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Application of a light-front coupled-cluster method to quantum electrodynamics

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A field-theoretic formulation of the exponential-operator technique is applied to a Hamiltonian eigenvalue problem in electrodynamics, quantized in light-front coordinates. Specifically, we consider the dressed-electron state, without positron contributions but with an unlimited number of photons, and compute its anomalous magnetic moment. A simple perturbative solution immediately yields the Schwinger result of $\alpha/2\pi$. The nonperturbative solution, which requires numerical techniques, sums a subset of corrections to all orders in α and incorporates additional physics.

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1. Introduction

Although the nonperturbative light-front coupled-cluster (LFCC) method [1] is intended for strongly coupled theories, where perturbation theory is of limited use, we explore its utility in the context of a gauge theory by considering the dressed-electron state in quantum electrodynamics (QED) [2]. The method requires the light-front coordinates of Dirac [3, 4], where the Hamiltonian evolves a state along the time direction $x^+ = t + z$. The spatial coordinates are $\underline{x} = (x^- \equiv t - z, \vec{x}_\perp \equiv (x,y))$. The light-front energy conjugate to the chosen time is $p^- \equiv E - p_z$, and the corresponding light-front momentum is $\underline{p} = (p^+ \equiv E + p_z, \vec{p}_\perp \equiv (p_x, p_y))$. In these coordinates, the fundamental Hamiltonian eigenvalue problem is $\mathscr{P}^-|\psi\rangle = \frac{M^2+P_\perp^2}{P^+}|\psi\rangle$. Ordinarily, this eigenvalue problem is solved approximately by a truncated Fock-space expansion of the eigenstate. The LFCC method solves the problem without Fock-space truncation by building the eigenstate as $|\psi\rangle = \sqrt{Z}e^T|\phi\rangle$ from a valence state $|\phi\rangle$ and an operator T that increases particle number while conserving any quantum numbers of the valence state. The constant Z is a normalization factor.

The valence state is then an eigenstate of an effective Hamiltonian $\overline{\mathscr{P}^-} = e^{-T}\mathscr{P}^- e^T$, which can be computed using a Baker–Hausdorff expansion $\overline{\mathscr{P}^-} = \mathscr{P}^- + [\mathscr{P}^-, T] + \frac{1}{2}[[\mathscr{P}^-, T], T] + \cdots$. The new eigenvalue problem is $\overline{\mathscr{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle$. When projected onto the valence and orthogonal sectors, it becomes

$$P_v\overline{\mathscr{P}^-}|\phi\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi\rangle, \quad (1-P_v)\overline{\mathscr{P}^-}|\phi\rangle = 0, \tag{1.1}$$

where P_v is the projection onto the valence sector.

A matrix element such as $\langle\psi_2|\hat{O}|\psi_1\rangle$ can be calculated, with $|\psi_i\rangle = \sqrt{Z_i}e^{T_i}|\phi_i\rangle$ and $Z_i = 1/\langle\phi_i|e^{T_i^\dagger}e^{T_i}|\phi_i\rangle$. We define $\overline{O_i} = e^{-T_i}\hat{O}e^{T_i} = \hat{O}_i + [\hat{O}_i, T] + \frac{1}{2}[[\hat{O}_i, T], T] + \cdots$ and, to avoid the infinite sum in the denominator,

$$\langle\tilde{\psi}_i| \equiv \langle\phi|\frac{e^{T_i^\dagger}e_i^T}{\langle\phi|e^{T_i^\dagger}e^{T_i}|\phi\rangle} = Z_i\langle\phi|e^{T_i^\dagger}e^{T_i} = \sqrt{Z_i}\langle\psi_i|e^{T_i}. \tag{1.2}$$

