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Chiral perturbation theory and final-state interactions in meson decays

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Final-state interactions in three-meson decays are of high importance for various reasons. From a close investigation of the corresponding Dalitz plots, we may learn something about meson–meson scattering, a prominent example of recent years being the extraction of pion–pion scattering lengths from the cusp effect in $K \to 3\pi$ decays. On the other hand, a precise analysis of rescattering effects is of high importance to understand the fundamental transition operators driving the decays, due to the way they enhance and shape the decay probabilities. A low-energy example for this occurs in the analysis of $\eta \to 3\pi$ decays, which play a central role in precision determinations of the light quark mass ratios. To obtain reliable descriptions of final-state interactions also at somewhat higher energies, one has to go beyond perturbative treatments and resort to dispersion-theoretical analyses, which we demonstrate for the examples of the decays $\omega \to 3\pi$ and $\phi \to 3\pi$.

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1. Introduction

A precise study of final-state interactions is increasingly becoming of high importance for our understanding of diverse aspects of hadronic particle decays. They can be of significance for various reasons: if final-state interactions are strong, they can significantly enhance decay probabilities; they can significantly *shape* the decay probabilities, most prominently through the occurrence of resonances; besides resonances, also new and non-trivial analytic structures can occur, such as threshold or cusp effects; and finally, they introduce strong phases or imaginary parts, the existence of which is e.g. a prerequisite for the extraction of CP-violating phases in weak decays.

The relation to the low-energy effective theory of chiral perturbation theory (ChPT) herein is at least two-fold: final-state interactions may give access to observables predicted by ChPT, as we will illustrate below for the case of the cusp effect in $K \to 3\pi$ decays that yields information on pion–pion scattering lengths; vice versa, rescattering effects may have to be considered to sufficient accuracy in order to exploit the relation of experimental observables to fundamental parameters of quantum chromodynamics as suggested by ChPT, and the decay $\eta \to 3\pi$ that gives access to light quark masses is an example for that. Finally, there have been various efforts to extend effective field theories to somewhat higher energies, including e.g. the lowest-lying vector resonances beyond the pseudo-Goldstone bosons of ChPT; we will show in the last section that dispersion relations incorporate strong, model-independent constraints also on the decays of these resonances.

2. Cusp effect in $K \to 3\pi$

In an investigation of the decay $K^\pm \to \pi^0\pi^0\pi^\pm$, the NA48/2 collaboration at CERN has observed a cusp, i.e. a sudden, discontinuous change in slope, in the decay spectrum with respect to the invariant mass squared of the $\pi^0\pi^0$ pair $d\Gamma/ds_3$, $s_3 = M^2_{\pi^0\pi^0}$ [1]. A first qualitative explanation was subsequently given by Cabibbo [2], who pointed out that a K^+ can, simplistically speaking, either decay "directly" into the $\pi^0\pi^0\pi^+$ final state, or alternatively decay into three charged pions $\pi^+\pi^+\pi^-$, with a $\pi^+\pi^-$ pair rescattering via the charge-exchange process into two neutral pions, compare Fig. 1. The loop (rescattering) diagram has a non-analytic piece proportional to

$$i v_\pm(s_3) = i\sqrt{1-\frac{4M^2_{\pi^+}}{s_3}} \,, \quad s_3 > 4M^2_{\pi^+} \,, \quad i v_\pm(s_3) = -\sqrt{\frac{4M^2_{\pi^+}}{s_3}-1} \,, \quad s_3 < 4M^2_{\pi^+} \,, \tag{2.1}$$

and as the charged pion is heavier than the neutral one by nearly 4.6 MeV, the (then real) loop diagram can interfere with the "direct" decay below the $\pi^+\pi^-$ threshold and produce a square-root-like singularity at $s_3 = 4M^2_{\pi^+}$, the cusp visible in the experimentally measured spectrum [1].

