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Unitarity and Analyticity Constraints on π-K Form Factors

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In this work we summarize our results on the technique of unitarity and analyticity bounds of the $\pi - K$ form factors. Using these tools stringent constraints are obtained on the shape parameters of the form factors, value of the scalar form factor at the second Callan-Treiman point and regions where possible zeros of the form factor can exist. Our work elegantly ties up inputs from the lattice, chiral perturbation theory and experimental information.

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1. Introduction

The results presented in this talk are those essentially present in our recent publication, ref. [1] with some updated information. In this talk, we present the method of unitarity bounds, recently reviewed in ref. [2], which employs the general principles of unitarity and analyticity in order to find stringent bounds on form factors. In particular, we will discuss applications to the pion-Kaon form factor which is an important laboratory for testing the consistency of chiral perturbation theory with modern inputs coming from lattice and with experimental information coming in the form of measurements of the modulus and phase of the form factor along parts of the unitarity cut. In this talk, we demonstrate that by deploying newly sharpened tools we are able to produce a coherent picture that tests also some experimental determinations of shape parameters of the form factors. We use these techniques to demonstrate that there are no nearby zeros which supports the conclusions of dispersive representations in the literature. Due to space constraints, only a very brief bibliography will be provided here and we direct the reader to our publications referred to above for a more complete one.

2. Formalism

The pion-Kaon form factors of interest appear in the semi-leptonic decay of the kaon and in the decay of the τ-lepton to a hadronic final state with a pion and kaon. The kaon semi-leptonic decay is described by the matrix element

$$\langle \pi^0(p')|\bar{s}\gamma_\mu u|K^+(p)\rangle = \frac{1}{\sqrt{2}}[(p'+p)_\mu f_+(t) + (p-p')_\mu f_-(t)], \tag{2.1}$$

where $f_+(t)$ is the vector form factor and the combination

$$f_0(t) = f_+(t) + \frac{t}{M_K^2 - M_\pi^2} f_-(t) \tag{2.2}$$

defines the scalar form factor. The shape parameters of interest appear in the expansion about $t=0$

$$f_k(t) = f_+(0)\left(1 + \lambda_k' \frac{t}{M_\pi^2} + \frac{1}{2}\lambda_k'' \frac{t^2}{M_\pi^4} + \cdots\right), \qquad k = 0,+ \tag{2.3}$$

In the above λ_k' is the slope and λ_k'' is the curvature parameter where $k = 0,+$. To improve the precision and to provide bounds on the shape parameters of the form factors, we use inputs coming from certain low-energy theorems, perturbative QCD, lattice computations and chiral perturbation theory, the phase and modulus of the form factors in the low-energy part of the unitarity cut.

The method of unitarity bounds starts with the consideration of an integral of the type:

$$\int_{t_+}^{\infty} dt\, \rho_{+,0}(t)|f_{+,0}(t)|^2 \leq I_{+,0}, \tag{2.4}$$

along the unitarity cut, whose upper bound is known from a dispersion relation, satisfied by a certain QCD correlator. For the scalar form factor this reads

$$\chi_0(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_0(-Q^2)\right] = \frac{1}{\pi}\int_{t_+}^{\infty} dt\, \frac{t\,\mathrm{Im}\Pi_0(t)}{(t+Q^2)^2}, \tag{2.5}$$

$$\mathrm{Im}\Pi_0(t) \geq \frac{3}{2}\frac{t_+t_-}{16\pi}\frac{[(t-t_+)(t-t_-)]^{1/2}}{t^3}|f_0(t)|^2, \tag{2.6}$$

with $t_\pm = (M_K \pm M_\pi)^2$. Analogous correlator involving the vector form factor is denoted by $\chi_1(Q^2)$.

