the abelian symmetries descending from the breaking of $SU(4)\to U(1)^3$ prevent tree-level couplings for the third generation, thus as in the case of $SU(5)$ we need to appeal to monodromies among the $U(1)$'s. Given that for the flipped model the highest accompanying symmetry is $SU(4)$, there are three possible choices for the monodromy group, namely S_3, $\mathscr{Z}_2\times\mathscr{Z}_2$ and $\mathscr{Z}_2$. The first two cases reduce the number of the available matter curves to two. The $\mathscr{Z}_2$ case gives exactly three matter curves. In the first two cases at least two families should reside on the same matter curve. Hierarchy is then generated by flux effects [49, 53, 73]. If we wish to accommodate the families on different matter curves, only the $\mathscr{Z}_2$ monodromy allows the possibility of distinct localization of the three families.

As an example, let us see how matter curves are organised in the case of $\mathscr{Z}_2$ monodromy. Assuming $t_i, i=1,\ldots,4$, with $\sum_{i=1}^4 t_i = 0$ we have the following correspondence between t_i and the representations[11]

$$4\to t_i,\ \bar{4}\to -t_i,\ 6\to t_i+t_j,\ i\neq j,\ \bar{6}\to -(t_i+t_j),\ i\neq j$$

Tables 7 and 8 show the flipped content for the case of $\mathscr{Z}_2$ monodromy. The resulting fermion mass textures and other phenomenological issues are discussed in [35].

[11]Note that, although $6\equiv\bar{6}$, we 'distinguish' them under the weights t_i $6/\bar{6}\to\pm(t_i+t_j)$. This will only result to a relabeling of the curves since $t_i+t_j = -(t_k+t_l)$ where all i,j,k,l differ.

$\bar{h} \in \bar{5}^i_{-2}, i=1,2,3,4$	$h \in 5^i_2, i=1,2,3,4$	$\theta_{ij} \in 1^{ij}_0$
$(\bar{5},\bar{6})_{-2} : \begin{cases} \bar{5}^{(1)}_{-2} : \{-t_1-t_2\} \\ \bar{5}^{(2)}_{-2} : \{-t_3-t_4\} \\ \bar{5}^{(3)}_{-2} : \{-t_{1,2}-t_3\} \\ \bar{5}^{(4)}_{-2} : \{-t_{1,2}-t_4\} \end{cases}$	$(5,6)_2 : \begin{cases} 5^{(1)}_2 : \{t_1+t_2\} \\ 5^{(2)}_2 : \{t_3+t_4\} \\ 5^{(3)}_2 : \{t_{1,2}+t_3\} \\ 5^{(4)}_2 : \{t_{1,2}+t_4\} \end{cases}$	$1^{ij} : \{t_i - t_j\}$

Table 8: Higgs curves, and their labeling under the four $SU(4)$ weights t_i .

7. Gauge coupling unification in F-theory models

The spectrum of the minimal supersymmetric extension of the Standard Model is consistent with a gauge coupling unification at a scale $M_{GUT} \sim 2\times 10^{16}$ GeV. In the simplest case, the SM gauge symmetry is embedded in the $SU(5)$ GUT with the SM matter content incorporated into $SU(5)$ multiplets. However, in a string derived $SU(5)$ model, one must confront the mismatch between M_{GUT} and the natural gravitational scale $M_{Pl} \sim 1.2\times 10^{19}$ GeV. We have pointed out earlier, that in F-theory it is possible to decouple gauge dynamics from gravity by restricting to compact surfaces S that are of del Pezzo type. The exact determination of the GUT scale however, may depend on the spectrum and other details of the chosen gauge symmetry and on the particular model. In F-theory $SU(5)$ we are examining here, there are several sources of threshold effects that have to be taken into account [12, 43, 44, 45, 46, 47, 69]. Thus, we encounter thresholds related to the flux mechanism which induce splitting of the gauge couplings at the GUT scale [12, 43]. A second source concerns threshold corrections generated from heavy KK massive modes [12, 46]. Furthermore, corrections to gauge coupling running arise due to the appearance of probe D3-branes generically present in F-theory compactifations and filling the $3+1$ non-compact dimensions while sitting at certain points of the internal manifold [47].

