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Factorization of chiral logarithms in the pion form factors

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The recently proposed hard pion chiral perturbation theory predicts that the chiral logarithms giving the leading quark mass dependent term in the pion form factors, factorize with respect to the energy dependence in the chiral limit. We have analyzed vector and scalar pion form factors $F_{V,S}(s)$ using standard chiral perturbation theory and dispersion relations. We show that this factorization property is valid for the elastic contribution to the dispersion integrals for $F_{V,S}(s)$ but it is violated starting at three loops when the inelastic four-pion channel opens up.

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1. Introduction

In ref. [1] Flynn and Sachrajda analyzed the $K_{\ell 3}$ decay in $SU(2)$ chiral perturbation theory (χPT) and argued that it is possible to predict the coefficient of the leading chiral log even when the final state pion is hard. This has nontrivial implications since it indicates that some chiral properties should emerge even outside the region of applicability of χPT. Bijnens and collaborators [2, 3, 4] then claimed that one can calculate such a chiral log even in more general processes where the pion is hard. This approach has been referred to as hard pion chiral perturbation theory (H$\pi\chi$PT). Here we focus on the application of this framework to the pion vector and scalar form factors, F_V and F_S. After expanding the *two-loop* χPT results [5] in the limit $M_\pi^2/s \ll 1$, it has indeed been found that [4]

$$F_{V,S}(s) = \overline{F}_{V,S}(s)\left(1+\alpha_{V,S}(s)L\right) + \mathscr{O}(M^2) , \tag{1.1}$$

in agreement with H$\pi\chi$PT. Here s is the momentum transfer squared and L stands for the leading chiral logarithm, defined as [1]

$$L \equiv \frac{M^2}{(4\pi F)^2} \ln \frac{M^2}{s} . \tag{1.2}$$

M^2 is proportional to the average up and down quark masses $\hat{m}$ ($M^2 = 2B\hat{m}$) and F is the pion decay constant in the chiral limit:

$$M_\pi^2 = M^2 + \mathscr{O}(M^4) , \qquad F_\pi = F + \mathscr{O}(M^2) . \tag{1.3}$$

$\overline{F}_{V,S}(s)$ denote the form factors for vanishing u and d quark masses. Using the dispersive representation of the form factors, we will explain the result in eq. (1.1) and show that factorization is violated by contributions of multipion intermediate states, which start at three loops. A more detailed discussion is contained in [6].

We denote $F_V(s)$ and $F_S(s)$ by the common symbol $F(s)$ (unless it is necessary to distinguish them), with normalization $F(0) = 1$. Both these form factors are analytic functions in the cut plane $[4M_\pi^2, \infty)$ and satisfy the following once subtracted dispersion relation

$$F(s) = 1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \, \frac{\mathrm{Im}\, F(s')}{s'(s'-s)} . \tag{1.4}$$

Unitarity relates in the elastic region the imaginary part of the form factor to the form factor itself and the $\pi\pi$ partial wave with the same quantum numbers:

$$\mathrm{Im}\, F(s) = \sigma(s) F(s) t^*(s) , \qquad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}} . \tag{1.5}$$

When s gets larger than the inelastic threshold, additional cuts involving several intermediate pions start to contribute. Correspondingly, the form factor can be split into an elastic and an inelastic part. In the next section we will show that the elastic term factorizes like in eq. (1.1) to *all* chiral orders (if we neglect inelastic contributions to the $\pi\pi$ partial waves). In sec. 3 we will illustrate our calculation of the lowest order inelastic contribution given by the four pion intermediate state. This yields leading chiral logs which do not respect H$\pi\chi$PT-factorization.

[1] Writing $\ln M^2/s$ as $\ln M^2/\mu^2 + \ln \mu^2/s$, one can then equivalently define L as μ^2-dependent as done in refs. [1, 4] with the second term contributing to the $\mathscr{O}(M^2)$ piece in eq. (1.1).

2. The elastic contribution to $F(s)$

In eq. (1.4) the terms proportional to L can be generated in a twofold way: either they are produced by the dispersive integration over s' or they are present in the integrand.

