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Static properties of light baryons in different large-N limits

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We investigate some static properties of light baryon ground states in three inequivalent large-N limits (with respect to the quark color representation): 't Hooft, QCD antisymmetric and QCD symmetric. Our framework is a constituent quark model with relativistic-type kinetic energy, stringlike confinement and one-gluon-exchange term, thus leading to well-defined results even for massless quarks. Moreover, two spin-dependent potentials are considered and treated as perturbations: the color magnetic interaction stemming from the one-gluon exchange process and the chiral (or Goldstone) boson exchange interaction. Our results confirm previous ones obtained by using either diagrammatic methods or constituent approaches for heavy quarks.

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1. Introduction

The large-N QCD approach is based on the replacement of the usual color group SU(3) by the group SU(N) with a large arbitrary value N, allowing a perturbative expansion of the theory in $1/N$ [1]. Taking into account the numerous successes obtained within this framework [2], it seems that the real QCD ($N = 3$) is not too different from the idealized world with large N. Current lattice calculations also strongly support this idea (see e.g. the review [3]).

In the original proposal by 't Hooft [1], denoted here QCD_F, the quarks are in the fundamental representation of SU(N) and the strong coupling constant α_S is such that the quantity $\alpha_0 = \alpha_S N \sim O(N^0)$. In this framework, a baryon is made of N quarks in the totally antisymmetric color singlet and the number of flavors remains finite while $N \to \infty$. It has been shown by diagrammatic methods that the baryon mass thus scales as N at the dominant order [4, 5].

Actually, the generalization of QCD to arbitrary numbers of colors is not unique, the main criterion being that the considered SU(N) gauge theory has to be equivalent to QCD when $N = 3$. For instance, a limit has been studied in which the quarks are in the two index antisymmetric representation of SU(N), which is equivalent to the fundamental representation for $N = 3$. Denoted here QCD_AS, that limit interestingly leads to a theory equivalent to $\mathcal{N} = 1$ supersymmetric Yang-Mills when one light quark is present, as shown in [6]. In this framework, a baryon is made of $N(N-1)/2$ quarks in the totally antisymmetric color singlet [7] and its mass is expected to scale as N^2 at the dominant order [8, 9]. In the same way, the quarks can also be considered in the two index symmetric representation of SU(N) [7]. Denoted here QCD_sym, this model is not equivalent to QCD_F for $N = 3$, but it is equivalent to some extent to QCD_AS when $N \to \infty$ [6]. In this case, a baryon is made of $N(N+1)/2$ quarks in the totally antisymmetric color singlet and its mass is expected to scale as N^2 at the dominant order [7]. Taking quarks in the two index symmetric representation is interesting since QCD-like theories with fermions in higher representations may be used in the so-called technicolor models [10].

In this work, our purpose is to compute the N-behavior of the mass for light baryons in the framework of a constituent quark model first suggested by Witten [4] (Hamiltonian with relativistic kinetic energy, stringlike confinement, and one-gluon-exchange term). Two spin-dependent potentials are also considered: the color magnetic interaction stemming from one-gluon exchanges [11] (see [12] for a review) and the chiral boson exchange interaction [13]. We focus only on the ground states containing solely u and d quarks. We have analytically proved that the static properties of light baryons scale as expected [14, 15]. All approximate solutions for the many-body Hamiltonians considered have been obtained using the auxiliary field method (AFM) [16, 17, 18, 19].

