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Phase quenching in finite-density QCD: models, holography, and lattice

Masanori Hanada*
KEK Theory Center, High Energy Accelerator Research Organization (KEK)
E-mail: hanada@post.kek.jp

Yoshinori Matsuo
KEK Theory Center, High Energy Accelerator Research Organization (KEK)
E-mail: ymatsuo@post.kek.jp

Naoki Yamamoto
Yukawa Institute for Theoretical Physics, Kyoto University
Institute for Nuclear Theory, University of Washington
Maryland Center for Fundamental Physics, Department of Physics, University of Maryland
E-mail: nyama@umd.edu

Finite-density QCD is difficult to study numerically because of the sign problem. We prove that, in a certain region of the phase diagram, the phase quenched approximation is exact to $O(N_f/N_c)$. It is true for any physical observables. We also consider the implications for the lattice simulations and find a quantitative evidence for the validity of the phase quenching from existing lattice QCD results at $N_c = 3$. Our results show that the phase-quench approximation is rather good already at $N_c = 3$, and the $1/N_c$ correction can be incorporated by the phase reweighting method without suffering from the overlap problem. We also show the same equivalence in effective models and holographic models.

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*Speaker.

PoS(Lattice 2012)072

1. Introduction

QCD at a finite baryon chemical potential and/or finite temperature is an important subject of study, which is crucial for understanding the early universe, the relativistic heavy ion collisions, and the dense matter inside the neutron stars. Although the lattice QCD simulations should play an important role for studying the strongly coupled parameter region of this theory, the notorious *sign problem* prevents us from a direct application of the simulation. In principle, a lattice simulation can be performed by using the phase-quenched ensemble. The effect of the phase can be taken into account by the phase reweighting method. However whether it is practical or not is not clear a priori; when the number of the flavors N_f is two, the phase quenched theory is the QCD with the isospin chemical potential[1], whose phase diagram is different from the original theory (for example the pion condensation takes place), and hence a severe overlap problem can appear.

Recently it turned out that the phase quenching and the phase reweighting are actually practically useful techniques. The first to emphasized this fact, albeit empirically, are probably Kogut and Sinclair [1]. They pointed out that various model calculations give the same answer for certain observables in the full and phase-quenched theories, as long as the pion condensation does not take place in the latter. They also pointed out that known lattice data give very similar results. Also Cohen [2] and Toublan [3] pointed out the similarity in the large-N_c limit. Recently these facts have been understood theoretically unified manner [4, 5, 6]. Actually there is an exact *equivalence* between the full and phase-quenched QCD at large-N_c, which provides a good approximation at $N_c = 3$.[2] The equivalence is a version of the large-N_c orbifold equivalence [7, 8], which was discovered through the study of the string theory. (For other interesting applications of the orbifold equivalence see e.g. [9].) In this paper, we briefly summarize the equivalence, show the lattice data which proves the equivalence can be seen already at $N_c = 3$, and point out the same equivalence holds for various effective models in the mean field approximation.

2. The (partial) equivalence between QCD$_B$ and QCD$_I$ in the large-N_c QCD

2.1 The equivalence between QCD$_B$ and QCD$_I$

Let us start with introducing the orbifold equivalence. First we choose the discrete symmetry P (subgroup of gauge, flavor, or spacetime symmetry) of the *parent* theory, which is the SO$(2N_c)$ or Sp$(2N_c)$ theory with the baryon chemical potential (SO$_B$ or Sp$_B$) in the present case. We then throw away all the degrees of freedom not invariant under P. This procedure is called the *orbifold projection*. After the projection, we obtain a new theory called the *daughter*. We consider two different projections, which give QCD with the baryon and isospin chemical potentials (QCD$_B$ and QCD$_I$) as daughters. The orbifold equivalence states that, in the large-N_c limit, correlation functions of operators $\mathscr{O}^{(p)}(A_\mu, \psi)$ invariant under P in the parent (called *neutral* operators) agree

[1]As we will see in Sec. 2.2, although this statement is correct for the partition function, there is a difference when one considers the physical observables.

