PROCEEDINGS OF SCIENCE

Three particles in a finite volume

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Within the non-relativistic potential scattering theory, we derive a generalized version of the Lüscher formula, which includes three-particle inelastic channels. We show that, even in the presence of the three-particle intermediate states, the discrete spectrum in a finite box is determined by the infinite-volume elements of the scattering S-matrix up to corrections, exponentially suppressed in large volumes.

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1. Introduction

The case of the elastic low-lying resonances on the lattice has been investigated in detail. Namely, the Lüscher formula [2] enables one to uniquely relate the discrete energy levels in a finite box to the elastic scattering phase shift in the infinite volume, measured at the same energy. This eventually opens the way for the extraction of the parameters of the elastic resonances – their masses and widths – in the lattice QCD (for illustration see, e.g., Refs. [3, 4]).

The case of the inelastic resonances however, is more complicated. Inelastic resonances, such as the Roper resonance $N(1440)$, have a significant decay rate into the three and more particle final states. *A priori,* in such a system one may expect significant finite-volume effects, which can not be evaluated by using the standard Lüscher approach. Thus, it is highly desirable to construct a framework that will allow one to systematically calculate the finite-volume effects, coming from the tree-body final states.

Formulating a counterpart of the Lüscher approach in a three-body case represents a major challenge. For this reason, at the first step, we want to simplify the problem as much as possible. Namely, we consider a non-relativistic quantum-mechanical model with coupled two-particle and three-particle channels. For three spinless non-identical particles with the masses $m_\alpha,\ \alpha=1,2,3$ the Hamiltonian of the model is given by a sum

$$\mathbf{H} \,=\, \mathbf{H}_0 + \mathbf{H}_{2\to 2} + \mathbf{H}_{2\to 3} \doteq \mathbf{H}_0 + \mathbf{H}_{\mathsf{I}}\,, \tag{1.1}$$

where $\mathbf{H}_0$ is a free Hamiltonian, $\mathbf{H}_{2\to 2}$ the pair interaction Hamiltonian and $\mathbf{H}_{2\to 3}$ describes the transition from two- to three-particle state $1+2\to 1+2+3$. Starting from the Faddeev formalism for three particles in the infinite volume the model can be reformulated [1] in a finite cubic box of a size L.

2. An alternative derivation of the Lüscher Equation via the splitting of the two-particle Green's function

In order to find the energy spectrum in a finite volume, one can try to solve the finite volume Faddeev equations numerically applying, e.g., the method of Ref. [6]. However, these equations contain potentials as well as off-shell two-body scattering matrices, which are model-dependent. The central question is, whether the predicted energy levels are also model-dependent. In other words, if two different potential models lead to the same S-matrix in the infinite volume, can the finite-volume spectra in these models be different.

In case of the two-particle elastic scattering, the answer is given by the Lüscher formula, which relates the finite-volume spectrum to the (on-shell) S-matrix element. Our aim is to rewrite the three-particle equations in a finite volume in a similar fashion, in terms of the on-shell S-matrix elements only.

To see this let us first consider a free finite volume two-particle Green's function in momentum space

$$G_0^L(\mathbf{k};z) = \frac{2\mu}{L^3}\sum_{\mathbf{p}}\frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})}{\mathbf{p}^2-q_0^2}\,, \tag{2.1}$$

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where μ is the reduced mass and $z = m_1 + m_2 + \dfrac{q_0^2}{2\mu}$. Applying the regular summation theorem [5] the above equation can be decomposed into an infinite volume contribution $G_\mathsf{K} = G_0 + G_\mathsf{U}$ and the finite volume corrections G_F