We focus here on two sources of thresholds, namely the ones induced by fluxes and those by KK-modes. Thresholds induced by the flux mechanism have been extensively analysed in recent literature [12, 43, 45]. There, it was shown that the $U(1)_Y$-flux induced splitting is compatible with the GUT embedding of the minimal supersymmetric standard model, provided that no extra matter other than color triplets is present in the spectrum. Thresholds originating from KK-massive modes have been discussed in [12] and were found to be related to a topologically invariant quantity, the Ray-Singer analytic torsion [81]. In F-theory, KK-massive modes exist for both the gauge and the matter fields. Taking also into account that several low energy effective models involve exotics in the light spectrum, it is possible that they might threaten the gauge coupling unification. Here it will be argued that under reasonable assumptions for the matter curve bundle structure, in a class of $SU(5)$ models the KK-massive modes do not have any effect on the unification[48]. Alternatively, one may implement the requirement of unification to constrain thresholds from KK modes of $SU(5)$ gauge and matter field [12, 45, 46, 69].

We start with the $SU(5)$ gauge multiplet under (4.15) and recall the fact that the massless exotics $R_{\pm 5/6}$ have been eliminated by imposing the condition $\chi(S, L^{5/6}) = 0$ (see eq. 4.16). At the one-loop level we write

$$\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2 k_a}{g_s^2} + b_a \log\frac{\Lambda^2}{\mu^2} + \mathscr{S}_a^{(g)}, \quad a = 3,2,Y\cdot \tag{7.1}$$

Here Λ is the gauge theory cutoff scale, $k_a = (1,1,5/3)$ are the normalization coefficients for the usual embedding of the Standard Model into $SU(5)$, g_s is the value of the gauge coupling at the high scale, and b_a the one-loop β-function coefficients. The massive modes in representations (4.15) induce threshold effects to the running of the gauge couplings denoted by $\mathscr{S}_a^{(g)}$. These can be written [12, 48] in terms of the Ray-Singer torsion $\mathscr{T}_i$

$$\mathscr{S}_a^{(g)} = \frac{4}{3} b_a^{(g)} \left(\mathscr{T}_{5/6} - \mathscr{T}_0\right) + 20\, k_a \mathscr{T}_{5/6}\cdot \tag{7.2}$$

We absorb the term proportional to k_a into a redefinition of g_s while the remaining part suggests that we can define M_{GUT} as [46]

$$M_{GUT} = e^{2/3\left(\mathscr{T}_{5/6} - \mathscr{T}_0\right)} M_C\cdot \tag{7.3}$$

Here we have associated the world volume factor $V_S^{-1/4}$ with the characteristic F-theory compactification scale M_C.

Next we will consider contributions arising from chiral matter and the Higgs fields transforming under the standard $10, \overline{10}$ and $5, \bar{5}$ non-trivial representations. We should mention that the $U(1)_Y$-flux introduced in order to break $SU(5)$ might eventually lead to incomplete $SU(5)$ representations, spoiling thus the gauge coupling unification. However, in the previous sections we have already discussed realistic cases where the matter fields add up to complete $SU(5)$ multiplets, so that the b_a^x-functions contribute in proportion to the coefficients k_a. Under the above assumptions, we may write threshold terms for the KK-states leaving in (6.4) representations as follows [46]

$$S_a^{\bar{5}} = -\frac{4}{3}\beta_a (\mathscr{T}_{-1/2} - \mathscr{T}_{1/3}) + k_a (2\cdot \mathscr{T}_{-1/2}) \tag{7.4}$$

$$S_a^{10} = +\frac{4}{3}\beta_a (\mathscr{T}_{-2/3} - \mathscr{T}_{1/6}) + k_a (6\cdot \mathscr{T}_{1/6})\cdot \tag{7.5}$$

with $\beta_a = \beta_{3,2,1} = \{\frac{3}{2}, 0, 1\}$ while $\mathscr{T}_{q_i}$ is the torsion and the indices refer to hypercharges. We now observe that the hypercharge differences not proportional to k_a in both Σ_{10} and $\Sigma_{\bar{5}}$ satisfy the same condition $q_i - q_j = -\frac{5}{6}$. Given this property and the fact that the torsion is a topologically invariant quantity, one could assume the existence of bundle structures for Σ_{10} and $\Sigma_{\bar{5}}$ so that the above differences vanish. Then, only the terms proportional to k_a remain which can be absorbed in a redefinition of the gauge coupling at M_{GUT}.