We then have

$$\langle\psi_2|\hat{O}|\psi_1\rangle = \sqrt{Z_1/Z_2}\langle\tilde{\psi}_2|\overline{O_2}e^{-T_2}e^{T_1}|\phi_1\rangle = \sqrt{Z_2/Z_1}\langle\tilde{\psi}_1|\overline{O_1^\dagger}e^{-T_1}e^{T_2}|\phi_2\rangle^*, \tag{1.3}$$

and, therefore,

$$\langle\psi_2|\hat{O}|\psi_1\rangle = \sqrt{\langle\tilde{\psi}_2|\overline{O_2}e^{-T_2}e^{T_1}|\phi_1\rangle\langle\tilde{\psi}_1|\overline{O_1^\dagger}e^{-T_1}e^{T_2}|\phi_2\rangle^*}. \tag{1.4}$$

In the diagonal case, this reduces to

$$\langle\psi|\hat{O}|\psi\rangle = \langle\tilde{\psi}|\overline{O}|\phi\rangle. \tag{1.5}$$

The $\langle\tilde{\psi}_i|$ can be shown to be left eigenstates of the effective Hamiltonian.

We apply this to QED in an arbitrary covariant gauge, for which the Pauli–Villars-regulated Lagrangian is [5]

$$\begin{aligned}\mathscr{L} = & \sum_{i=0}^{2}(-1)^i\left[-\frac{1}{4}F_i^{\mu\nu}F_{i,\mu\nu} + \frac{1}{2}\mu_i^2 A_i^\mu A_{i\mu} - \frac{1}{2}\zeta\left(\partial^\mu A_{i\mu}\right)^2\right] \\ & + \sum_{i=0}^{2}(-1)^i\overline{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - e\overline{\psi}\gamma^\mu\psi A_\mu.\end{aligned} \tag{1.6}$$

Here the fundamental physical ($i=0$) and Pauli–Villars ($i=1$) fields appear in null combinations

$$\psi=\sum_{i=0}^{2}\sqrt{\beta_i}\psi_i,\;\; A_\mu=\sum_{i=0}^{2}\sqrt{\xi_i}A_{i\mu},\;\; F_{i\mu\nu}=\partial_\mu A_{i\nu}-\partial_\nu A_{i\mu}. \tag{1.7}$$

The coupling coefficients ξ_i and β_i are constrained by

$$\xi_0=1,\;\; \sum_{i=0}^{2}(-1)^i\xi_i=0,\;\; \beta_0=1,\;\; \sum_{i=0}^{2}(-1)^i\beta_i=0. \tag{1.8}$$

To fix ξ_2 and β_2, we require chiral symmetry restoration in the zero-mass limit [6] and a zero photon mass [7]. The light-front Hamiltonian, without antifermion terms, is then found to be [5]

$$\begin{aligned}\mathscr{P}^-=&\sum_{is}\int d\underline{p}\frac{m_i^2+p_\perp^2}{p^+}(-1)^i b_{is}^\dagger(\underline{p})b_{is}(\underline{p})+\sum_{l\lambda}\int d\underline{k}\frac{\mu_{l\lambda}^2+k_\perp^2}{k^+}(-1)^l\varepsilon^\lambda a_{l\lambda}^\dagger(\underline{k})a_{l\lambda}(\underline{k}) \\ &+\sum_{ijl\sigma s\lambda}\int dy d\vec{k}_\perp\int\frac{d\underline{p}}{\sqrt{16\pi^3 p^+}}\Big\{h_{ijl}^{\sigma s\lambda}(y,\vec{k}_\perp)a_{l\lambda}^\dagger(y,\vec{k}_\perp;\underline{p})b_{js}^\dagger(1-y,-\vec{k}_\perp;\underline{p})b_{i\sigma}(\underline{p}) \\ &\qquad+h_{ijl}^{\sigma s\lambda *}(y,\vec{k}_\perp)b_{i\sigma}^\dagger(\underline{p})b_{js}(1-y,-\vec{k}_\perp;\underline{p})a_{l\lambda}(y,\vec{k}_\perp;\underline{p})\Big\},\end{aligned} \tag{1.9}$$

with $\varepsilon^\lambda=(-1,1,1,1)$ and the $h_{ijl}^{\sigma s\lambda}$ known vertex functions.