$$\begin{aligned} G_0^L(\mathbf{k};z) &= 2\mu\left\{\frac{1}{\mathbf{k}^2 - q_0^2 - i0} - i\pi\Delta(\mathbf{k}^2, q_0^2) + \Delta(\mathbf{k}^2, q_0^2)\sum_{lm} Y_{lm}^*(\hat{\mathbf{k}})\frac{2}{\nu^{l+1}} Z_{lm}(1;\nu^2)\right\} \\ &\doteq G_0(\mathbf{k};z) + G_\mathsf{U}(\mathbf{k};z) + G_\mathsf{F}(\mathbf{k};z) \doteq G_\mathsf{K}(\mathbf{k};z) + G_\mathsf{F}(\mathbf{k};z)\,, \end{aligned} \tag{2.2}$$

where $\nu = q_0 L/(2\pi)$, and $Z_{lm}(1;\nu^2)$ stands for the Lüscher zeta-function

$$Z_{lm}(1;\nu^2) = \lim_{\lambda\to\infty}\left\{\sum_{\mathbf{n}\in\mathbb{Z}^3}\theta(\lambda^2 - \mathbf{n}^2)\frac{\mathscr{Y}_{lm}(\mathbf{n})}{\mathbf{n}^2 - \nu^2} - \delta_{l0}\delta_{m0}\sqrt{4\pi}\lambda\right\}, \qquad \lambda = \frac{\Lambda L}{2\pi}\,. \tag{2.3}$$

The splitting of the Green's function is understood in terms of a distribution which is integrated with a regular function (such as a potential) and the quantity $\Delta(\mathbf{p}^2, q_0^2)$ is defined in the following manner

$$-i\pi\int\frac{d^3\mathbf{p}}{(2\pi)^3}\Delta(\mathbf{p}^2, q_0^2)\Phi(\mathbf{p}) = \frac{\sqrt{-q_0^2 - i0}}{(4\pi)^{3/2}}\Phi_0(q_0) f(q_0^2/\mu^2), \tag{2.4}$$

which acts similar to the usual δ-Dirac distribution in the sense that it projects the quantities $G_\mathsf{U}(\mathbf{k};z)$ and $G_\mathsf{F}(\mathbf{k};z)$ on the energy shell. The function f serves the purpose of a regulator such that it is $f(q_0^2/\mu^2) = 1$ for $q_0^2 \geq 0$ and effectively cuts the contributions for $-q_0^2 > \mu^2$. Note that the presence of the cutoff function in a two-particle case is not essential, because q_0 is an external parameter. For this reason, one may, e.g., make a simple choice $f(x) = 1$ here.

Next, we define

$$\begin{aligned} \mathbf{T} &= (-\mathbf{V}) + (-\mathbf{V})\mathbf{G}_0\mathbf{T}\,, \\ \mathbf{K} &= (-\mathbf{V}) + (-\mathbf{V})(\mathbf{G}_0 + \mathbf{G}_\mathsf{U})\mathbf{K} = \mathbf{T} + \mathbf{T}\mathbf{G}_\mathsf{U}\mathbf{K}\,, \\ \mathbf{T}^L &= (-\mathbf{V}) + (-\mathbf{V})(\mathbf{G}_0 + \mathbf{G}_\mathsf{U} + \mathbf{G}_\mathsf{F})\mathbf{T}^L = \mathbf{K} + \mathbf{K}\mathbf{G}_\mathsf{F}\mathbf{T}^L\,. \end{aligned} \tag{2.5}$$

It is immediately seen that $\mathbf{T}$ and $\mathbf{K}$ are the two-body T- and K-matrices in the infinite volume, respectively. After carrying out the partial-wave expansion and integrating over the angles, the last line of Eq. (2.5) can be solved on shell by requiring that the resulting system of linear equations is singular. This procedure yields the Lüscher formula in terms of the on-shell K-matrix which is related to the infinite volume scattering phase by

$$\tan\delta_l(q_0) = \frac{\mu p}{2\pi} K_l(q_0, q_0; z)\,. \tag{2.6}$$

3. Splitting of the three-particle Green's function

In analogy to the two-particle case discussed above we can apply the same idea of the splitting into the infinite and finite volume contributions to the three-particle Green's function. Due to the presence of a third particle and an additional momentum which is also summed over the derivation

