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Recent results from QCD sum rule analyses based on the maximum entropy method

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QCD sum rules are analyzed using the maximum entropy method, which makes it possible to have direct access to the spectral functions of two-point correlators. In the traditional analysis of QCD sum rules, one employs the "pole + continuum" ansatz for the physical region of the spectral function and extracts the position and residue of the low-lying pole. Our Bayesian approach puts less restrictions on the functional form of the spectral function, and thus provides an ideal tool for analyzing hadronic spectral functions and their modification at finite temperature or density. In these proceedings, after a brief description of the method, our results obtained in the ρ-meson and nucleon channels will be reviewed.

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1. Introduction

QCD sum rules have long been used to describe the physical properties of hadrons directly from QCD [1, 2]. While this approach was ofter quite successful, it also has its limitations. Firstly, the uncertainties that are involved in the calculation of the operator product expansion (OPE) are quite large, mainly coming from the possible range of values of the condensates and the unknown contributions from the higher order terms that have been neglected due to the truncation of the OPE. Secondly, it has been necessary to introduce some simple ansatz for parametrizing the spectral function in the QCD sum rule studies carried out so far. For example, it is most common to assume the "pole + continuum" functional form, where the pole represents the hadron in question and the continuum stands for the excited and scattering states that contribute to the spectral function. This ansatz works well in certain cases, but it may not always be a valid description of the true spectral function.

The arguments pointed out above indicate that some improved analysis procedure of QCD sum rules could be useful for obtaining more general results, which are less dependent on a-priori assumptions. As will be shown in these proceedings, the maximum entropy method (MEM) is suitable for such a task. This is so because MEM provides the most probable spectral function directly from the OPE data and their uncertainties, without having to use assumptions about its explicit form. To verify this claim, we have firsly studied the light vector meson channel with isospin 1 (containing the ρ) as a first test [3]. Then, we have investigated the nucleon channel with and without parity projection [4, 5], and quarkonium spectral functions at both zero and finite temperature [6, 7], confirming the usefulness of our novel approach. In this article we will, after a short account of the MEM procedure and its concrete application to QCD sum rules, review our obtained results of the light quark sector (ρ meson and nucleon), while the quarkonia channels will be covered in the article by K. Suzuki of these proceedings.

2. Formalism

2.1 QCD sum rules

In QCD sum rules [1, 2] one makes use of the analytic properties of a general correlator:

$$\Pi(q^2) = i \int d^4x e^{iqx} \langle 0|T[J(x)J^{\dagger}(0)]|0\rangle. \tag{2.1}$$

Here, $J(x)$ is some mesonic or baryonic operator, built form quark or/and gluon fields, which couples to the hadron of interest. The correlator can be rewritten as a dispersion relation, which connects the imaginary part of $\Pi(q)$ with its values in the deep euclidean region ($-q^2 \to \infty$), where it is possible to systematically carry out the OPE. This dispersion relation is obtained as:

$$\Pi(q^2) = \frac{1}{\pi} \int_0^{\infty} ds \frac{\mathrm{Im}\Pi(s)}{s-q^2}. \tag{2.2}$$

After calculating the OPE of $\Pi(q^2)$, the Borel transformation, symbolically denoted as $\widehat{L_M}$, is applied to both sides of Eq.(2.2), giving the following expression for $G_{OPE}(M) \equiv \widehat{L_M}[\Pi^{OPE}(q^2)]$, which depends only on M, the so-called Borel mass:

$$G_{OPE}(M) = \frac{1}{M^2} \int_0^{\infty} ds e^{-s/M^2} \rho(\sqrt{s}). \tag{2.3}$$

Here, $\mathrm{Im}\Pi(s) \equiv \pi\rho(\sqrt{s})$ have been used. Note however that Eq.(2.3) is by no means the only possible sum rule that can be constructed from the correlator of Eq.(2.1). Using the analyticity, one can in fact formulate a much more general class of sum rules with an arbitrary kernel [4]. The virtue of the Borel kernel shown above is that it is simple, strongly suppresses the large-energy part of the spectral function $\rho(\sqrt{s})$ and improves the convergence of the OPE. However, depending on the channel to be investigated other kernels might be more suitable. As will be shown later in these proceedings, this tunred out to be the case in our study of the nucleon channel [4, 5].