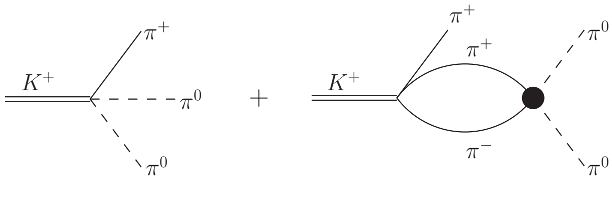

Figure 1: "Direct" and "rescattering" contribution to the decay $K^+ \to \pi^0\pi^0\pi^+$. The black dot marks the charge-exchange $\pi\pi$ scattering vertex proportional to the scattering lengths at threshold.

What is more, the strength of this cusp is proportional to the charge-exchange $\pi\pi$ scattering amplitude at threshold, that is, to the combination of scattering lengths $a_0^0 - a_0^2$ (up to isospin-breaking corrections), for which a very precise theoretical prediction exists [3]. An investigation of this cusp effect could therefore lead to a new method to determine the $\pi\pi$ scattering lengths, provided a theoretical framework can be devised that allows to match the precision of the data.

The aim of the development of such a theoretical representation of the $K \to 3\pi$ decay amplitude is to parameterize it *directly* in terms of scattering lengths (and higher-order threshold parameters) as well as $K \to 3\pi$ tree-level coupling constants that replace the conventional, polynomial-type Dalitz-plot parameters. This is in marked contrast to ChPT, where the scattering lengths are calculated perturbatively in an expansion in quark masses, and more akin to the theory of hadronic atoms [4], where a non-relativistic effective field theory (NREFT) is used to relate decay widths and energy-level shifts to scattering amplitudes at threshold. A similar NREFT has been developed for the analysis of cusp effects [5, 6, 7] (see also the review in [8]), with the difference that relativistic recoil corrections are fully retained, and only particle-pair creation is neglected to avoid mass-renormalization effects. The effective theory is then constructed as a double-expansion in $\pi\pi$ threshold parameters (collectively denoted by a) and a non-relativistic parameter $\varepsilon \propto |\mathbf{p}_\pi|/M_\pi$. This expansion has been performed completely up to $\mathscr{O}(a^0\varepsilon^4, a^1\varepsilon^5, a^2\varepsilon^4)$ [7], that is to two loops, including the appropriate number of derivative interactions. Via two-loop effects, the decay amplitude even shows a sub-leading dependence on a_0^2, although with reduced precision.

Finally, radiative corrections have a surprisingly large impact on the scattering-length extraction from the cusp effect [9]: Coulomb-photon-exchange inside the charged-pion loop in Fig. 1 modifies the analytic structure near threshold according to

$$i\, v_\pm(s_3) \longrightarrow i\, v_\pm(s_3) - \frac{\alpha}{2}\log\big(-v_\pm^2(s_3)\big) + \dots \; , \tag{2.2}$$

so while suppressed by the fine-structure constant $\alpha = e^2/4\pi$, radiative corrections induce a logarithmic singularity near the charged-pion threshold, right where the sensitivity of the amplitude to $\pi\pi$ rescattering is largest. Taking these into account, the NA48/2 collaboration found [10]

$$\begin{aligned} a_0^0 - a_0^2 &= 0.2571 \pm 0.0048_{\text{stat}} \pm 0.0025_{\text{syst}} \pm 0.0014_{\text{ext}} \; , \\ a_0^2 &= -0.024 \pm 0.013_{\text{stat}} \pm 0.009_{\text{syst}} \pm 0.002_{\text{ext}} \; , \end{aligned} \tag{2.3}$$

in an analysis of $K^\pm \to \pi^\pm\pi^0\pi^0$ decays, which agrees beautifully with the theoretical prediction $a_0^0 - a_0^2 = 0.265 \pm 0.004$, $a_0^2 = -0.0444 \pm 0.0010$ [3]. Similar cusp effects have also been predicted in other decay modes such as $K_L \to 3\pi^0$, $\eta \to 3\pi^0$ [6], and $\eta' \to \eta\pi^0\pi^0$ [11].