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of the splitting is somewhat more complicated. We start with the definition of the momentum space three-particle Green's function in a channel $\alpha = 1,2,3$ (in order to ease the notations, we suppress the channel index α in all momenta)

$$G^L_{0\alpha}(\mathbf{k},\mathbf{l};z) \;=\; \frac{1}{L^6}\sum_{\mathbf{pq}} \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})(2\pi)^3\delta^3(\mathbf{q}-\mathbf{l})}{M+\dfrac{\mathbf{p}^2}{2M_\alpha}+\dfrac{\mathbf{q}^2}{2\mu_\alpha}-z}\,,$$

$$\mathbf{q} \;=\; \tilde{\mathbf{p}}+\frac{m_\beta}{m_\beta+m_\gamma}\,\mathbf{p}\,, \qquad (\mathbf{p},\tilde{\mathbf{p}}) = \frac{2\pi}{L}\,(\mathbf{n},\tilde{\mathbf{n}})\,, \quad \mathbf{n},\tilde{\mathbf{n}}\in\mathbb{Z}^3\,, \tag{3.1}$$

where $\mu_\alpha = m_\beta m_\gamma/(m_\beta+m_\gamma)$ is the reduced mass and the summation is carried out over the integers $\mathbf{n},\tilde{\mathbf{n}}$. The first step consists in rewriting of the above equation in such a form that it partly resembles an effective two-body Green's function

$$G^L_{0\alpha}(\mathbf{k},\mathbf{l};z) \;=\; \frac{1}{L^3}\sum_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\,\frac{2\mu_\alpha}{L^3}\sum_{\mathbf{p}_1}\frac{(2\pi)^3\delta^3(\mathbf{q}-\mathbf{l})}{\mathbf{q}^2-q^2_{0\alpha}} \tag{3.2}$$

with

$$q^2_{0\alpha} = 2\mu_\alpha\left(z-M-\frac{\mathbf{p}^2}{2M_\alpha}\right), \qquad \nu_\alpha = \frac{q_{0\alpha}L}{(2\pi)}\,, \qquad \mathbf{a} = \frac{m_\beta}{(m_\beta+m_\gamma)}\,\frac{\mathbf{p}L}{(2\pi)}\,. \tag{3.3}$$

Afterwards one may proceed in a similar manner outlined in the previous section and make use of the regular summation theorem. Then the three-particle Green's function can be decomposed [1] into the infinite-volume contribution $G_{\mathsf{K}\alpha} = G_{0\alpha}+G_{\mathsf{U}\alpha}$ and the finite volume corrections $G_{\mathsf{F}\alpha}$

$$\begin{aligned}
G^L_{0\alpha}(\mathbf{k},\mathbf{l};z) \;&=\; \frac{1}{M+\dfrac{\mathbf{k}^2}{2M_\alpha}+\dfrac{\mathbf{l}^2}{2\mu_\alpha}-z-i0} - i\pi\Delta\left(\mathbf{l}^2,q^2_{0\alpha}\right) \\
&+\; \frac{1}{L^3}\sum_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\,\Delta\left(\mathbf{l}^2,q^2_{0\alpha}\right)\sum_{lm}Y^*_{lm}(\hat{\mathbf{l}})\,\frac{2}{\nu^l_\alpha}Z^{\mathbf{a}}_{lm}(1;\nu^2_\alpha) \\
&\doteq\; G_{0\alpha}(\mathbf{k},\mathbf{l};z)+G_{\mathsf{U}\alpha}(\mathbf{k},\mathbf{l};z)+G_{\mathsf{F}\alpha}(\mathbf{k},\mathbf{l};z) \\
&\doteq\; G_{\mathsf{K}\alpha}(\mathbf{k},\mathbf{l};z)+\frac{1}{L^3}\sum_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p},\mathbf{l};z)\,,
\end{aligned} \tag{3.4}$$