The first possible mechanism is that the chiral logs are generated by the lower integration boundary $s = 4M_\pi^2$, which goes to zero in the chiral limit. The power of s' in $\mathrm{Im}\,F(s)$ determines the presence of the logarithm. Expanding both the form factor and the $\pi\pi$ amplitude, one obtains

$$F(s,M_\pi^2) = 1 + \mathcal{O}(s) + \mathcal{O}(M^2)\ , \quad t(s,M_\pi^2) = c_1 M^2 + c_2 s + \mathcal{O}(s^2) + \mathcal{O}(M^4)\ , \tag{2.1}$$

with c_1 and c_2 numerical constants. Substituting eqs. (2.1) into eq. (1.4) we obtain

$$F(s) = 1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sigma(s')}{s'(s'-s)}\left(c_1 M^2 + c_2 s' + \mathcal{O}(p^4)\right)\ . \tag{2.2}$$

Three type of integrals should then be evaluated. The first is the well-known loop function $\bar{J}(s)$ which has the following expansion in $M^2/s \ll 1$,

$$\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sigma(s')}{s'(s'-s)} = 16\pi \bar{J}(s) = \frac{1}{\pi}\left[1 + \ln\frac{M^2}{-s} + \frac{2M^2}{s}\left(1 - \ln\frac{M^2}{-s}\right) + \mathcal{O}\left(\frac{M^4}{s^2}\right)\right]\ . \tag{2.3}$$

The chiral logarithms are produced from the lower integration boundary. Therefore, for the remaining integrals we can introduce an M_π-independent upper cut-off Λ^2, which allows us to interchange integration and expansion for small M. The second type of integral is then given by

$$\frac{s}{\pi}\int_{4M_\pi^2}^{\Lambda^2} ds' \frac{\sigma(s')}{(s'-s)} = \bar{d}_1(s,\Lambda^2) - \frac{2}{\pi}M^2 \ln\frac{M^2}{s} + \mathcal{O}(M^2) \tag{2.4}$$

and the third is the integration over terms of $\mathcal{O}(p^4)$. If these are proportional to s'^2 then they are enough suppressed to not generate a leading chiral log. Otherwise these terms go at least like M_π^2, producing contributions which are beyond our accuracy. Putting all pieces together we obtain

$$F(s) = \overline{F}(s) + 16\pi F^2 (c_1 - 2c_2) L + \mathcal{O}(M^2)\ . \tag{2.5}$$

The constants c_1 and c_2 are related to the $\pi\pi$ scattering lengths and volumes characterizing the threshold expansion [7]. For the vector form factor the relevant parameters are $c_1 = -a_1^1 + \mathcal{O}(M^2)$, and $c_2 = a_1^1/4 + \mathcal{O}(M^2)$. For the scalar form factor we must use instead $c_1 = a_0^0/M^2 - b_0^0 + \mathcal{O}(M^2)$ and $c_2 = b_0^0/4 + \mathcal{O}(M^2)$ leading to

$$\alpha_V = 16\pi F^2\left(-\frac{3a_1^1}{2}\right) = -1\ , \quad \alpha_S = 16\pi F^2\left(\frac{a_0^0}{M^2} - \frac{3b_0^0}{2}\right) = -\frac{5}{2}\ , \tag{2.6}$$

which reproduces the known result [5, 4].

From what we have discussed so far, it is clear that the dispersive integration may only generate a chiral log at $\mathcal{O}(p^2)$. We shall now consider leading chiral logs at higher chiral orders. We are interested in terms proportional to $s^{n-1}L$ at order p^{2n}, which arise only if the integrand itself contains already a chiral log.

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Let us start from the unitarity relations for the form factors order by order:

$$\begin{aligned}\mathrm{Im}\,F^{(2)}(s) &= \sigma(s)t^{(2)}(s) \\ \mathrm{Im}\,F^{(4)}(s) &= \sigma(s)\left[t^{(4)*}(s)+F^{(2)}(s)t^{(2)}(s)\right] \\ \vdots\quad & \quad\vdots\end{aligned} \tag{2.7}$$