We can now use the conformal map $t \to \tilde{z}(t)$

$$\tilde{z}(t) = \frac{\sqrt{t_+} - \sqrt{t_+ - t}}{\sqrt{t_+} + \sqrt{t_+ - t}}, \tag{2.7}$$

that maps the cut t-plane onto the unit disc $|z| < 1$ in the $z \equiv \tilde{z}(t)$ plane, with t_+ mapped onto $z = 1$, the point at infinity to $z = -1$ and the origin to $z = 0$.[1]

Using this map, we cast eqn.(2.4) into a canonical form, incorporate phase and modulus information as well as the Callan-Treiman relations and finally employ a determinant for obtaining bounds on the shape parameters and for finding regions of excluded zeros in the complex t-plane [2].

3. Inputs

The essential inputs of our formalism, the vector $\chi_1(Q^2)$ and the scalar $\chi_0(Q^2)$ correlators, can be calculated in perturbative QCD up to order α_s^4 for $Q^2 >> \Lambda_{QCD}^2$ We get $\chi_1(2\,\mathrm{GeV}) = (343.8 \pm 51.6) \times 10^{-5}\,\mathrm{GeV}^{-2}$ and $\chi_0(2\,\mathrm{GeV}) = (253 \pm 68) \times 10^{-6}$. An improvement can be achieved when we implement theoretical and experimental information into the formalism of unitarity bounds. The first improvement comes when we use the the value of vector form factor at zero momentum transfer. Recent determinations from the lattice give $f_+(0) = f_0(0) = 0.964(5)$. We can also use two low-energy theorems, namely soft-pion and soft-kaon theorems, for the improvement of the bounds on the slope and curvature parameters in the scalar case. The soft-pion theorem relates the value of the scalar form factor at first Callan-Treiman (CT) point $\Delta_{K\pi} \equiv M_K^2 - M_\pi^2$ to the ratio F_K/F_π of the decay constants:

$$f_0(\Delta_{K\pi}) = F_K/F_\pi + \Delta_{CT}. \tag{3.1}$$

Recent lattice evaluations with N_f = 2 + 1 flavors of sea quarks give $F_K/F_\pi = 1.193 \pm 0.006$. In the isospin limit, $\Delta_{CT} = -3.1 \times 10^{-3}$ to one loop and $\Delta_{CT} \simeq 0$ to two-loops in chiral perturbation theory

At $\overline{\Delta}_{K\pi}(= -\Delta_{K\pi})$, a soft-kaon result relates the value of the scalar form factor to F_π/F_K

$$f_0(-\Delta_{K\pi}) = F_\pi/F_K + \overline{\Delta}_{CT}. \tag{3.2}$$

A calculation in ChPT to one-loop in the isospin limit gives $\overline{\Delta}_{CT} = 0.03$, but the higher order ChPT corrections are expected to be larger in this case. As discussed in [1], due to the poor knowledge of $\overline{\Delta}_{CT}$, the low-energy theorem eqn.(3.2) is not useful for further constraining the shape of the $K_{\ell 3}$ form factors at low energies. On the other hand, we obtain from the same machinery, the stringent bound on the quantity $\overline{\Delta}_{CT}$ which is $-0.046 \leq \overline{\Delta}_{CT} \leq 0.014$.

[1] The associated mathematical theory is that of Hardy Spaces and Analytic Interpolation Theory, see Ref.[2] and also Irinel Caprini, talk given at this workshop.

Further improvement of the bounds can be achieved if the phase of the form factor along the elastic part of the unitarity cut is known from an independent source. In our calculations we use below t_{in} recent high precision phase-shift parametrizations (see ref. [1] for details). Above t_{in} we take $\delta(t)$ to be Lipschitz continuous, i.e., a smooth function approaching π at high energies. The results are independent of the choice of the phase for $t > t_{in}$. We can further improve the bounds if the modulus of the form factor is known along the unitarity cut, $t \leq t_{in}$: we can shift the branch point from $t_{\pm}$ to t_{in} by subtracting the low energy integral from the integral Eq. (2.4). In order to estimate the low-energy integral, which is the value of the integral contribution from t_+ to t_{in},we use the Breit-Wigner parameterizations of $|f_+(t)|$ and $|f_0(t)|$ in terms of the resonances given by the Belle Collaboration for fitting the rate of $\tau \to K\pi\nu$ decay. The above leads to the value $31.4 \times 10^{-5}\,\text{GeV}^{-2}$ for the vector form factor and 60.9×10^{-6} for the scalar form factor.