3. Dalitz-plot parameters in $\eta \to 3\pi$

With a representation for final-state interactions up to two loops at hand, including all effects due to the different masses of charged and neutral pions, one may try to find whether this can also be used to investigate other decays into three pions. The special interest in the decay $\eta \to 3\pi$ derives from ChPT: the decay violates isospin symmetry, and electromagnetic effects have been shown to be strongly suppressed [12, 13, 14], such that it offers potentially clean access to the quark mass difference $m_u - m_d$. Indeed, at leading order in the chiral expansion, the amplitude for the charged final state $\eta \to \pi^+\pi^-\pi^0$ is given by

$$\mathscr{M}_c^{\mathrm{LO}}(s,t,u) = \frac{B(m_d - m_u)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}\right\} , \tag{3.1}$$

where $s = (p_{\pi^+} + p_{\pi^-})^2$, $t = (p_{\pi^-} + p_{\pi^0})^2$, $u = (p_{\pi^+} + p_{\pi^0})^2$, $3s_0 = s+t+u = M_\eta^2 + 3M_\pi^2$. At first order in isospin breaking, the corresponding amplitude for the neutral final state $\eta \to 3\pi^0$ is given by $\mathscr{M}_n(s,t,u) = -\mathscr{M}_c(s,t,u) - \mathscr{M}_c(t,u,s) - \mathscr{M}_c(u,s,t)$. However, large higher-order corrections render the extraction of the normalization of the amplitude, and therefore of information on the light quark masses, from experimental data difficult; in order to do this reliably, clearly one has to achieve a very good description of the Dalitz plot distribution (compare also other current theoretical efforts in Refs. [15, 16]). The latter is, for $\eta \to 3\pi$, conventionally described as an expansion around its center in terms of the normalized variables[1]

$$x = \frac{t-u}{\sqrt{3}R_\eta} , \quad y = \frac{s_0 - s}{R_\eta} , \quad z = x^2 + y^2 , \quad R_\eta = \frac{2}{3}M_\eta(M_\eta - 3M_\pi) , \tag{3.2}$$

defining the leading Dalitz plot parameters a, b, d, and α according to

$$|\mathscr{M}_c(x,y)|^2 = |\mathscr{N}_c|^2\big\{1 + ay + by^2 + dx^2 + \dots\big\} , \quad |\mathscr{M}_n(z)|^2 = |\mathscr{N}_n|^2\big\{1 + 2\alpha z + \dots\big\} . \tag{3.3}$$

In particular the neutral slope parameter α has been a point of major concern over the last few years, as shown in Fig. 2: while various very precise experiments converge beautifully to a value of $\alpha = (-31.7 \pm 1.6) \times 10^{-3}$, the parameter-free prediction in ChPT at one-loop is positive [18], as is the central value at two loops [19]; a dispersive calculation matched to ChPT at least leads to a negative sign [20]. Matching the NREFT tree-level couplings to one-loop ChPT, we find the following decomposition of α [17]:

$$\alpha = \big(+10.7_{\text{tree}} + 12.4_{\text{1-loop}} - 44.1_{\text{2-loop}} - 6.0_{\text{higher}} - 0.6_{\text{iso-break}}\big) \times 10^{-3} = (-24.6 \pm 4.9) \times 10^{-3} , \tag{3.4}$$

where higher-order corrections are estimated by single-channel "bubble-sum" resummation and yield, besides different $\pi\pi$ scattering parameterizations, the quoted uncertainty; higher orders in isospin-breaking are very small. The NREFT power counting helps explain these seemingly surprising numbers: as the one-loop contributions to the amplitude are purely imaginary in this scheme, one- and two-loop corrections appear at the same order $\mathscr{O}(a^2\varepsilon^2)$, only higher loops are suppressed; both are enhanced in ε versus the tree-level terms, which are $\mathscr{O}(\varepsilon^4)$. The total result is marginally compatible with the experimental determination.