[2]Previously we argued the equivalence is restricted to a class of observables. As we will see, however, the equivalence holds for any observables. We thank F. Karsch for a valuable critical comment, which made us revisit the issue and led to more precise statement.

with those of the operators $\mathscr{O}^{(d)}(A_\mu^{\rm proj},\psi^{\rm proj})$ that consist of projected fields in the daughter:

$$\langle \mathscr{O}_1^{(p)}\mathscr{O}_2^{(p)}\cdots\rangle_p = \langle \mathscr{O}_1^{(d)}\mathscr{O}_2^{(d)}\cdots\rangle_d. \tag{2.1}$$

Here we take the coupling constants as $g^2_{\rm SU}=g^2_{\rm SO}=g^2_{\rm Sp}$, where the 't Hooft coupling $g^2_{\rm SU}N_c$ is kept finite. The field theoretic proof was given in [8] for a class of theories, which can be generalized to various cases. For QCD$_B$, QCD$_I$, SO$_B$ and Sp$_B$, a couple of evidences of nonperturbative equivalence were also provided by the weak-coupling analysis in QCD and QCD-like theories at high density limit [5], low-energy effective theories [10], chiral random matrix models [5] and holographic models [11].

In order to build a projection from SO$_B$ to QCD$_B$, we use the $\mathbb{Z}_4$ discrete symmetries of SO$_B$ generated by $J_c=-i\sigma_2\otimes 1_{N_c}$ (1_N is an $N\times N$ identity matrix) and $\omega=e^{i\pi/2}\in U(1)_B$. We require the gauge field $A^{\rm SO}_{\mu,ab}$ and the fermion $\psi^{\rm SO}_{\alpha,a}$ to be invariant under the following $\mathbb{Z}_2$ transformation embedded in the gauge and U$(1)_B$ transformation [4],

$$A^{\rm SO}_{\mu,ab} = (J_c)_{aa'}A^{\rm SO}_{\mu,a'b'}(J_c^{-1})_{b'b}, \tag{2.2}$$

$$\psi^{\rm SO}_{\alpha,a} = \omega(J_c)_{aa'}\psi^{\rm SO}_{\alpha,a'}. \tag{2.3}$$

From these projection conditions, it turns out that the daughter is QCD$_B$. The projection symmetry breaks down in the BEC/BCS crossover region (diquark condensation region) of SO$_B$, because the U$(1)_B$ symmetry is broken to $\mathbb{Z}_2$ there.

One can also construct the projection from SO$_B$ to QCD$_I$ for even N_f by choosing another $\mathbb{Z}_2$ symmetry [4, 5],

$$A^{\rm SO}_{\mu,ab} = (J_c)_{aa'}A^{\rm SO}_{\mu,a'b'}(J_c^{-1})_{b'b}, \tag{2.4}$$

$$\psi^{\rm SO}_{\alpha,af} = (J_c)_{aa'}\psi^{\rm SO}_{\alpha,a'f'}(J_i^{-1})_{f'f}, \tag{2.5}$$

where $J_i=-i\sigma_2\otimes 1_{N_f/2}$ generates $\mathbb{Z}_4$ subgroup of SU(2) isospin symmetry and the projection condition for the gauge field is the same as (2.2). In this case, the isospin symmetry used for the projection is unbroken everywhere, and so the orbifold equivalence holds including the BEC/BCS region of the phase diagram. Therefore, through the equivalence with SO$_B$, we obtain the equivalence between QCD$_B$ and QCD$_I$ outside the BEC/BCS region of the latter; the phase quenching is exact for neutral sectors in this region.

Let us consider the $1/N_c$ corrections to QCD$_B$ and QCD$_I$. In the 't Hooft large-N_c limit, expectation values of gluonic operators trivially agree because the fermions are not dynamical. Now consider finite-N_c, say $N_c=3$ and $N_f=2$. Then the largest correction to the 't Hooft limit comes from one-fermion-loop planar diagrams, which, as we have seen, do not distinguish μ_B and μ_I. Therefore the difference of expectation values of gluonic operators is at most $(N_f/N_c)^2$ (two-fermion-loop planar diagrams). In particular, the deconfinement temperatures, which are determined by the Polyakov loop, agree up to corrections of this order. A similar observation was made in [3] by a perturbative argument.