2. The dressed-electron state

The right and left-hand valence states ($\overline{\mathscr{P}^-}$ is not Hermitian!) are $|\phi_a^\pm\rangle=\sum_i z_{ai}b_{i\pm}^\dagger(\underline{P})|0\rangle$ and $\langle\tilde{\phi}_a^\pm|=\langle 0|\sum_i\tilde{z}_{ai}b_{i\pm}(\underline{P})$. We approximate the T operator with the simplest form

$$\begin{aligned}T\;=&\sum_{ijls\sigma\lambda}\int dy d\vec{k}_\perp\int\frac{d\underline{p}}{\sqrt{16\pi^3}}\sqrt{p^+}t_{ijl}^{\sigma s\lambda}(y,\vec{k}_\perp)a_{l\lambda}^\dagger(yp^+,y\vec{p}_\perp+\vec{k}_\perp) \\ &\times b_{js}^\dagger((1-y)p^+,(1-y)\vec{p}_\perp-\vec{k}_\perp)b_{i\sigma}(\underline{p}).\end{aligned} \tag{2.1}$$

The effective Hamiltonian $\overline{\mathscr{P}^-}$ can then be constructed [2]. From this effective Hamiltonian, the right and left-hand valence-sector equations become, for $a=0,1$,

$$m_i^2 z_{ai}^\pm+\sum_j I_{ij}z_{aj}^\pm=M_a^2 z_{ai}^\pm \quad\text{and}\quad m_i^2\tilde{z}_{ai}^\pm+\sum_j(-1)^{i+j}I_{ji}\tilde{z}_{aj}^\pm=M_a^2\tilde{z}_{ai}^\pm, \tag{2.2}$$

with M_a the ath eigenmass and the self-energy given by

$$I_{ji}=(-1)^i\sum_{i'ls\lambda}(-1)^{i'+l}\varepsilon^\lambda\int\frac{dy d\vec{k}'_\perp}{16\pi^3}h_{ji'l}^{\sigma s\lambda *}(y,\vec{k}_\perp)t_{ii'l}^{\sigma s\lambda}(y,\vec{k}_\perp). \tag{2.3}$$

The valence eigenvectors are orthonormal and complete in the following sense:

$$\sum_i(-1)^i\tilde{z}_{ai}^\pm z_{bi}^\pm=(-1)^a\delta_{ab}\;\;\text{and}\;\;\sum_a(-1)^a z_{ia}^\pm\tilde{z}_{ja}^\pm=(-1)^i\delta_{ij}. \tag{2.4}$$

The t functions satisfy the projection of the effective eigenvalue problem onto one-electron/one-photon states, orthogonal to $|\phi\rangle$, which gives [2]

$$\sum_i (-1)^i z_{ai}^{\pm} \left\{ h_{ijl}^{\pm s\lambda}(y,\vec{k}_\perp) + \frac{1}{2} V_{ijl}^{\pm s\lambda}(y,\vec{k}_\perp) + \left[\frac{m_j^2+k_\perp^2}{1-y} + \frac{\mu_{l\lambda}^2+k_\perp^2}{y} - m_i^2\right] t_{ijl}^{\pm s\lambda}(y,\vec{k}_\perp) \right. \\ \left. + \frac{1}{2}\sum_{i'} \frac{I_{ji'}}{1-y} t_{ii'l}^{\pm s\lambda}(y,\vec{k}_\perp) - \sum_{j'} (-1)^{i+j'} t_{j'jl}^{\pm s\lambda}(y,\vec{k}_\perp) I_{j'i} \right\} = 0, \tag{2.5}$$

with the vertex correction

$$V_{ijl}^{\sigma s\lambda}(y,\vec{k}_\perp) = \sum_{i'j'l'\sigma's\lambda'} (-1)^{i'+j'+l'} \varepsilon^{\lambda'} \int \frac{dy' d\vec{k}_\perp'}{16\pi^3} \frac{\theta(1-y-y')}{(1-y')^{1/2}(1-y)^{3/2}} \\ \times h_{jj'l'}^{ss'\lambda'*}\left(\frac{y'}{1-y}, \vec{k}_\perp' + \frac{y'}{1-y}\vec{k}_\perp\right) t_{i'j'l}^{\sigma's'\lambda}\left(\frac{y}{1-y'}, \vec{k}_\perp + \frac{y}{1-y'}\vec{k}_\perp'\right) t_{ii'l'}^{\sigma\sigma'\lambda'}(y',\vec{k}_\perp'). \tag{2.6}$$

