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Definitions of a static SU(2) color triplet potential

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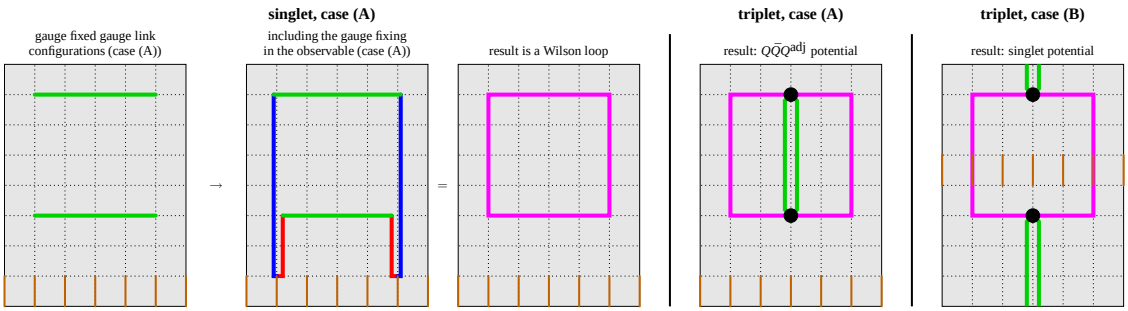
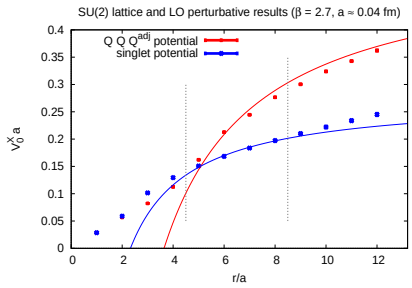
We discuss possibilities and problems to non-perturbatively define and compute a static color triplet potential in SU(2) gauge theory. Numerical lattice results are presented and compared to analytical perturbative results.





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Calculating the static potential in SU(2) gauge theory: basic principle

The calculation of the singlet static potential is usually based on trial states $|\Phi^{\text{sing}}\rangle \equiv \bar{Q}(-r/2)U(-r/2;+r/2)Q(+r/2)|\Omega\rangle$, while for the triplet static potential typically $|\Phi^{\text{trip},a}\rangle \equiv \bar{Q}(-r/2)U(-r/2;s)\sigma^a U(s;+r/2)Q(+r/2)|\Omega\rangle$ is suggested or used (cf. e.g. [1]). Here $\pm r/2 \equiv (0,0,\pm r/2)$, Q and $\bar{Q}$ are static quark/antiquark operators, U are spatial parallel transporters (on a lattice products of links) and σ^a denote Pauli matrices acting in color space. From the asymptotic behavior of the corresponding temporal correlation function the static potential $V_0^X(r)$, $X \in \{\text{singlet},\text{triplet}\}$ can be extracted.

Lattice computations without gauge fixing

On the lattice the singlet correlation function is proportional to Wilson loops, $\langle\Phi^{\text{sing}}(t_2)|\Phi^{\text{sing}}(t_1)\rangle \propto W(r,\Delta t)$, $\Delta t = t_2 - t_1$, from which the singlet potential can be determined (cf. the figure on page 1, blue dots). Since the triplet correlation function is not gauge invariant, one obtains $\langle\Phi^{\text{trip},a}(t_2)|\Phi^{\text{trip},a}(t_1)\rangle = 0$ and cannot determine a triplet potential.

Lattice computations in temporal gauge

Temporal gauge $A_0^g = 0$ in the continuum corresponds to temporal links $U_0^g(t,\mathbf{x}) = 1$ on a lattice. These links gauge transform according to $U_0(t,\mathbf{x}) \to U_0^g(t,\mathbf{x}) = g(t,\mathbf{x})U_0(t,\mathbf{x})g^\dagger(t+a,\mathbf{x})$, where $g(t,\mathbf{x}) \in \text{SU(2)}$. On a lattice with finite periodic temporal extension it is not possible to realize temporal gauge everywhere. There will be a slice of links, where $U_0 \neq 0$ (in the following wlog. $U_0^g(t=0,\mathbf{x}) \neq 1$, while $U_0^g(t=1\ldots T-1,\mathbf{x}) = 1$; T is the periodic temporal extension of the lattice). A possible choice for the corresponding gauge transformation $g(t,\mathbf{x})$ is
$g(t=2a,\mathbf{x}) = U_0(t=a,\mathbf{x})$,
$g(t=3a,\mathbf{x}) = g(t=2a,\mathbf{x})U_0(t=2a,\mathbf{x}) = U_0(t=a,\mathbf{x})U_0(t=2a,\mathbf{x})$,
$g(t=4a,\mathbf{x}) = g(t=3a,\mathbf{x})U_0(t=3a,\mathbf{x}) = U_0(t=a,\mathbf{x})U_0(t=2a,\mathbf{x})U_0(t=3a,\mathbf{x})$, ...