In order to get the real part, the dispersive integral has to be performed. The one loop result of the form factor is then

$$F^{(2)}(s)=\overline{F}^{(2)}(s)+\alpha L+\mathscr{O}(M^2)\ , \tag{2.8}$$

where the coefficient of the log was calculated in eq. (2.6). At the next order (*i.e.* at two loops) the only source of leading chiral logs is the integrand. The term containing L has as coefficient exactly the absoptive part of the form factor at one chiral order lower times α. Therefore, at two loops the form factor can be written as

$$F(s)=\left(1+\overline{F}^{(2)}(s)\right)(1+\alpha L)+\overline{F}^{(4)}(s)+\mathscr{O}(M^2)+\mathscr{O}(p^6)\ . \tag{2.9}$$

i.e. in the form predicted by H$\pi\chi$PT. For the elastic contribution to $F(s)$, the same argument can be repeated in the same way order by order. At every new step the terms threatening factorization are the contributions to $\mathrm{Im}\,F^{(n)}$ arising from the $\pi\pi$ scattering amplitude at the same order. In [6], using Roy equations [8], we calculated these terms and found that no leading chiral logs appear from the $\pi\pi$ scattering amplitude. However, at order p^6, the four-pion intermediate states contribute to $\mathrm{Im}\,F(s)$ and we will now show that these yield leading chiral logarithms which are responsible for the breakdown of factorization, leading to the three-loop result,

$$F(s)=\left(1+\overline{F}^{(2)}(s)+\overline{F}^{(4)}(s)\right)(1+\alpha L)+\alpha_{\mathrm{inel}}(s)L+\overline{F}^{(6)}(s)+\mathscr{O}(M^2)+\mathscr{O}(p^8)\ . \tag{2.10}$$

3. The contribution from inelastic channels

The lowest-order inelastic contribution to $F(s)$ is given by three-loop diagrams with four intermediate pions, see fig. (1). We evaluated them by means of the following dispersion relation with the lower integration boundary given by the four-pion threshold and $\mathrm{Im}\,F_{\mathrm{inel}}$ written in terms of the phase space integral which follows from unitarity:

$$F_{\mathrm{inel}}(s)=\frac{s}{\pi}\int_{16M_\pi^2}^{\infty}ds'\,\frac{1}{s'(s'-s)}\int d\Phi_4(s;p_1,p_2,p_3,p_4)\,F_{4\pi}\cdot T_{6\pi}^*. \tag{3.1}$$

Here $d\Phi_4$ is the phase space for four pions of momenta $p_1,\ldots,p_4$. $F_{4\pi}$ denotes the current-4π vertex and $T_{6\pi}$ is the four-pion-to-two-pion scattering amplitude. Chiral logarithms are either produced by the dispersive integration or by the integral in $d\Phi_4$. Since $\mathrm{Im}\,F_{\mathrm{inel}}$ is of $\mathscr{O}(p^6)$, the integration over s' cannot yield terms proportional to s^2L, as can be seen by applying the same arguments as in sec. 2. Therefore we need to concentrate only on the phase space integral.

Chiral logarithms are produced by integrations over intermediate momenta with mass-dependent boundaries. We find that, in order to calculate the terms proportional to L, we can expand the

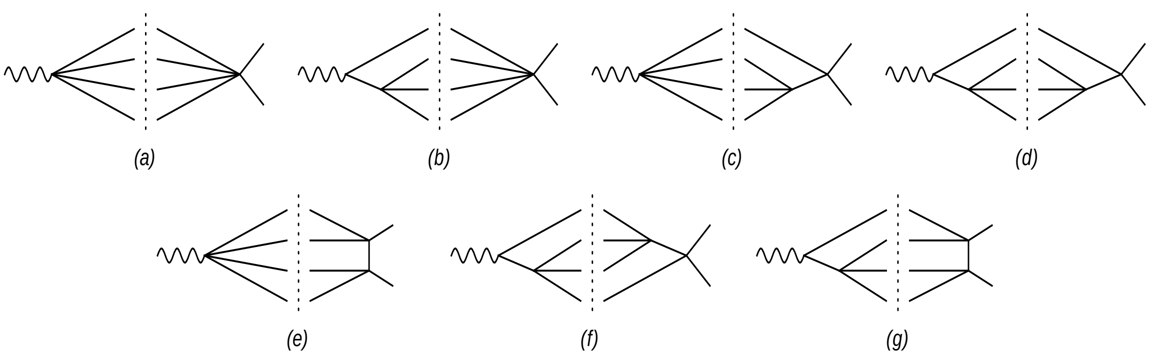

Figure 1: Three-loop cut diagrams contributing to the scalar $F_{\rm inel}(s)$ at order p^6.

integrand for small M and keep only the relevant terms. We used the following phase space parametrization since it allows us to perform analytical integrations after this expansion.