2. The auxiliary field method

The AFM allows to treat Hamiltonians with the following form

$$H = \sum_{i=1}^{M} \sqrt{\boldsymbol{p}_i^2 + m^2} + \sum_{i=1}^{M} U(|\boldsymbol{r}_i - \boldsymbol{R}|) + \sum_{i<j=1}^{M} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \tag{2.1}$$

where $\sum_{i=1}^{M} \boldsymbol{p}_i = \boldsymbol{0}$ and $\boldsymbol{R}$ is the center of mass coordinate [16]. $U(x)$ is a one-body potential and $V(x)$ is a two-body potential. The method relies on the knowledge of the analytic solution of a particular M-body Hamiltonian. In practice, only the following one is usable

$$H_{\text{ho}} = \frac{1}{2m}\sum_{i=1}^{M} \boldsymbol{p}_i^2 + k\sum_{i=1}^{M}(\boldsymbol{r}_i - \boldsymbol{R})^2 + \rho \sum_{i<j=1}^{M} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2. \tag{2.2}$$

Its eigenvalues are given by

$$E_{\text{ho}} = \sqrt{\frac{2}{m}(k + M\rho)}\, Q \tag{2.3}$$

with

$$Q = \sum_{i=1}^{M-1} (2n_i + l_i) + \frac{3}{2}(M-1). \tag{2.4}$$

It can then be shown that the AFM solution of the Hamiltonian (2.1) is obtained by solving the following set of equations [16, 17, 18, 19]

$$\begin{cases} E_{\text{AFM}} = M\,T(p_0) + M\,U\left(\frac{r_0}{M}\right) + C_M\,V\left(\frac{r_0}{\sqrt{C_M}}\right) \\ p_0 = \dfrac{Q}{r_0} \\ M\,p_0 T'(p_0) = r_0 U'\left(\frac{r_0}{M}\right) + \sqrt{C_M}\,r_0 V'\left(\frac{r_0}{\sqrt{C_M}}\right) \end{cases} \tag{2.5}$$

with $T(p_0) = \sqrt{p_0^2 + m^2}$ and $C_M = \frac{M(M-1)}{2}$. The global quantum number Q is given by (2.4). The AFM eigenstates are written as the product of harmonic oscillator states for each internal variables $\boldsymbol{x}_i$

$$|\phi\rangle = \prod_{i=1}^{M-1} |n_i, l_i, \lambda_i, \boldsymbol{x}_i\rangle \quad \text{with} \quad \lambda_i = \sqrt{\frac{i}{i+1} M\,Q}\,\frac{1}{r_0}. \tag{2.6}$$

These states have a good parity. It is possible to fix the total angular momentum and the symmetry, but this task can be very complicated.

3. Light baryon Masses

The spin-independent Hamiltonian considered here for n_q quarks was first proposed by Witten [4]. In natural units, $\hbar = c = 1$, it is given by

$$H_B = \sum_{i=1}^{n_q} \sqrt{\boldsymbol{p}_i^2} + \frac{C_q}{C_\square}\sigma \sum_{i=1}^{n_q} |\boldsymbol{r}_i - \boldsymbol{R}| + F_{qq}\frac{\alpha_0}{N} \sum_{i<j=1}^{n_q} \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{3.1}$$

The kinematic term is written under the spinless Salpeter form with a vanishing mass, $m_q = 0$, for the quarks, since only u and d flavors are considered here. The confinement is inspired from the QCD string model in which each quark generates a straight flux tube whose tension is proportional to its quadratic color Casimir operator C_q, with the fundamental string tension $\sigma \sim \mathrm{O}(1)$. Then, the flux tubes have to meet in one or several points such that the total energy contained in those flux tubes is minimal. When the number of quarks tends towards infinity, an unique junction point identified to the center of mass $\boldsymbol{R}$ is a good approximation [14]. The short-range part of the potential is ensured by the one-gluon exchange interaction whose color dependence is given by the factor $F_{qq} = \langle \boldsymbol{\lambda}_q^c \cdot \boldsymbol{\lambda}_q^c/4\rangle$.

Table 1: Color parameters for various large-N limits.