As a next step, one usually takes the "pole + continuum" ansatz as a crude parametrization for the spectral function,

$$\rho(\sqrt{s}) \simeq |\lambda|^2 \delta(s - m^2) + \Theta(s - s_{\mathrm{th}}) \frac{1}{\pi} \mathrm{Im}\Pi^{\mathrm{OPE}}(s), \tag{2.4}$$

plugs it into the sum rules and carries out some sort of fitting to determine the parameters $|\lambda|^2$, m and s_{th}. This procedure works well as long as Eq.(2.4) is a qualitatively correct description of the physical spectral function, but this is of course not necessarily always the case. Therefore, a more general analysis method would be desirable, which does not depend on the "pole + continuum" ansatz of Eq.(2.4). For this purpose we will employ the maximum entropy method (MEM) to directly extract the spectral function from the sum rules.

2.2 The Maximum Entropy Method (MEM)

In this section, we shortly discuss the essential ideas of the MEM method. For more details, consult for instance [3, 8, 9].

The problem to be solved with the help of MEM is the following. One is interested in some function $\rho(\omega)$, but has only limited information about an integral over $\rho(\omega)$:

$$G_{OPE}(M) = \int_0^{\infty} d\omega K(M,\omega) \rho(\omega), \tag{2.5}$$

where $K(M,\omega)$ is the kernel and corresponds to

$$K(M,\omega) = \frac{2\omega}{M^2} e^{-\omega^2/M^2} \tag{2.6}$$

in the case of the sum rules of Eq.(2.3). Here, the variable ω is defined as $\omega = \sqrt{s}$. If $G_{\mathrm{OPE}}(M)$ is known only with limited accuracy or is only calculable in a finite range of the Borel mass M, the problem of obtaining $\rho(\omega)$ from $G_{OPE}(M)$ is ill-posed and cannot be solved analytically.

The MEM approach now makes use of Bayes' theorem, by which additional information about $\rho(\omega)$ such as positivity and its asymptotic behavior at small or large energies can be incorporated into the analysis in a systematic way and by which one then can deduce the most probable from of $\rho(\omega)$. Bayes' theorem is given as

$$P[\rho|GH] = \frac{P[G|\rho H]P[\rho|H]}{P[G|H]}, \tag{2.7}$$

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where the prior knowledge of $\rho(\omega)$ is denoted as H and $P[\rho|GH]$ stands for the conditional probability of $\rho(\omega)$ given $G_{OPE}(M)$ and H. Ignoring the constant term $P[G|H]$ in the denominator as it does not depend on $\rho(\omega)$ and maximizing the remaining functional will give the most probable spectral function. $P[G|\rho H]$ is called the "likelihood function" and can be obtained as

$$\begin{aligned} P[G|\rho H] =& e^{-L[\rho]}, \\ L[\rho] =& \frac{1}{2(M_{\max}-M_{\min})}\int_{M_{\min}}^{M_{\max}} dM \frac{\left[G_{OPE}(M)-G_{\rho}(M)\right]^2}{\sigma^2(M)}. \end{aligned} \tag{2.8}$$

$G_{OPE}(M)$ is determined from the OPE of the two-point function and $G_{\rho}(M)$ is defined as on the right hand side of Eq.(2.5). Therefore, it implicitly depends on $\rho(\omega)$. The function $\sigma(M)$ describes the uncertainty of $G_{OPE}(M)$ at Borel mass M, which stems from our limited knowledge of the vacuum condensates which appear at higher orders of the OPE and of the fundamental parameters of QCD, such as the quark masses or the coupling constant.

$P[\rho|H]$ on the other hand is called the "prior probability" and is written down as

$$\begin{aligned} P[\rho|H] &= e^{\alpha S[\rho]}, \\ S[\rho] &= \int_0^{\infty} d\omega \Big[\rho(\omega) - m(\omega) - \rho(\omega)\log\Big(\frac{\rho(\omega)}{m(\omega)}\Big)\Big], \end{aligned} \tag{2.9}$$

where $S[\rho]$ is known as the Shannon-Jaynes entropy. The function $m(\omega)$, introduced in Eq.(2.9) is the so-called "default model". In the case of no available data $G_{OPE}(M)$ or infinitely large error $\sigma(M)$, the MEM procedure will just give $m(\omega)$ for $\rho(\omega)$ because this function maximizes $P[\rho|H]$. The default model can therefore be used to incorporate known information about $\rho(\omega)$ into the analysis.

Assembling all the terms discussed, we arrive at the final form for the probability $P[\rho|GH]$:

$$\begin{aligned} P[\rho|GH] \propto& P[G|\rho H]P[\rho|H] \\ =& e^{Q[\rho]}, \\ Q[\rho] \equiv& \alpha S[\rho] - L[\rho]. \end{aligned} \tag{2.10}$$

It is now just a numerical problem to obtain the form of $\rho(\omega)$ that maximizes $Q[\rho]$ and is the most probable $\rho(\omega)$ given $G_{OPE}(M)$ and H. For this task, the Bryan algorithm [10] has proven to be efficient in a large number of MEM studies and we use a slightly modified version of it to extract the spectral functions shown in the following sections.