where due to the additional momentum we have to deal with the Lüscher zeta-function in the moving frame

$$Z^{\mathbf{a}}_{lm}(1;\nu^2_\alpha) = \lim_{\lambda\to\infty}\left\{\sum_{\mathbf{n}\in\mathbb{Z}^3}\theta(\lambda^2-(\mathbf{n}+\mathbf{a})^2)\,\frac{\mathscr{Y}_{lm}(\mathbf{n}+\mathbf{a})}{(\mathbf{n}+\mathbf{a})^2-\nu^2_\alpha} - \delta_{l0}\delta_{m0}\sqrt{4\pi}\lambda\right\}. \tag{3.5}$$

Again the splitting in Eq. (3.4) is understood as a distribution which is integrated with a regular function of the momentum variables $\mathbf{k}$ and $\mathbf{l}$ and $\Delta\left(\mathbf{l}^2,q^2_{0\alpha}\right)$ projects the quantities $G_{\mathsf{U}\alpha}$ and $G_{\mathsf{F}\alpha}$ on the energy shell. Further, since the variables $\mathbf{k}$ and $\mathbf{p}$ are not restricted from above, the argument of the distribution Δ can become positive and arbitrarily large, independent of the choice of the variables $\mathbf{l}^2$ and z. This means that one has to necessarily deal with the analytic continuation of

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the pertinent amplitudes below threshold. This problem occurs merely in the three-particle case and serves us as justification to introduce the regulator function f which cuts the sub-threshold contribution for the momenta $-q_{0\alpha}^2 > \mu^2$. Thus, assuming the range of the potentials much smaller than the inverse of the lightest mass in the system, one does not expect to encounter any singularities in the analytic continuation for $-q_{0\alpha}^2 < \mu^2$.

Using the representation of the three-particle Green's function, given in Eq. (3.4), one may try to apply the splitting, as it was done in the two-particle case directly to the three-particle T-matrix. However, in the three-particle case, due to the presence of the disconnected contributions - the diagrams where a spectator particle propagates freely, when the other two particles interact - one accounts new complications. Namely, these diagrams generate the factor $L^3\delta_{\mathbf{p}_\alpha\mathbf{q}_\alpha}$ ($(2\pi)^3\delta^3(\mathbf{p}_\alpha-\mathbf{q}_\alpha)$) in a finite (infinite) volume which is not a regular function and is L-dependent (in a finite volume). Consequently, the splitting in the T - matrix, can be applied here first after these disconnected contributions are removed. Below we briefly discuss the way it can be done.

4. The three-particle counterpart of the Lüscher formula

In order to derive the three-particle analog of the Lüscher formula, we have to deal with the three-body equations in the presence of the three-particle force[1]. To this end, one may use, e.g., the formalism described in the papers [7]. However, in the presence of the Kronecker-δ, contained in the disconnected parts, the use of the splitting procedure for the three-particle propagator according to Eq. (3.4) can not be justified mathematically. In order to circumvent this problem, we act by using a trial and error method. Namely, we first apply the splitting in the three-body Lippmann-Schwinger equations, as if there were no disconnected parts, and further use the Faddeev trick. In the resulting equations, the disconnected terms, containing the δ-functions, emerge in a finite volume. At the next stage, we discard these singular terms by hand, thus making a *conjecture* about the correct form of the equations. At the final step, we check this conjecture explicitly, by considering the multiple-scattering series that emerge from the resulting equations, and showing that this series coincides with the original multiple-scattering series in a finite volume.