	$\mathrm{QCD_F}$	$\mathrm{QCD_{AS}}$	$\mathrm{QCD_{Sym}}$
n_q	N	$\frac{N(N-1)}{2}$	$\frac{N(N+1)}{2}$
$C_q/C_\square$	1	$2\frac{N-2}{N-1}$	$2\frac{N+2}{N+1}$
F_{qq}	$-\frac{N+1}{2N}$	$-\frac{2}{N}$	$-\frac{2}{N}$

Such a Hamiltonian can be solved by the AFM. We find [14]

$$r_0 = \sqrt{\frac{C_\square}{C_q\,\sigma}\left(n_q\,Q + \binom{n_q}{2}^{3/2} F_{qq}\,\frac{\alpha_0}{N}\right)}, \tag{3.2}$$

and upper bounds are given by

$$M_B = 2\sqrt{\frac{C_q}{C_\square}\sigma\left(n_q\,Q + \binom{n_q}{2}^{3/2} F_{qq}\,\frac{\alpha_0}{N}\right)}. \tag{3.3}$$

The various parameters appearing in these formulae are given in Table 1. The ground state masses for the three large-N limits considered are plotted on Fig 1. It appears that the masses scale as $\mathrm{O}(n_q)$ up to subleading corrections. It can be also shown that the baryon radius and the contribution of strange quarks to the mass scale as $\mathrm{O}(1)$ [14]. Moreover, large-N baryons lie on Regge trajectories as in the real QCD world.

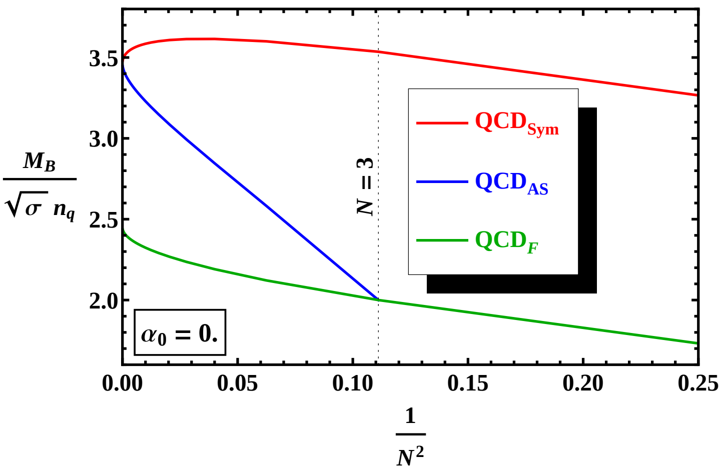

Figure 1: Baryon ground state masses per quark in $\sqrt{\sigma}$ unit for three large-N limits, with $\alpha_0 = 0$. In order to guide the eyes, N is considered as a continuous variable. Masses are slightly lowered when $\alpha_0 \neq 0$.

4. Spin contributions

The spin contributions for light baryons can be estimated by the quantity $\delta_S = m_\Delta - m_N$. In the real world, $\delta_S \approx 0.3\, m_N$. These effects are then quite large. Nevertheless, it is expected that $\delta_S \sim 1/n_q$ in the large-N limits. So, the contributions of spin-dependent parts of the Hamiltonian will be computed with the perturbation theory. Generally, it is considered that $\delta_S \propto 1/m_q^2$, but this is a too crude nonrelativistic approximation. In this work, we use a better ansatz and replace m_q by $E_q = \langle \sqrt{\boldsymbol{p}^2 + m_q^2} \rangle$, in which the mean value is computed with the state considered [12]. It can be shown that $E_q \approx \sqrt{p_0^2 + m_q^2}$, with p_0 given by (2.5).