2.2 More equivalence between QCD$_B$ and phase-quenched QCD

It is often said that QCD$_I$ and the phase quenched QCD are the same. However, although this statement is correct for the partition function, there is a difference when one considers the physical

PoS(Lattice 2012)072

observables. In QCD_I, the propagators of up and down quarks are $D^{-1}(+\mu)$ and $D^{-1}(-\mu)$. On the other hand, in the phase-quenched QCD, it is natural to take both $D^{-1}(+\mu)$. (In the terms of the lattice QCD simulation, the configuration are generated by using QCD_I, while the same operators as QCD_B are used for the measurement.) Therefore the expectation values of the chiral condensate, the baryon density, and the isospin density are calculated as follows:

	QCD_B	QCD_I	phase-quenched QCD
chiral condensate	$2\langle \mathrm{Tr}D^{-1}(\mu)\rangle_B$	$\langle \mathrm{Tr}D^{-1}(\mu)+\mathrm{Tr}D^{-1}(-\mu)\rangle_I$	$2\langle \mathrm{Tr}D^{-1}(\mu)\rangle_I$
baryon density	$2\langle \mathrm{Tr}\gamma^0 D^{-1}(\mu)\rangle_B$	$\langle \mathrm{Tr}\gamma^0 D^{-1}(\mu)+\mathrm{Tr}\gamma^0 D^{-1}(-\mu)\rangle_I$	$2\langle \mathrm{Tr}\gamma^0 D^{-1}(\mu)\rangle_I$
isospin density	0	$\langle \mathrm{Tr}\gamma^0 D^{-1}(\mu)-\mathrm{Tr}\gamma^0 D^{-1}(-\mu)\rangle_I$	0

Here $\langle\ \rangle_B$ and $\langle\ \rangle_I$ are the expectation values with QCD_B and QCD_I ensembles, respectively. We can easily see the chiral condensate in QCD_I and the phase-quenched QCD take the same because of the charge-conjugation invariance of the QCD_I ensemble. Therefore, the orbifold equivalence (the chiral condensate in QCD_B = the chiral condensate in QCD_I) tells us it is not affected by the phase quenching. For the baryon density, let us remind $\langle \mathrm{Tr}\gamma^0 D^{-1}(-\mu)\rangle_I = -\langle \mathrm{Tr}\gamma^0 D^{-1}(+\mu)\rangle_I$, again because of the charge-conjugation invariance of the QCD_I ensemble. Therefore, the isospin density in QCD_I and the baryon density in the phase-quenched QCD take the same value. (Also the baryon density in QCD_I becomes zero.) By combining it with the orbifold equivalence (the baryon density in QCD_B = the isospin density in QCD_I), we conclude that the phase quenching does not affect the expectation value of the baryon density. The same argument holds for other observables too, and the orbifold equivalence leads to the exactness of the phase quenching for any observable to $O(N_f/N_c)$.

Let us provide a heuristic argument which can be generalized to any number of flavors and any value of the quark chemical potentials. Let us assume that the overlap problem is not severe, as the argument based on the orbifold equivalence shows, and consider why the determinant phase does not modify the expectation values. In the large-N_c limit, the quantum fluctuation is suppressed. In the phase quenched simulation, with a standard 't Hooft counting, properly normalized operators like $\bar{\psi}\psi/N_c$ fluctuates only $O(N_f/N_c)$. In other words, the histograms of the properly normalized quantities have a very sharp peak of the width $O(N_f/N_c)$. On the other hand, the phase factor is of order one[3], and it cannot change drastically around the peak. Therefore the correction is at most of order N_f/N_c.

3. Evidence from lattice simulations at $N_c = 3$

We have seen that the phase quenching is exact to $O(N_f/N_c)$. But the standard 't Hooft counting does not tell us the expansion coefficients. That motivates us to look at lattice data of $N_c = 3$ QCD. In the following we summarize lattice studies which compared QCD_B and the phase-quenched QCD.

- In [12], Nakamura et al studied two-flavor QCD_B and phase-quenched QCD by using staggered fermions. The QCD_B is obtained by the phase reweighting. The bare quark mass is

[3] Because the expectation value of the phase factor $\langle e^{i\mathrm{Im}S}\rangle$ is real, the one-point function $\langle \mathrm{Im}S\rangle$ is zero, and hence the leading contribution to the average phase comes from the connected two-point function of the imaginary part of the action, $\langle(\mathrm{Im}S)^2\rangle_{conn.}$, which is of order one. We thank Y. Hidaka for a clear explanation on this point.