Finally, we wish to show the significance of the imaginary parts in the decay amplitudes. Expanding the *amplitudes* (as opposed to their squared moduli) around the Dalitz plot center,

$$\mathscr{M}_c(x,y) = \mathscr{N}_c\big\{1 + \bar{a}y + \bar{b}y^2 + \bar{d}x^2 + \dots\big\} , \quad \mathscr{M}_n(z) = \mathscr{N}_n\big\{1 + \bar{\alpha}z + \dots\big\} , \tag{3.5}$$

comparison to Eq. (3.3) immediately demonstrates $a = 2\mathrm{Re}\,\bar{a}$, $b = |\bar{a}|^2 + 2\mathrm{Re}\,\bar{b}$, $d = 2\mathrm{Re}\,\bar{d}$, $\alpha = \mathrm{Re}\,\bar{\alpha}$. The isospin relation between $\mathscr{M}_c$ and $\mathscr{M}_n$ then leads to [19]

$$\alpha = \frac{1}{4}\Big(b + d - \frac{a^2}{4} - (\mathrm{Im}\,\bar{a})^2\Big) < \frac{1}{4}\Big(b + d - \frac{a^2}{4}\Big) , \tag{3.6}$$

[1] Here and in the following, for simplicity we neglect corrections induced by the pion-mass difference, which are meticulously traced in Ref. [17]. Numerical results shown here refer to the exact relations.

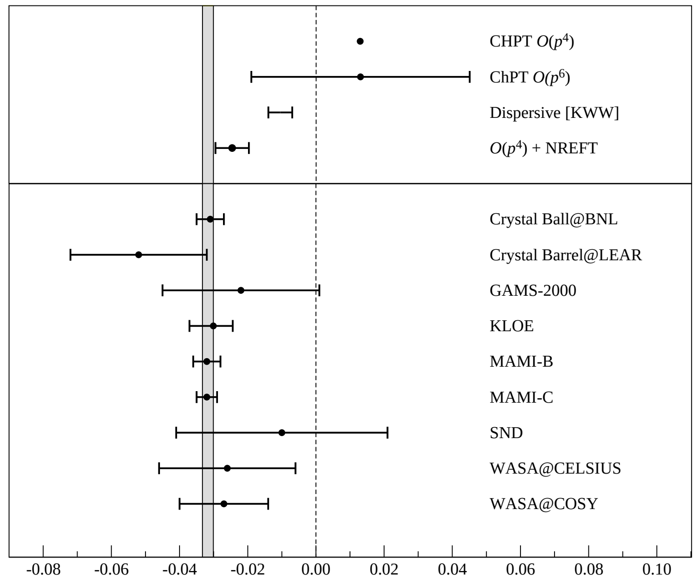

Figure 2: Comparison of values for the slope parameter α. Top: theoretical predictions. Bottom: experimental determinations. The gray shaded area is the particle data group average. Figure taken from Ref. [17].

hence there is only an *inequality* between the Dalitz plot parameters of both channels. However, the imaginary part of $\bar{a}$ is generated purely by final-state interactions and thereby determined by the same tree-level couplings that fix the real parts: we can reformulate this relation as [17]

$$\alpha = \frac{1}{4}\left(b + d - \frac{a^2}{4}\right) - \zeta_1(1+\zeta_2 a)^2 \ , \quad \zeta_1 = 0.050 \pm 0.005 \ , \quad \zeta_2 = 0.225 \pm 0.003 \ , \tag{3.7}$$

with $\zeta_{1/2}$ determined purely by $\pi\pi$ rescattering. The most precise experimental measurement of the charged Dalitz plot parameters [21], however, is compatible with $\zeta_1 = 0$ or no imaginary part in $\bar{a}$ at all, at clear odds with Eq. (3.7). Our analysis therefore points towards a significant tension between the measured parameters of the two different $\eta \to 3\pi$ final states.

4. Dispersion relations for $\omega \to 3\pi$ and $\phi \to 3\pi$

When proceeding to three-pion decays of somewhat heavier mesons, like those of the lightest isoscalar vector mesons, ω and ϕ, it is obvious that perturbative treatments of final-state interactions are doomed to fail, as the influence of the ρ resonance is already significant (ω) or even falls inside the Dalitz plot (ϕ). A method to resum rescattering effects between all three pions non-perturbatively is given by dispersion relations. One starts by decomposing the amplitude $\mathscr{M}(s,t,u)$ according to

$$\mathscr{M}(s,t,u) = i\,\varepsilon_{\mu\nu\alpha\beta} n^\mu p^\nu_{\pi^+} p^\alpha_{\pi^-} p^\beta_{\pi^0} \mathscr{F}(s,t,u) \ , \tag{4.1}$$