$$d\Phi_4 = \frac{1}{(4\pi)^6}\frac{\lambda^{1/2}(s,q^2,M_\pi^2)}{2s}\frac{\lambda^{1/2}(q^2,k^2,M_\pi^2)}{2q^2}\frac{\lambda^{1/2}(k^2,M_\pi^2,M_\pi^2)}{2k^2}\,d\Omega_k\,d\Omega_q\,d\Omega_s\,\frac{dq^2}{2\pi}\frac{dk^2}{2\pi} \tag{3.2}$$

in terms of the Källén function $\lambda(x,y,z) \equiv x^2+y^2+z^2-2(xy+xz+yz)$. Here $k = p_1+p_2$ and $q = p_1+p_2+p_3$. Ω_s is the solid angle in the center-of-mass frame of the two final pions, Ω_q is the solid angle in the frame where $\vec{q}=0$ and Ω_k is the solid angle in the frame where $\vec{k}=0$. The advantage of using this parametrization is that each λ function contains M_π^2 as an argument, which enables us to expand all factors and perform all the integrations analytically, at least for the diagrams (a), (b), (c) and (d).

The integration range of the kinematic variables k^2 and q^2 is determined by the delta function which ensures momentum conservation:

$$4M_\pi^2 \le k^2 \le (\sqrt{q^2}-M_\pi)^2\ ,\quad 9M_\pi^2 \le q^2 \le (\sqrt{s}-M_\pi)^2\ . \tag{3.3}$$

We stress that chiral logarithms stem from both upper and lower integration boundaries due to the functional form of the phase space.

Let us now consider the scalar form factor. For the diagrams (a) to (d) we have been able to determine analytically both the values in the chiral limit and the coefficients of L for ${\rm Im}\,F_{\rm inel}(s)$. The latter ones are given by

$$\delta_i\,\frac{s^2}{(4\pi F)^4}\quad \text{with}\quad \delta_a=-\frac{20}{3}\pi,\quad \delta_b=\frac{55}{6}\pi,\quad \delta_c=\frac{25}{3}\pi\quad\text{and}\quad \delta_d=-\frac{9}{2}\pi\ . \tag{3.4}$$

For the remaining diagrams the structure is too complicated to perform all integrations analytically. We computed the corresponding coefficients for the leading chiral logs numerically. With the same numerical routine we were able to reproduce the coefficients δ_i for the diagrams (a) to (d) within one per cent.

The contribution from all seven graphs leads to our determination of the coefficient of L in ${\rm Im}\,F_{\rm inel}(s)$ at three loops:

$$\delta\,\frac{s^2}{(4\pi F)^4}\quad\text{with}\quad \delta\sim -1.2\,\pi \tag{3.5}$$

We are finalizing checks on this number and will provide an estimate of the uncertainty in ref. [6]. We stress that the pion mass dependence resulting from the sum of the seven diagrams is definitely not compatible with a vanishing coefficient for the chiral logarithm.

After performing the dispersive integral in eq. (3.1), for $\alpha_{\rm inel}$ in eq. (2.10) we get

$$\alpha_{\rm inel}(s) = \left[C(\mu^2) + \delta \times \left(\ln\frac{\mu^2}{s} + i\pi\right)\right] \frac{s^2}{(4\pi F)^4} \tag{3.6}$$

where $C(\mu^2)$ is a combination of LECs compensating the μ-dependence. Assuming that $C(\mu = 1\,\text{GeV})$ is of natural size, comparing the logarithms from the elastic and the inelastic part at three loops, we find that the factorization breaking effect is about 10% in the range for s of interest.

We stress that at $\mathscr{O}(p^8)$ in the form factors there will be contributions from four-pion intermediate cuts in the $\pi\pi$ scattering amplitude $T^{(6)}$. These can be additional sources of chiral logs.

4. Conclusions

Applying dispersion relations and standard chiral perturbation theory to the calculation of vector and scalar pion form factors, we have shown to what extent the factorization of leading chiral logs for $M_\pi^2/s \ll 1$ conjectured in H$\pi\chi$PT is valid. Elastic contribution to the dispersive integrals does lead to factorization while the inelastic part violates it. This implies that starting at three loops H$\pi\chi$PT does not reproduce the correct leading quark mass dependence.

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