Non-perturbative computations (lattice), singlet potential:

The trial states $|\Phi^{\text{sing}}\rangle$ are gauge invariant. Therefore, the result is identical to the result without gauge fixing (cf. the figure on page 1, blue dots).

Gauge transforming the temporal links to $U_0^g(t,\mathbf{x}) = 1$ and computing

$$\langle\Phi^{\text{sing}}(t_2)|\Phi^{\text{sing}}(t_1)\rangle = \left\langle \text{Tr}\Big(U^g(t_1,-r/2;t_1,+r/2)U^g(t_2,+r/2;t_2,-r/2)\Big)\right\rangle \quad (1)$$

(here we assume $1 \leq t_1 < t_2 < T$, "case (A)") is equivalent to consider the manifestly gauge invariant observable

$$\langle\Phi^{\text{sing}}(t_2)|\Phi^{\text{sing}}(t_1)\rangle = \left\langle \text{Tr}\Big(U(t_1,-r/2;t_1,+r/2)\underbrace{g^\dagger(t_1,+r/2)g(t_2,+r/2)}_{U(t_1,+r/2;t_2,+r/2)}$$
$$U(t_2,+r/2;t_2,-r/2)\underbrace{g^\dagger(t_2,-r/2)g(t_1,-r/2)}_{U(t_2,-r/2;t_1,-r/2)}\Big)\right\rangle = W(r,\Delta t) \quad (2)$$

(cf. the figure on page 1). Similar considerations yield the same result for "case (B)", $0 = t_1 < t_2 < T$ or $1 \leq t_2 < t_1 < T$. This technique of transforming a non-gauge invariant observable into an equivalent manifestly gauge invariant observable will be helpful for interpreting the triplet potential.

A helpful theoretical tool to understand, which states are contained in a correlation function, is the transfer matrix formalism (cf. e.g. [2, 3]). Without gauge fixing the transfer matrix is $\hat{T} = e^{-Ha}$, $\hat{T}|\psi^{(n)}\rangle = \lambda^{(n)}|\psi^{(n)}\rangle$, $\lambda^{(n)} = e^{-E^{(n)}a}$ (lattice discretization errors neglected), where $E^{(n)}$ are the energies of gauge invariant states (e.g. the vacuum, glueballs). Similarly the transfer matrix in temporal gauge is $\hat{T}_0 = e^{-H_0 a}$, $\hat{T}_0|\psi_0^{(n)}\rangle = \lambda_0^{(n)}|\psi_0^{(n)}\rangle$. In temporal gauge remaining gauge degrees of freedom are time-independent gauge transformations $g(\mathbf{x})$. One can show $[\hat{T}_0, g(\mathbf{x})] = 0$, i.e. eigenstates of $\hat{T}_0$ can be classified according to SU(2) color quantum numbers $(j(\mathbf{x}), m(\mathbf{x}))$ at each $\mathbf{x}$. $\lambda_0^{(n)} = e^{-E_0^{(n)}a}$, where $E_0^{(n)}$ are the energies of the gauge invariant states already mentioned as well as of additional non-gauge invariant states with $j(\mathbf{x}) \neq 0$. Such states can be interpreted as states containing static color charges (= static quarks)[1]. One can derive

$$\langle\Phi^{\text{sing}}(t_2)|\Phi^{\text{sing}}(t_1)\rangle = \sum_k e^{-V_k^{\text{sing}}(r)\Delta t} \sum_m e^{-\mathscr{E}_m(T-\Delta t)} \sum_{\alpha,\beta} \left|\langle k, \alpha\beta|\hat{U}_{\alpha\beta}(-r/2;+r/2)|m\rangle\right|^2, \quad (3)$$

where $\alpha \equiv m(-r/2) = \pm 1/2$ and $\beta \equiv m(+r/2) = \pm 1/2$ are color indices at $\pm r/2$. As expected this correlation function is suited to extract the common singlet potential $V_0^{\text{sing}}(r)$.