To partially diagonalize in flavor, we define $C_{abl}^{\pm s\lambda}(y,\vec{k}_\perp) = \sum_{ij}(-1)^{i+j} z_{ai}^{\pm}\tilde{z}_{bj}^{\pm} t_{ijl}^{\pm s\lambda}(y,\vec{k}_\perp)$. With analogous definitions for H, I, and V, we have

$$\left[M_a^2 - \frac{M_b^2+k_\perp^2}{1-y} - \frac{\mu_{l\lambda}^2+k_\perp^2}{y}\right] C_{abl}^{\pm s\lambda}(y,\vec{k}_\perp) \\ = H_{abl}^{\pm s\lambda}(y,\vec{k}_\perp) + \frac{1}{2}\left[V_{abl}^{\pm s\lambda}(y,\vec{k}_\perp) - \sum_{b'} \frac{I_{bb'}}{1-y} C_{ab'l}^{\pm s\lambda}(y,\vec{k}_\perp)\right] \tag{2.7}$$

to be solved simultaneously with the valence sector equations, which depend on C/t through the self-energy matrix I. Notice that the physical mass M_b has replaced the bare mass in the kinetic energy term, without any need for sector-dependent renormalization [8].

In order to compute matrix elements, such as appear in the computation of form factors, we need the left-hand eigenstate. The dual to $\langle\tilde{\psi}| = \sqrt{Z}\langle\psi|e^T$ is a right eigenstate of $\overline{\mathscr{P}^-}^\dagger$

$$|\widetilde{\psi}_a^\sigma(\underline{P})\rangle = |\widetilde{\phi}_a^\sigma(\underline{P})\rangle + \sum_{jls\lambda}\int dy d\vec{k}_\perp \sqrt{\frac{P^+}{16\pi^3}} l_{ajl}^{\sigma s\lambda}(y,\vec{k}_\perp) a_{l\lambda}^\dagger(y,\vec{k}_\perp;\underline{P}) b_{js}^\dagger(1-y,-\vec{k}_\perp;\underline{P})|0\rangle, \tag{2.8}$$

The flavor-diagonal left-hand wave functions are $D_{abl}^{\pm s\lambda}(y,\vec{k}_\perp) \equiv \sum_j (-1)^j z_{bj}^s l_{ajl}^{\pm s\lambda}(y,\vec{k}_\perp)$. They satisfy the coupled equations [2]

$$\left[M_a^2 - \frac{M_b^2+k_\perp^2}{1-y} - \frac{\mu_{l\lambda}^2+k_\perp^2}{y}\right] D_{abl}^{\sigma s\lambda}(y,\vec{k}_\perp) \\ = \tilde{H}_{abl}^{\sigma s\lambda}(y,\vec{k}_\perp) + W_{abl}^{\sigma s\lambda}(y,\vec{k}_\perp) - \sum_{b'} J_{b'a}^\sigma \tilde{H}_{b'bl}^{\sigma s\lambda *}(y,\vec{k}_\perp), \tag{2.9}$$

where $W_{abl}^{\sigma s\lambda}$ is a vertex-correction analog of $V_{abl}^{\sigma s\lambda}$, though linear in D, and J_{ba}^σ is a self-energy analog of I_{ba}. Solutions for M_a, z_{ai}^σ, $\tilde{z}_{ai}^\sigma$, and $C_{abl}^{\sigma s\lambda}$ are used as input.