Once $\rho_{\alpha}(\omega)$ maximizing $Q[\rho]$ for a fixed value of α is found, this parameter is integrated out by averaging $\rho(\omega)$ over a range of values of α. The explicit procedure for this step is discussed in [3]. The result of this last integration then leads to our final result $\rho_{\rm out}(\omega)$.

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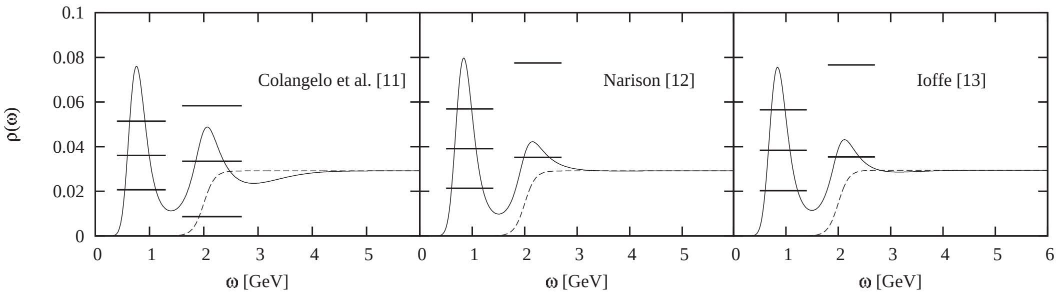

Figure 1: Results of the MEM analysis using OPE data. The dashed lines show the used default model, while the horizontal bars stand for the values of the spectral function, averaged over the peaks and the corresponding errors. For the two figures on the right, the lower error bars of the second peak are not shown because they lie below $\rho(\omega) = 0$.

3. Analysis of the ρ-meson sum rule

The explicit calculation of $G_{\rm OPE}(M)$ for the ρ-meson channel leads to:

$$\begin{aligned} G_{\rm OPE}(M) =& \frac{1}{4\pi^2}\Big(1+\eta(\alpha_s)\Big) \\ &+ \Big(2m\langle\overline{q}q\rangle + \frac{1}{12}\big\langle\frac{\alpha_s}{\pi}G^2\big\rangle\Big)\frac{1}{M^4} \\ &- \frac{112\pi}{81}\alpha_s\kappa\langle\overline{q}q\rangle^2\frac{1}{M^6} + \dots, \\ \eta(\alpha_s) =& \frac{\alpha_s}{\pi} + 0.154\alpha_s^2 - 0.372\alpha_s^3 + \dots \end{aligned} \tag{3.1}$$

In the above equation α_s represents the strong coupling constant, m is the (averaged) quark mass of the u- and d-quark, and $\langle\overline{q}q\rangle$ stands for the corresponding quark condensate. Furthermore, the gluon condensate $\big\langle\frac{\alpha_s}{\pi}G^2\big\rangle$ is a shorthand notation for $\big\langle\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\mu\nu}\big\rangle$, while κ is a parameter expressing the (potential) breaking of the vacuum saturation approximation, which would give $\kappa = 1$. Various estimates of the condensate values and their ranges exist, and we have used those given in three recent reviews: [11, 12, 13]. The explicit values can be found in [3].

The MEM analysis of $G_{\rm OPE}(M)$ was carried out using the default model

$$m(\omega) = \frac{1}{4\pi^2}\Big(1+\frac{\alpha_s}{\pi}\Big)\frac{1}{1+e^{(\omega_0-\omega)/\delta}}, \tag{3.2}$$

with $\omega_0 = 2.0$ GeV and $\delta = 0.1$ GeV, which leads to reasonable results at both low and high energy. For a detailed discussion concerning this specific choice, see again [3]. Here, we just mention that the above default model is constructed to have the physically correct behavior at both low and high energy. As the sum rules used here do not contain much information on the spectral function at very low (around $\omega = 0$) and high ($\omega \gtrsim 3$ GeV) energy, this kind of default model is necessary for the MEM output to behave reasonably in these energy regions. The results are shown in Fig. 1. It clearly seen that all three data sets give a significant lowest peak, corresponding to the ρ-meson resonance. Next, fitting the spectral functions of Fig. 1 to a relativistic Breit-Wigner peak plus a second order polynomial background, we determine the coupling strength F_ρ from our obtained spectral function. All these results are summarized in Table 1.

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Table 1: The final results for the three employed parameter sets. The errors were determined from a MEM test analysis of mock data.