The baryon wavefunction is a color singlet completely antisymmetrical. The ground state is given by $|\mathrm{GS}\rangle = \prod_{i=1}^{M-1} |0,0,\lambda_i,\boldsymbol{x}_i\rangle$ which is completely symmetrical. As we consider only states with the same value for the spin S and the isospin I, the spin-isospin part of the wavefunction has the following symmetry

$$[z] = \underbrace{\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array}}_{\frac{n_q-2S}{2}\ \text{columns}} \overbrace{\begin{array}{|c|}\hline \; \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \; \\ \hline \end{array}}^{2S\ \text{columns}} . \tag{4.1}$$

One can check that the SU(2) dimension of this representation is $2S+1$, while for the permutation group S_{n_q} we have

$$d[\mathrm{S}_{n_q}]_{[z]} = \frac{2S+1}{n_q+1} \binom{n_q+1}{(n_q-2S)/2}. \tag{4.2}$$

The spin-spin interaction coming from the one-gluon exchange (OGE) interaction is well known [11, 12]. The form retained is

$$W_{ij}^{\mathrm{OGE}} = -\frac{A}{E_q^2} \frac{\alpha_0}{N} \delta^3(\boldsymbol{r}_{ij}) \frac{\boldsymbol{\lambda}_i^c \cdot \boldsymbol{\lambda}_j^c}{4} \boldsymbol{s}_i \cdot \boldsymbol{s}_j, \tag{4.3}$$

with $A = 8\pi/3$ and $E_q \approx p_0$ since $m_q = 0$. $\boldsymbol{\lambda}_i^c \cdot \boldsymbol{\lambda}_j^c/4$ is the color exchange operator and $\boldsymbol{s}_i \cdot \boldsymbol{s}_j$ is the spin exchange operator. As this potential is treated as a perturbation, the delta distribution gives a finite result.

Using the AFM solution given in the previous section, it is possible to compute the contribution of the spin-spin interaction $W^{\mathrm{OGE}} = \sum_{i<j=1}^{n_q} W_{ij}^{\mathrm{OGE}}$ for the various large-N limits. When $N \to \infty$, we obtain

$$\langle W_{\mathrm{F}}^{\mathrm{OGE}} \rangle = \frac{\alpha_0 A}{2\pi^{3/2}} \sqrt{\frac{\sigma}{6(12-\sqrt{2}\,\alpha_0)}} \left[\frac{S(S+1)}{N} - \frac{3}{4} \right] + \mathrm{O}\left(\frac{1}{N}\right), \tag{4.4}$$

$$\begin{aligned} \langle W_{\mathrm{AS}}^{\mathrm{OGE}} \rangle &= \langle W_{\mathrm{Sym}}^{\mathrm{OGE}} \rangle \\ &= \frac{\alpha_0 A}{\pi^{3/2}} \sqrt{\frac{\sigma}{6(6-\sqrt{2}\,\alpha_0)}} \left[\frac{S(S+1)}{N^2/2} - \frac{3}{4} \right] + \mathrm{O}\left(\frac{1}{N}\right). \end{aligned} \tag{4.5}$$

For each limit, we see that $\delta_S \sim \boldsymbol{S}^2/n_q$, as expected. But, it appears other terms which are S-independent. These particulars contributions are unavoidable in the framework of a potential model. They are not really disturbing to fit the parameters of the mass formulae since they can be absorbed in the various terms of the usual large-N expansions, such as

$$M_B = c_0 N + c_1 + c_2 \frac{\boldsymbol{S}^2}{N} + \dots \tag{4.6}$$

The chiral boson exchange (CBE) mechanism, also known as Goldstone boson exchange mechanism, has been proposed as an alternative to the one-gluon exchange process [13]. Based on the approximate chiral symmetry of QCD, it can yield very good baryon spectra. As our purpose is only to check the N-dependence of this interaction, we will consider the simplest representation of the most important component of the potential that is mediated by the octet of pseudoscalar bosons. In the SU(3)$_F$ invariant limit, we take

$$W_{ij}^{\rm CBE} = \frac{B}{E_q^2} g^2 V^{\rm CBE}(\boldsymbol{r}_{ij}) \frac{\boldsymbol{\lambda}_i^f \cdot \boldsymbol{\lambda}_j^f}{4} \boldsymbol{s}_i \cdot \boldsymbol{s}_j, \tag{4.7}$$

with $B = 1/(3\pi)$ and $E_q \approx p_0$. The coupling constant is given by $g = m_u \dfrac{g_A}{f_\pi}$, and the radial part has the following form