We will assume that matter resides on at most genus one ($g = 1$) matter curves (see example discussed in section 4.1 as well as in [13]) with chiral matter forming complete $SU(5)$ multiplets. For $g = 1$ in particular, according to Ray-Singer[81, 48], the analytic torsion is

$$\mathscr{T}_v \equiv \mathscr{T}_{z=u-\tau v} = \ln\left|\frac{e^{\pi i \tau v^2}\vartheta_1(u-\tau v, \tau)}{\eta(\tau)}\right| \tag{7.6}$$

For $v \to v-1$, making use of known theta-function identities we observe

$$\mathscr{T}_{v-1} \equiv \mathscr{T}_{z=u-\tau(v-1)} = \ln\left|\frac{e^{\pi i\tau(v-1)^2}\vartheta_1(u-\tau(v-1),\tau)}{\eta(\tau)}\right| = \mathscr{T}_{z=u-\tau v}\,. \tag{7.7}$$

In order to use this result, we need to make a proper identification of the hypercharge q_i. Considering now two successive hypercharge values q_i, q_j such that $|q_i - q_j| = \frac{5}{6}$ and using the association

$$v_i = \frac{q_i}{|q_i - q_j|} \tag{7.8}$$

we get the identification

$$\mathscr{T}_{u-\tau v_i} \leftrightarrow \mathscr{T}_{q_i}.$$

With this embedding we can easily see that the differences $\mathscr{T}_{-2/3} - \mathscr{T}_{1/6} = 0$ and $\mathscr{T}_{-1/2} - \mathscr{T}_{1/3} = 0$ so that threshold corrections vanish and unification is retained. Thus, adding matter contributions of complete $SU(5)$ representations to (7.1), while assuming that the Higgs-triplet pair decouples at M_X we finally get

$$\frac{16\pi^2}{g_a^2(\mu)} = \frac{16\pi^2}{g_G^2}k_a + (b_a^{(g)} + b_a)\log\frac{M_{GUT}^2}{\mu^2} + b_a^T\log\frac{M_{GUT}^2}{M_X^2} \tag{7.9}$$

where b_a^T, b_a MSSM beta functions with and without the triplet-pair contribution and the GUT value of the gauge g_G coupling is related to the string g_s coupling by

$$\frac{16\pi^2}{g_G^2} = \frac{16\pi^2}{g_s^2} + 20\mathscr{T}_{5/6} + 6\mathscr{T}_{1/6} + 2\mathscr{T}_{1/3}.$$

8. Summary and recent progress

In the previous sections we have presented techniques for the construction of F-theory $SU(5)$ models and analysed ways and novel mechanisms for symmetry breaking and doublet triplet splitting. In F-theory, important properties of the effective field theory model depend on the specific geometry of the compact space and the internal fluxes. Thus, we have investigated how the triplet-doublet splitting problem for example can be solved by judicious choice of fluxes in order to split the $SU(5)$ Higgs fiveplets [13, 33, 36, 37].

Several other important issues of GUT models have been successfully treated in their F-theory analogues. Thus, it has been suggested [82] that unwanted proton decay operators can be avoided through the incorporation of an R-symmetry by invoking symmetries of the manifold and of the fluxes [35, 36]. Alternative ways have also been presented based on the abelian factors [42, 65, 57, 83] and the extension of the $SU(5)$ gauge group to flipped $SU(5)\times U(1)_X$ discussed in the previous section.

Further progress has also been made towards the computation of the Yukawa couplings and the determination of the fermion mass spectrum[53, 73, 84]. We have already explained that in F-theory chiral matter is localised along the intersections of the surface S with other 7-branes, while Yukawa couplings are formed when three of these curves intersect at a single point on S. Their computation relies on the knowledge of the profile of the wavefunctions of the states participating in the intersection. When a specific geometry is chosen for the internal space (and in particular for the GUT surface) these profiles are found by solving the corresponding equations of motion [11]. Besides, the precise knowledge of the common gauge coupling value at the GUT scale is crucial for the determination of the Yukawa couplings involved in the calculation of the fermion mass spectrum [46]. Then, their values are obtained by computing the integral of the overlapping wave-functions at the triple intersections. Despite this important success towards a reliable computation of undetermined parameters of GUTs and the Standard Model, yet a lot of work is required to formulate a complete picture of an F-theory derived effective low energy model; because a theory, no matter how beautiful it is, has to face the relentless test of the experimental proof.

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