PoS(Lattice 2012)072

$am = 0.05$ on a $8^3 \times 4$ lattice. The chiral condensate and the Polyakov loop are computed for $a\mu = 0.1$ and 0.2. They found a perfect agreement between QCD$_B$ and the phase-quenched QCD within numerical errors.

- In the right panel of Fig. 1 and the left panel of Fig. 4 of [13], the free energy at various temperatures between $0.5T_c$ and $1.1T_c$ are plotted as functions of Q. By putting these plots on top of each other, one can see a very nice agreement near the critical temperature and $Q \lesssim 100$. It clearly shows the validity of the phase quenching. It should also be remarked that the corrections are still tiny for $N_f = 8$, a larger number of flavors than $N_f = 2+1$ in the real world.

- The large-N_c equivalence holds for the imaginary baryon and isospin chemical potentials, $(\mu_u, \mu_d) = (i\mu_{\text{img}}, i\mu_{\text{img}})$ and $(\mu_u, \mu_d) = (i\mu_{\text{img}}, -i\mu_{\text{img}})$, without any modification. As a result, the chiral condensates $\langle\bar{\psi}\psi\rangle_B$ and $\langle\bar{\psi}\psi\rangle_I$ take the same value at finite imaginary potentials as long as the projection symmetries are unbroken.

  In [15], the chiral critical temperatures $T_c(\mu)$ in two-flavor QCD were exploited by the extrapolations from the imaginary chemical potential, by using a fitting ansatz

$$\frac{T_c(\mu)}{T_c(0)} = 1 + a_1 \left(\frac{\mu}{\pi T}\right)^2. \tag{3.1}$$

  They found [15] $a_1 = -0.470(13)$ for μ_I and $a_1 = -0.522(10)$ for μ_B, which provide a nice quantitative agreement already at $N_c = 3$.

- Let us consider the Taylor expansion method, in which the expectation value of an observable is expanded in powers of μ/T,

$$\langle\mathscr{O}\rangle_{B,I} = \sum_{n=0}^{\infty} c_n^{B,I} \left(\frac{\mu}{T}\right)^n. \tag{3.2}$$

  Taylor coefficients c_n^B and c_n^I, which are functions of the temperature T, can be determined by the simulation at $\mu = 0$. The large-N_c equivalence tells that the coefficients agree in the large-N_c limit: $\lim_{N_c\to\infty} c_n^B = \lim_{N_c\to\infty} c_n^I$.

  In [14], the coefficient c_2^B and c_2^I for the chiral condensate and the pressure of the quark-gluon gass have been calculated[4] in two-flavor QCD. Although the difference between c_2^B and c_2^I are not very small for $T < T_c$ in the chiral symmetry broken (and confinement) phase, they agree exceptionally well for $T \gtrsim T_c$. The origin of the difference for $T < T_c$ may come from the contributions of thermally excited mesons which large-N_c is suppressed at large-N_c. On the other hand, for $T > T_c$, fundamental degrees of freedom are deconfined quarks and gluons rather than baryons or mesons, where the difference between QCD$_B$ and QCD$_I$ becomes much smaller and the large-N_c equivalence is very well satisfied even at $N_c = 3$.

[4]For odd n, c_n^B and c_n^I vanish, and the first nontrivial μ-dependences appear in c_2^B and c_2^I. Although c_n^B ($n \geq 4$) have been calculated, c_n^I ($n \geq 4$) have not been calculated in [14]. (Note that, for $n \geq 4$, they use the same symbol c_n^I for another quantity.)

4. The equivalence in the effective models

As an example of the effective models, we consider the Nambu–Jona-Lasinio (NJL) model. In order to simplify the discussion, we consider the chiral limit. The starting point is the Lagrangian with the U(N_c) color current interaction with N_f flavors,

$$\mathscr{L}_{\rm NJL} = \bar{\psi}_f \left(\gamma^\mu \partial_\mu + \mu_f \gamma^4\right) \psi_f + \frac{G}{N_c} J^{(\rm U)}_{\mu A} J^{(\rm U)}_{\mu A}, \tag{4.1}$$

where $J^{(\rm U)}_{\mu A} = \bar{\psi}_f \gamma_\mu T^A_{\rm U} \psi_f$ and $T^A_{\rm U}$ are the U(N_c) color generators and summation is taken over repeated indices. The coupling constant G is taken to be of order N_c^0. One rewrites it keeping only the interactions in the scalar and pseudoscalar channels after Fierz transformations:

$$\begin{aligned} \mathscr{L}_{\rm NJL} &= \bar{\psi} \left(\gamma^\mu \partial_\mu + \mu_f \gamma^4\right) \psi + \mathscr{L}_{\rm int}, \\ \mathscr{L}_{\rm int} &= \frac{G}{N_c} \left[(\bar{\psi}_f \psi_{f'})(\bar{\psi}_{f'} \psi_f) + (\bar{\psi}_f i\gamma^5 \psi_{f'})(\bar{\psi}_{f'} i\gamma^5 \psi_f) \right]. \end{aligned} \tag{4.2}$$

In the Lagrangians (4.1) and (4.2), the invariance under U(N_c) gauge symmetry and $\mathrm{U}(N_f)_L \times U(N_f)_R$ flavor symmetry are manifest. Here we ignore the effect of instantons or the $\mathrm{U}(1)_A$ anomaly which explicitly breaks the $\mathrm{U}(1)_A$ symmetry, because it is subleading in $1/N_c$. (From the viewpoint of the orbifold equivalence, there is no reason for the exactness of the phase quenching at the level of mean-field approximation (MFA) if we take into account the $1/N_c$-suppressed instanton effects. However, even if we incorporate them, the phase quenching for the chiral condensate turns out to be exact within the NJL model [6].) For SO($2N_c$) theory, we can construct the corresponding NJL model in the same manner, by using the SO($2N_c$) current. The proof of the large-N_c orbifold equivalence applies to the NJL model, by starting with the NJL model for SO_B and by using similar projection conditions as the previous section [6].

When one considers the large-N_c limit, one should set up the correct $1/N_c$-counting scheme which reproduce the correct $1/N_c$-scaling in the large-N_c QCD. The quark ψ has N_c colors so that a closed color loop gives a factor of N_c. The coupling constant of the four-fermi interaction should be taken as $O(N_c^{-1})$, and furthermore, the form of possible four-fermi interactions are restricted; in other words only the interactions which have origins in QCD are allowed. In this setup, the right $1/N_c$-counting follows and we can use the same proof of the orbifold equivalence as the large-N_c QCD [6].

Now let us see the relationship between the large-N_c limit and the MFA. We first perform the Hubbard-Stratonovich transformation by introducing auxiliary fields corresponding to the fermion bilinears, $\sigma = \bar{\psi}\psi$ and $\pi_a = \bar{\psi} i\gamma_5 \tau_a \psi$, and then integrate out fermions to obtain the partition funcion

$$Z \equiv e^{-W} = \int d\sigma d\pi e^{-I(\sigma,\pi)}. \tag{4.3}$$

Here $I(\sigma,\pi)$ is the bosinized effective action

$$I(\sigma,\pi) = N_c \left[-\mathrm{Tr}\log D + \frac{1}{G} \int d^4x (\sigma^2 + \pi_a^2) \right], \tag{4.4}$$

with $D = \gamma^\mu \partial_\mu + 2\sigma_A + 2\pi_A$. It describes mesons σ and π. Because of an overall factor N_c in the action, the $1/N_c$-expansion is equivalent to the expansion with respect to meson loops. The leading order corresponds to the saddle-point approximation, or equivalently the conventional MFA where the auxiliary fields are replaced by the expectation value (i.e., the mean-field). In order to go beyond the MFA, we have to take into account meson-loops order by order.

Similar arguments hold also for various other theories, such as linear sigma model [6], Polyakov-Nambu-Jona-Lasinio model [6], Polyakov-quark-meson model [6], chiral random matrix model [5], Sakai-Sugimoto model [6] and D3/D7 model [11].

5. Conclusion and Outlook

We have seen the phase quenching is exact to $O(N_f/N_c)$, outside the pion condensation of the phase quenched theory. In other words, the effect of the phase is $1/N_c$-suppressed, and hence the reweighting method works without being threatened by the overlap problem. Previous lattice studies confirm the effect of the phase is small already at $N_c = 3$. We have also shown the exactness of the phase quenching in effective models, which had been realized by explicit calculations and used to justify the reweighting method, can be understood in a unified manner, from the point of view of the large-N_c equivalence. Now the phase reweighting method has a theoretical justification, and hence it is important to study the QCD phase diagram by using it.

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