	Colangelo *et al.* [11]	Narison [12]	Ioffe [13]	Experiment
m_ρ	0.76 ± 0.07 GeV	0.84 ± 0.12 GeV	0.84 ± 0.10 GeV	0.77 GeV
F_ρ	0.174 ± 0.039 GeV	0.190 ± 0.053 GeV	0.187 ± 0.050 GeV	0.141 GeV

4. Analysis of the parity projected nucleon sum rule

In QCD sum rules of the nucleon channel, it has been proposed to consider the "old fashioned" correlation function for properly implementing the parity projection [15]:

$$\begin{aligned} \Pi^{\text{for}}(q_0) &= i \int d^4x e^{iqx} \theta(x_0) \langle 0 | T[\eta(x)\overline{\eta}(0)] | 0 \rangle \Big|_{\vec{q}=0} \\ &\equiv \gamma_0 \Pi_1^{\text{for}}(q_0) + \Pi_2^{\text{for}}(q_0), \end{aligned} \tag{4.1}$$

where η is an interpolating field carrying the quantum number of the nucleon. Using the parity projection operators $P^{\pm} \equiv \frac{1}{2}(\gamma_0 \pm 1)$, one can construct a correlator that contains contributions of only positive or negative parity states, as

$$\begin{aligned} \frac{1}{2}\text{Tr}\big[P^{\pm}\Pi^{\text{for}}(q_0)\big] &= \Pi_1^{\text{for}}(q_0) \pm \Pi_2^{\text{for}}(q_0) \equiv \Pi^{\pm}(q_0) \\ &= -\int_0^{\infty} \rho_{\pm}(m) \frac{1}{q_0 - m + i\varepsilon} dm. \end{aligned} \tag{4.2}$$

Making use of the fact that the forward correlator is analytic in the upper half of the complex q_0 plane, we then get the parity projected sum rule:

$$\int_{-\infty}^{\infty} dq_0 \frac{1}{\pi} \text{Im}\Big[\Pi^{\pm}_{\text{OPE}}(q_0)\Big] W(q_0) = \int_0^{\infty} dq_0 \rho^{\pm}_{\text{Phys.}}(q_0) W(q_0). \tag{4.3}$$

Here, $\Pi^{\pm}_{\text{OPE}}(q_0)$ is calculated by the OPE in the deep Euclidean region, $\rho^{\pm}_{\text{Phys.}}(q_0)$ stands for the physical spectral function of positive and negative parity and $W(q_0)$ is an arbitrary analytic function in the upper half of the imaginary plane and real on the real axis.

Although the contributions of both parity states are now separated, we face further problems such as the large α_s corrections and large contribution of the continuum, which lower the reliability of the sum rule analysis [16]. Following the method proposed by Ioffe and Zyablyuk [17], who have constructed a new class of sum rules by using the phase rotated complex variable $q^2 e^{i\theta}$ instead of the real q^2, it is however possible to improve this situation. One advantage of this approach lies in its ability to suppress certain terms of the OPE by choosing some specific value of θ. To apply the above idea to the parity projected sum rules, we use the phase-rotated kernel:

$$W(q_0, s, \tau, \theta) dq_0 = \frac{1}{\sqrt{4\pi\tau}} \text{Re}\left[q_0 e^{-i\theta} \exp\Big(-\frac{(q_0^2 e^{-2i\theta} - s)^2}{4\tau} \Big) e^{-i\theta} dq_0 \right]. \tag{4.4}$$

With this kernel we obtain the specific form of $G^{\text{for}\,\pm}_{\text{OPE}}(s, \tau, \theta)$ which is defined as the left hand side of Eq.(4.3) [4]:

$$\begin{aligned} G^{\text{for}\,\pm}_{\text{OPE}}(s, \tau, \theta) &= \big(C_0 + \frac{\alpha_s}{\pi} C_{0\alpha_s}(\theta)\big) \cos 5\theta + C_4 \langle \frac{\alpha_s}{\pi} G^2 \rangle \cos\theta + \cdots \\ &\pm \Big[(C_3 + \frac{\alpha_s}{\pi} C_{3\alpha_s}) \langle \overline{q}q \rangle \cos 2\theta + C_5 \langle \overline{q} g \sigma \cdot G q \rangle + \cdots \Big], \end{aligned} \tag{4.5}$$