By combining with the values $I_{+,0}$, we obtain the new upper bound on the integral Eq. (2.4) from t_{in} to ∞, $I'_+ = (312 \pm 69) \times 10^{-5}\text{GeV}^{-2}$ and $I'_0 = (192 \pm 90) \times 10^{-6}$.

4. Results

4.1 Shape parameters

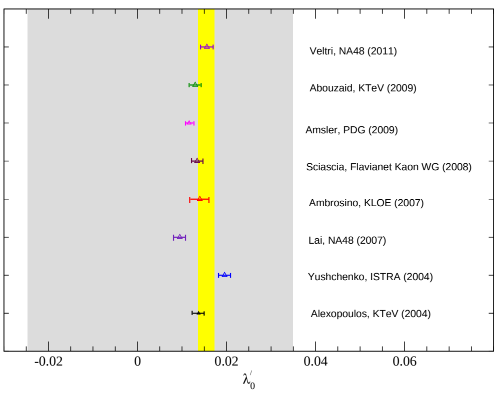

Figure 1: The allowed range for the slope of the scalar form factor, when we include phase, modulus and the CT constraint (yellow band). The grey band shows the range without the CT constraint.

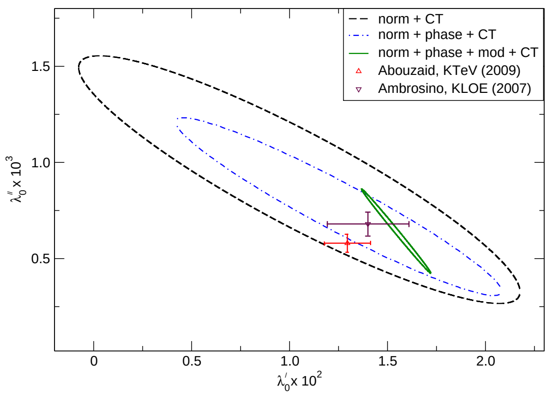

Figure 2: Allowed domain for the slope and curvature of the scalar form factor, using the normalization $f_+(0) = 0.962$, the value $f_0(\Delta_{K\pi}) = 1.193$, and phase and modulus information up to $t_{in} = (1\,\text{GeV})^2$.

In Figs. 1 and 2, the constraints for the scalar form factor are represented together with experimental information from various experiments. As shown in Fig. 1, the slope λ'_0 of the scalar form factor, predicted by NA48 (2007) is not consistent with our predictions (yellow band) which are obtained by taking into account the phase, modulus as well as the CT constraint. Nevertheless our predicted range for the slope is well-respected by the recent 2011 analysis by NA48 [3] .

The value of λ'_0 for this new determination by the NA48 reads $\lambda'_0 = (15.6 \pm 1.2 \pm 0.9) \times 10^{-3}$. On the theoretical side, the prediction of ChPT to two loops gives $\lambda'_0 = (13.9^{-0.4}_{+1.3} \pm 0.4) \times 10^{-3}$ and $\lambda''_0 = (8.0^{-1.7}_{+0.3}) \times 10^{-4}$ which are consistent with our results within errors as shown in Fig. 2. For the central value of the slope λ'_0 given above, the range of λ''_0 is $(8.24 \times 10^{-4}, 8.42 \times 10^{-4})$. The predictions $\lambda_0' = (16.00 \pm 1.00) \times 10^{-3}$, $\lambda_0'' = (6.34 \pm 0.38) \times 10^{-4}$ are obtained from dispersion relations. Comparison of the experimental results with our constraints for the vector form factor were discussed in ref. [1]. We also note that the theoretical predictions $\lambda_+' = (24.9 \pm 1.3) \times 10^{-3}$, $\lambda_+'' = (1.6 \pm 0.5) \times 10^{-3}$ obtained from ChPT to two loops, and $\lambda_+' = (26.05^{+0.21}_{-0.51}) \times 10^{-3}$, $\lambda_+'' = (1.29^{+0.01}_{-0.04}) \times 10^{-3}$, and $\lambda_+' = (25.49 \pm 0.31) \times 10^{-3}$, $\lambda_+'' = (1.22 \pm 0.14) \times 10^{-3}$ obtained from dispersion relations are consistent with the constraint. For more results, see [1].