Non-perturbative computations (lattice), triplet potential:

Again one has to distinguish the two cases (A) and (B), which this time yield different results. When including the gauge fixing in the observable, one finds that (s,t_1) and (s,t_2), the spacetime positions of the "triplet generators" σ^a, are connected by an adjoint static propagator: $\text{Tr}(\sigma^a U(t_1,s;t_2,s)\sigma^b U(t_2,s;t_1,s))$. Within the transfer matrix formalism one can derive for case (A)

$$\langle\Phi^{\text{trip},a}(t_2)|\Phi^{\text{trip},a}(t_1)\rangle = \sum_{\alpha,\beta} \left|\langle k, \alpha\beta, m(s) = a|\hat{U}_{\alpha\beta,a}(-r/2;s;+r/2)|m\rangle\right|^2 \quad (4)$$

and for case (B)

$$\langle\Phi^{\text{trip},a}(t_2)|\Phi^{\text{trip},a}(t_1)\rangle = \sum_k e^{-V_k^{\text{sing}}(r)\Delta t} \sum_m e^{-\mathscr{E}_m^{Q\text{adj}}(T-\Delta t)} \sum_{\alpha,\beta} \left|\langle k, \alpha\beta|\hat{U}_{\alpha\beta,a}(-r/2;s;+r/2)|m, m(s) = a\rangle\right|^2. \quad (5)$$

The conclusion is that one can either extract a three-quark potential (one quark at $+r/2$, one antiquark at $-r/2$, one adjoint quark at s) (case (A)) or the ordinary singlet potential (case (B)).

Perturbative calculations in Lorenz gauge

Most perturbative calculations of the static potential are carried out in Lorenz gauge $\partial_\mu A_\mu = 0$. The leading order result for trial states $|\Phi^{\text{sing}}\rangle$ is $V_0^{\text{sing}}(r) = -3g^2/16\pi r$, i.e. an attractive singlet

[1] We use the following notation of energy eigenvalues $E_0^{(n)}$: (1) gauge invariant states, i.e. no static quarks: $\mathscr{E}_n$ ($j(\mathbf{x}) = 0$ for all $\mathbf{x}$); (2) a static quark/antiquark at $-r/2$ and at $+r/2$: $V_n^{\text{sing}}(r)$ ($j(-r/2) = j(+r/2) = 1/2$); (3) an adjoint static quark at s: $\mathscr{E}_n^{Q\text{adj}}$ ($j(s) = 1$); (4) a static quark/antiquark at $-r/2$ and at $+r/2$, an adjoint static quark at s: $V_n^{Q\bar{Q}Q\text{adj}}(r)$ ($j(-r/2) = j(+r/2) = 1/2$, $j(s) = 1$).

potential. This result can be compared to the non-perturbative lattice result (in any gauge), since the trial state is gauge invariant. To perform a precise matching of lattice and perturbative static potentials, higher orders (NNLO or NNNLO) are required (cf. e.g. [4, 5] for recent work on this topic), but nevertheless qualitative agreement is found (cf. the figure on page 1, blue dots and blue line). The leading order result for trial states $|\Phi^{\text{trip},a}\rangle$ is $V_0^{\text{trip}}(r) = +g^2/16\pi r$, i.e. a repulsive triplet potential. Note, however, that in Lorenz gauge a transfer matrix does not exist, which renders a physical interpretation difficult. One can also calculate the gauge invariant triplet diagram obtained by using temporal gauge (cf. the figure on page 1, "triplet, case (A)"). Then one obtains $V_0^{Q\bar{Q}Q^{\text{adj}}}(r) = -9g^2/16\pi r$ (for $s = 0$), i.e. an attractive three-quark potential. Again qualitative agreement with the lattice result is found (cf. the figure on page 1, red dots and red line).

Conclusions

The singlet potential corresponds to a gauge invariant trial state $\bar{Q}(-r/2)U(-r/2;+r/2)Q(+r/2)|\Omega\rangle$. It is the same in any gauge and its interpretation as a static quark antiquark potential is clear.

The triplet potential corresponding to trial states $\bar{Q}(-r/2)U(-r/2;s)\sigma^a U(s;+r/2)Q(+r/2)|\Omega\rangle$ is different, when using different gauges: (1) without gauge fixing it cannot be calculated/computed; (2) in temporal gauge it corresponds to a three-quark potential and not to a potential between a quark and an antiquark in a color triplet state, i.e. the name "triplet potential" is misleading; (3) in Lorenz gauge a perturbative calculation yields a repulsive potential; since a transfer matrix does not exist, the physical interpretation is unclear.

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