Symbolically, the result for the effective two-body K-matrix can be written as follows

$$\mathbf{K}^L = \mathbf{K}_{2\to2} + \mathbf{K}_{2\to3}(\mathbf{G}_\mathsf{F} + \mathbf{G}_\mathsf{F}\mathbf{R}_\mathsf{F}\mathbf{G}_\mathsf{F})\mathbf{K}_{3\to2}\,, \tag{4.1}$$

where $\mathbf{K}_{i\to j}$ denote the pertinent K-matrix elements in the infinite volume[2], $\mathbf{G}_\mathsf{F}$ stands for the finite-volume part of the three-particle Green's function (see Eq. (3.4)), and the quantity $\mathbf{R}_\mathsf{F}$ is given by

$$\mathbf{R}_\mathsf{F} = \sum_{\mu,\nu=1}^{4}\mathbf{R}_{\mu\nu} + \sum_{\alpha=1}^{3}\theta_\alpha\,, \qquad \mathbf{R}_{4\beta} = \theta_4\mathbf{G}_\mathsf{F}\left(\theta_\beta + \sum_{\gamma=1}^{3}\mathbf{R}_{\gamma\beta}\right), \qquad \mathbf{R}_{44} = \theta_4 + \theta_4\mathbf{G}_\mathsf{F}\sum_{\gamma=1}^{3}\mathbf{R}_{\gamma4}\,,$$

$$\mathbf{R}_{\alpha\beta} = \theta_\alpha\mathbf{G}_\mathsf{F}\left(\sum_{\gamma=1}^{3}(1-\delta_{\alpha\gamma})\mathbf{R}_{\gamma\beta} + \mathbf{R}_{4\beta}\right), \qquad \mathbf{R}_{\alpha4} = \theta_\alpha\mathbf{G}_\mathsf{F}\sum_{\gamma=1}^{3}(1-\delta_{\alpha\gamma})\mathbf{R}_{\gamma4} + \theta_\alpha\mathbf{G}_\mathsf{F}\mathbf{R}_{44}\,. \tag{4.2}$$

[1]Although our model in Eq. (1.1) does not implicitly include the three particle force it can be added without much effort. In fact in our calculations [1] we show that an induced three -body force arises naturally.

[2]Above threshold, the three-body K-matrix coincides with the one defined, e.g., in Refs. [8, 9].

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Here, $\mu,\nu = 1,\cdots,4$, whereas $\alpha,\beta,\gamma = 1,\cdots,3$ and

$$\theta_\mu = \mathbf{K}_\mu + \mathbf{K}_\mu \mathbf{G}_\mathsf{F} \theta_\mu\,, \qquad \mathbf{K}_\mu = (-\mathbf{V}_\mu) + (-\mathbf{V}_\mu)(\mathbf{G}_0 + \mathbf{G}_\mathsf{U})\mathbf{K}_\mu\,, \qquad \mathbf{K}_{3\to 3} = \sum_{\mu=1}^{4} \mathbf{K}_\mu\,. \quad (4.3)$$

Note that in definition of $\mathbf{R}_{\alpha\beta}$, the terms of the type $\theta_\alpha \mathbf{G}_\mathsf{F} \theta_\beta$ with $\alpha \neq \beta$ have been omitted. Physically, such terms correspond to the finite-volume corrections in the disconnected diagrams and emerge, if one faithfully applies the splitting procedure even to the disconnected piece. This omission can be justified by showing that Eq. (4.2) produces – diagram by diagram – the correct splitting of the infinite- and finite-volume parts in the multiple-scattering series. The physical meaning of this prescription is very transparent. The finite-volume corrections emerge only in the loop diagrams. However, due to the presence of the Kronecker-delta in the disconnected diagrams, a first iteration of the disconnected diagrams in the Faddeev equations gives a connected diagram without a loop. Its explicit expression is identical in a finite and the infinite volumes. The loops (and, consequently, the finite-volume corrections) emerge first in the second iteration.

5. Concluding Remarks

In this letter we have sketched the steps of the derivation of the three-body counterpart of the Lüscher formula. We find that the fundamental property of the finite-volume spectrum, which follows from this formula, is that the spectrum is completely determined by the S-matrix elements for the transitions $2\to 2$, $2\to 3$ and $3\to 3$ in the infinite volume. Consequently, two different potential models with the same S-matrix elements lead to the same spectra up to the exponentially suppressed corrections.

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