3. Anomalous magnetic moment

We compute the anomalous moment a_e from the spin-flip matrix element [9] of the current $J^+ = \overline{\psi}\gamma^+\psi$ coupled to a photon of momentum q in the Drell–Yan ($q^+ = 0$) frame [10]

$$16\pi^3\langle\psi_a^\sigma(\underline{P}+\underline{q})|J^+(0)|\psi_a^\pm(\underline{P})\rangle = 2\delta_{\sigma\pm}F_1(q^2) \pm \frac{q^1 \pm iq^2}{M_a}\delta_{\sigma\mp}F_2(q^2). \tag{3.1}$$

In the limit of infinite Pauli–Villars masses, and with $M_0 = m_e$, the electron mass, we find [2]

$$\begin{aligned} F_1(q^2) \;=\; \frac{1}{\mathcal{N}}\Bigg[1 + \sum_s \int \frac{dyd\vec{k}_\perp}{16\pi^3}\Bigg\{ &\sum_{\lambda=\pm} l_{000}^{\pm s\lambda *}(y,\vec{k}_\perp - y\vec{q}_\perp)t_{000}^{\pm s\lambda}(y,\vec{k}_\perp) \\ &- \sum_{\lambda=0}^{3}\varepsilon^\lambda l_{000}^{\pm s\lambda *}(y,\vec{k}_\perp)t_{000}^{\pm s\lambda}(y,\vec{k}_\perp)\Bigg\}\Bigg] \end{aligned} \tag{3.2}$$

and

$$F_2(q^2) = \pm\frac{2m_e}{q^1 \pm iq^2}\frac{1}{\mathcal{N}}\sum_s\sum_{\lambda=\pm}\int\frac{dyd\vec{k}_\perp}{16\pi^3}l_{000}^{\mp s\lambda *}(y,\vec{k}_\perp - y\vec{q}_\perp)t_{000}^{\pm s\lambda}(y,\vec{k}_\perp), \tag{3.3}$$

with

$$\mathcal{N} = 1 - \sum_s\sum_{\lambda=0,3}\varepsilon^\lambda\int\frac{dyd\vec{k}_\perp}{16\pi^3}l_{000}^{\pm s\lambda *}(y,\vec{k}_\perp)t_{000}^{\pm s\lambda}(y,\vec{k}_\perp). \tag{3.4}$$

A second term is absent in F_2 because l and t are orthogonal for opposite spins. The $q^2 \to 0$ limit can be taken, to find $F_1(0) = 1$ and

$$a_e = F_2(0) = \pm m_e\sum_{s\lambda}\varepsilon^\lambda\int\frac{dyd\vec{k}_\perp}{16\pi^3}y l_{000}^{\mp s\lambda *}(y,\vec{k}_\perp)\left(\frac{\partial}{\partial k^1} \mp i\frac{\partial}{\partial k^2}\right)t_{000}^{\pm s\lambda}(y,\vec{k}_\perp). \tag{3.5}$$

As a check, we can consider a perturbative solution

$$t_{000}^{\sigma s\lambda} = l_{000}^{\sigma s\lambda} = h_{000}^{\sigma s\lambda}\Big/\left[m_e^2 - \frac{m_e^2 + k_\perp^2}{1-y} - \frac{\mu_{l\lambda}^2 + k_\perp^2}{y}\right]. \tag{3.6}$$

Substitution into the expression for a_e gives immediately the Schwinger result [11] $\alpha/2\pi$, in the limit of zero photon mass, for any covariant gauge.

4. Summary

The LFCC method provides a nonperturbative approach to bound-state problems in quantum field theories without truncation of the Fock space and without the uncanceled divergences and spectator dependence that such truncation can cause. The approximation is instead a truncation of the operator T that generates contributions from higher Fock states. It is systematically improvable through the addition of more terms to T, with increasing numbers of particles created and annihilated.

To complete the application to the dressed-electron state, we need to solve numerically the coupled systems that determine the t and l functions and to use these solutions to compute the anomalous moment. Within the arbitrary-gauge formulation, we can test directly for gauge dependence [5]. A more complete investigation of QED would include consideration of the dressed-photon state, contributions from electron-positron pairs to the dressed-electron state, and true bound states such as muonium and positronium. These will provide some guidance for applications to quantum chromodynamics, particularly in extensions of the holographic model for mesons [12].

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