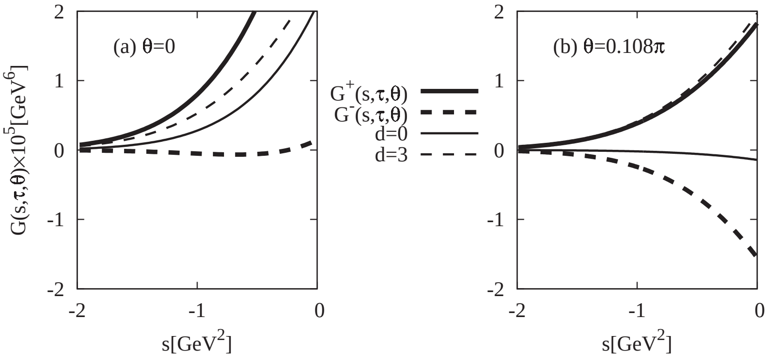

Figure 2: (a) $G_{\rm OPE}^{\rm for\,\pm}(s,\tau)$, dimension 0 (perturbative) and dimension 3 (chiral condensate) terms at $\tau = 0.5[\mathrm{GeV}^4]$ and $\theta = 0$. The thick lines denote $G_{\rm OPE}^{\rm for\,\pm}(s,\tau)$ and the thin lines are the dimension 0 and 3 terms. (b) Same as for (a), but for $\theta = 0.108\pi$.

where $C_0, C_{0\alpha}(\theta), C_3, C_{3\alpha_s}, C_4$ and C_5 are numerical coefficients. Choosing a suitable value for the phase θ, the ratio of α_s corrections to leading terms at dimension 0 ($\frac{C_{0\alpha_s}}{C_0}$) is reduced significantly. Specifically this is ratio is suppressed from 90 % at $\theta = 0$ to 5 % at $\theta = 0.108\pi$. The perturbative and chiral condensate terms and $G^{\pm}(s,\tau)$ at $\theta = 0$ and $\theta = 0.108\pi$ are given in Fig. 2. It can be seen in the figure that in the phase rotated sum rule, the contribution of the perturbative term is strongly reduced and the $\langle \overline{q}q \rangle$ term is dominant. We also find that the difference of the OPE data between the positive parity and negative parity states is caused by the chiral condensate in Fig. 2.

Carrying out the analysis using the OPE data $G_{\rm OPE}^{\rm for\,\pm}(s,\tau,\theta = 0.108\pi)$ with the help of MEM, we obtain the corresponding spectral functions. The results are shown in Fig. 3. In the positive parity spectral function, peaks are found at 970 MeV and 1930 MeV. As can be inferred from the error bars, the lowest peak which corresponds to the nucleon ground state is statistically significant, while the second one is not. For negative parity, peaks appear at 1540 MeV and 1840 MeV. As for the positive parity case, the second peak is not statistically significant. The lowest peak appears close to the experimentally observed lowest negative parity state N(1535). However, we can not conclude that this peak only contains the contribution of the N(1535) due to its large width and

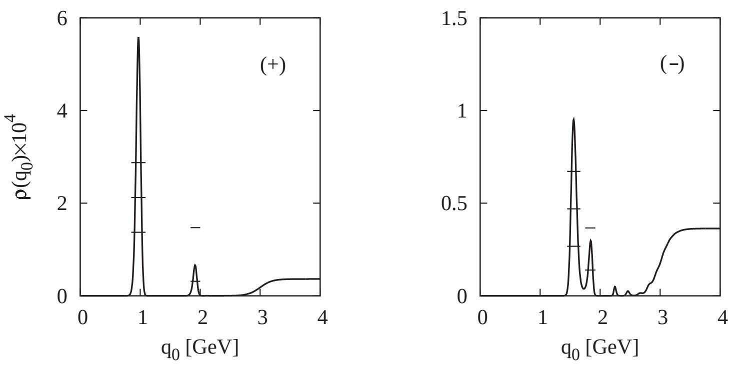

Figure 3: The positive (left) and negative parity (right) spectral functions extracted from MEM analyses of the OPE data $G_{\rm OPE}^{\rm for\,\pm}(s,\tau,\theta = 0.108\pi)$. The parity of the corresponding spectral functions is shown on the top right corner of each figure.

the next lying state N (1650). Related to this topic, it is important to note that the MEM analysis only has a limited resolution and thus will generate some artifical width to any peak and generally only give smeared output spectral functions. This fact could cause the N(1650) to merge with the N(1535), leaving only a single peak. Therefore our conclusion to be drawn from this analysis is that some negative parity excited state exists near 1540 MeV.

5. Summary and conclusions

The MEM technique has been applied to QCD sum rules for the ρ-meson and nucleon channels, showing that this novel approach makes it possible to extract the spectral functions from the sum rules without making strong assumptions about their explicit form. These findings are promising and encourage us to expand this method to other channels and to the investigation of the properties of hadrons at finite temperature or density.

Acknowledgments

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