where n^μ is the polarization vector of the decaying ω/ϕ. Due to Bose symmetry, only partial waves of odd angular momentum contribute; neglecting discontinuities of F- and higher partial waves, $\mathscr{F}(s,t,u)$ can be further decomposed as $\mathscr{F}(s,t,u) = \mathscr{F}(s) + \mathscr{F}(t) + \mathscr{F}(u)$. The unitarity relation for $\mathscr{F}(s)$, assuming elastic final-state interactions, then leads to the following expression for the discontinuity of $\mathscr{F}(s)$:

$$\operatorname{disc}\mathscr{F}(s) = 2i\big\{\mathscr{F}(s)+\hat{\mathscr{F}}(s)\big\}\times\theta\big(s-4M_\pi^2\big)\times\sin\delta_1^1(s)e^{-i\delta_1^1(s)} \,, \tag{4.2}$$

where $\delta_1^1(s)$ is the $\pi\pi$ P-wave phase shift. Were it not for the *inhomogeneities* $\hat{\mathscr{F}}(s)$, Eq. (4.2) would correspond to the discontinuity equation of the vector form factor, which is solved by the Omnès function

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds'\frac{\delta_1^1(s')}{s'(s'-s)}\right\} . \tag{4.3}$$

The function $\hat{\mathscr{F}}(s)$ is given by angular averages over $\mathscr{F}$ according to

$$\hat{\mathscr{F}}(s) = 3\big\langle(1-z^2)\mathscr{F}\big\rangle(s) \,, \qquad \big\langle z^n f\big\rangle(s) = \frac{1}{2}\int_{-1}^1 dz\, z^n f\Big(\tfrac{1}{2}\big(3s_0-s+z\kappa(s)\big)\Big) \,,$$
$$s_0 = \frac{1}{3}\big(M_V^2+3M_\pi^2\big) \,, \qquad \kappa(s) = \lambda^{1/2}(M_V^2,M_\pi^2,s)\sqrt{1-\frac{4M_\pi^2}{s}} \,, \tag{4.4}$$

where $\lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$, and M_V is the mass of the decaying vector meson. The angular integration including the $\kappa(s)$ function is non-trivial and generates a complex analytic structure, including three-particle cuts due to the fact that ω and ϕ are unstable and decay [22]. The analog to the Omnès solution (4.3) are then integral equations involving the inhomogeneities

$$\mathscr{F}(s) = \Omega_1^1(s)\left\{a+\frac{s}{\pi}\int_{4M_\pi^2}^\infty\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathscr{F}}(s')}{|\Omega_1^1(s')|(s'-s)}\right\} , \tag{4.5}$$

with the subtraction constant a. The number of subtractions is chosen such that the dispersion integral is guaranteed to converge.

Equations (4.4) and (4.5) can be solved iteratively: starting from an arbitrary input function $\mathscr{F}(s)$, we can calculate the inhomogeneity $\hat{\mathscr{F}}(s)$ according to Eq. (4.4), from which a new $\mathscr{F}(s)$ is obtained from Eq. (4.5); the procedure is stopped once a fixed point of the iteration is reached with sufficient accuracy. In the example discussed here, see Eq. (4.5), the subtraction constant works as an overall normalization factor of the solution; we match it to the partial decay width, but note that a *normalized* Dalitz plot distribution is subsequently a pure prediction. While the result is independent of the starting function, for the case at hand, we choose $\mathscr{F}(s) = \Omega_1^1(s)$ in order to allow us to quantify crossed-channel effects (generated by the iteration) in a plausible way.

Figure 3 shows the result of such an iteration for the decay $\phi\to 3\pi$: it converges fast, with the third iteration already all but indistinguishable from the final result. The difference to the starting point of the iteration, the Omnès function without any crossed-channel rescattering, is however very significant. The picture for $\omega\to 3\pi$ (not shown here) is qualitatively very similar, with convergence reached even faster (after two iterations, see Ref. [22]).