$$V^{\rm CBE}(\boldsymbol{x}) = \Lambda^2 \frac{e^{-\Lambda x}}{x} - 4\pi\delta^3(\boldsymbol{x}), \tag{4.8}$$

where $\Lambda \sim {\rm O}(1)$ is the degenerate pseudoscalar meson mass. $\boldsymbol{\lambda}_i^f \cdot \boldsymbol{\lambda}_j^f/4$ is the flavor exchange operator. In our model, $m_u \sim {\rm O}(1)$ is the effective u mass. Strong indications exist, indicating that the vector axial coupling constant $g_A \sim {\rm O}(1)$ [20, 21, 22]. Moreover, we take [21, 23]

$$f_\pi(N) = \sqrt{\frac{n_q}{3}} f_\pi(3) \quad \text{with} \quad f_\pi(3) \sim {\rm O}(1), \tag{4.9}$$

where $f_\pi(3) = 131$ MeV is the pion decay constant. Note that for this potential, no one-gluon exchange is considered ($\alpha_0 = 0$).

As in the previous section, it is possible to compute the contribution of the spin-spin interaction $W^{\rm CBE} = \sum_{i<j=1}^{n_q} W_{ij}^{\rm CBE}$ for the various large-N limits. Denoting by $W(\Lambda,\lambda)$ the mean value of $V^{\rm CBE}$ for an oscillator state with size λ, we obtain for $N \to \infty$,

$$\begin{aligned}
\langle W_{\rm F}^{\rm CBE} \rangle &= \frac{5 g_A^2 B}{8 f_\pi^2(3)} \left| W(\Lambda, \sqrt{\sigma/2}) \right| \left[\frac{S(S+1)}{N} - \frac{9}{20} N \right] + {\rm O}(1), \\
\langle W_{\rm AS}^{\rm CBE} \rangle &= \langle W_{\rm Sym}^{\rm CBE} \rangle \\
&= \frac{5 g_A^2 B}{8 f_\pi^2(3)} \left| W(\Lambda, \sqrt{\sigma}) \right| \left[\frac{S(S+1)}{N^2/2} - \frac{9}{20} \frac{N^2}{2} \right] + {\rm O}(N).
\end{aligned}$$

For each limit, we see again that $\delta_S \sim \boldsymbol{S}^2/n_q$ for $N \to \infty$, and that terms appear which are S-independent. As in the previous case, they are not really disturbing to fit the parameters of the mass formulae.

5. Concluding remarks

Even if approximate analytical results are computed in this work, it has been shown in [14] that the N-behavior of the solutions are correct when $N \to \infty$. Within our model, the results are the same for the QCD_F and QCD_AS schemes for $N = 3$, while they are the same for the QCD_AS and QCD_Sym schemes for $N \to \infty$. This is in agreement with the results in [24], where it has been shown that predictions for baryon mass relations obtained with QCD_F and QCD_AS limits are both in agreement with experimental data.

We focus only on the ground states $S = I$ containing u and d quarks. It is then not possible to disentangle the contributions coming from spin and isospin. However, it could be interesting to study in future works other states with $S \neq I$, since isospin-dependent operators could play an important role for the masses of some multiplets [25].

The main result of this work is that the S-dependent mass term for light baryons is proportional to $\boldsymbol{S}^2/n_q$ when $N \to \infty$, as already shown from diagrammatic methods, mostly valid for heavy quarks. It is obtained for both the one-gluon exchange mechanism and the chiral boson exchange potential, despite their different origins. These interactions yield also S-independent contributions which behave very differently. From our point of view, it is not possible to prefer one interaction with respect to the other on the basis of our results. For instance, the S-independent contributions can be absorbed in various terms of usual baryon mass formulae. We think that our work validates our approach to study baryons in various large-N limits. Since approximate analytical baryon eigenfunctions are available with our method, a lot of observables can *a priori* be computed.

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