The resulting Dalitz plots for both $\omega\to 3\pi$ and $\phi\to 3\pi$ are shown in Fig. 4, normalized by the P-wave phase space factor, using the kinematical variables x and y defined in analogy to Eq. (3.2). Comparison to the experimental $\phi\to 3\pi$ Dalitz plot of Ref. [23] shows that crossed-channel effects improve the reduced χ^2 from $1.71\ldots 2.06$ (with $\mathscr{F}(s) = a\,\Omega_1^1(s)$) to $1.17\ldots 1.50$; further improvement and perfect agreement with the data can be achieved by introducing an additional subtraction

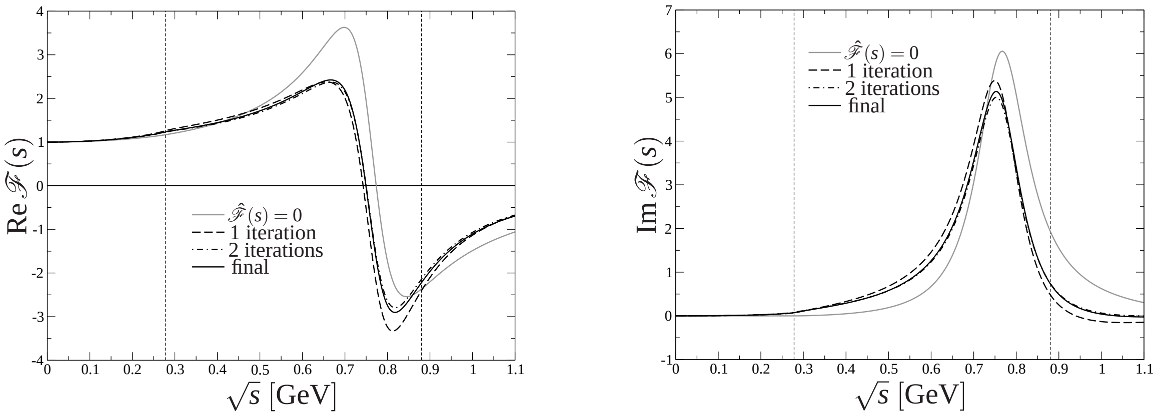

Figure 3: Successive iteration steps of real (left panel) and imaginary (right panel) part of the amplitude $\mathscr{F}(s)$ for $\phi \to 3\pi$. The vertical dashed lines denote the physical region of the decay.

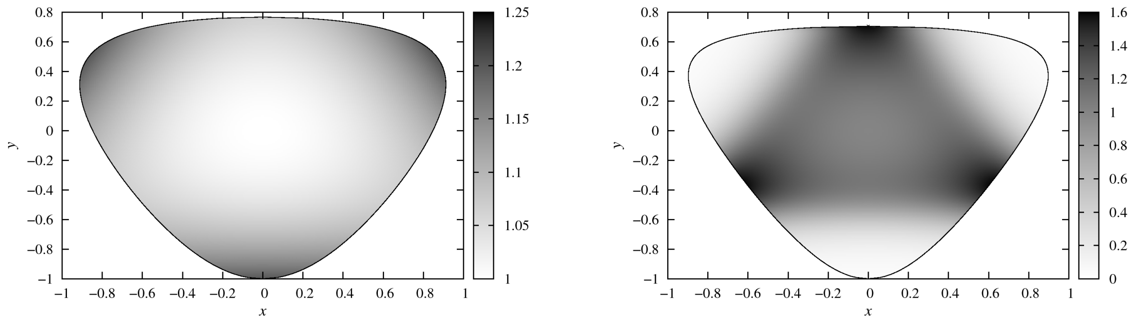

Figure 4: Dalitz plots for $\omega \to 3\pi$ (left) and $\phi \to 3\pi$ (right), normalized by the P-wave phase space.

constant in Eq. (4.5). The $\omega \to 3\pi$ and $\phi \to 3\pi$ decay amplitudes constructed in Ref. [22] have subsequently also been used as input in a dispersive analysis of the transition form factors as measured in the decays $\omega \to \pi^0\ell^+\ell^-$, $\phi \to \pi^0\ell^+\ell^-$ [24].

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