4.2 Zeros

We also apply the technique to find regions on the real axis and in the complex t-plane where zeros are excluded. The knowledge of zeros is of interest, for instance, for the dispersive methods (Omnès-type representations) and for testing specific models of the form factors. In particular, an interesting discussion is presented on the role of zeros in ref. [4]. As can be seen from Figs. 3 and 4, nearby zeros distorts the form factors when dispersive parameterizations are used while the zeros far away have no influence on the form factors. Our results rule out nearby zeros of the type considered in ref. [4] for purposes of illustration. In the case of the vector form factors, simple zeros are excluded in the interval $-0.31\,\text{GeV}^2 \leq t_0 \leq 0.23\,\text{GeV}^2$ of the real axis, while for the scalar form factor the range with no zeros is $-0.91\,\text{GeV}^2 \leq t_0 \leq 0.48\text{GeV}^2$. If we also impose the Callan-Treiman constraint, the scalar form factor cannot have simple zeros in the range $-1.81\,\text{GeV}^2 \leq t_0 \leq 0.93\,\text{GeV}^2$. We can also extend our technique to derive regions in the complex plane where the form factors can not have zeros. The formalism rules out zeros in the physical region of the kaon semileptonic decay. In the case of complex zeros, we have obtained a rather large region where they cannot be present. For the scalar and vector form factors, Figs. 5 and 6 respectively show the region where complex zeros are excluded. For more results, see [1].

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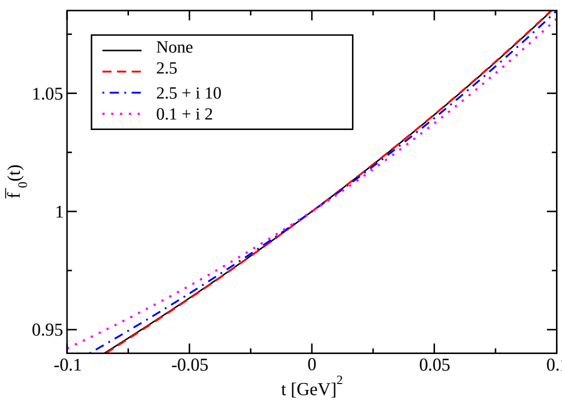

Figure 3: Influence of timelike zeros (from ref. [4])

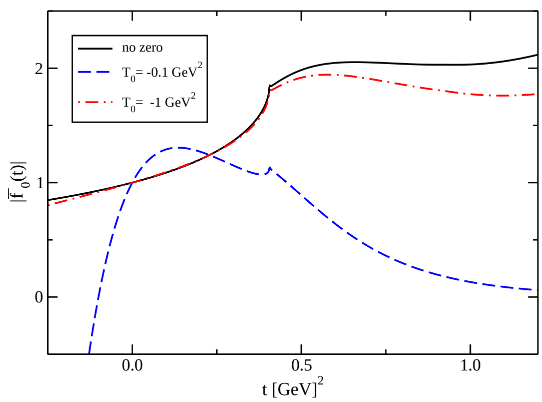

Figure 4: Influence of spacelike zeros (from ref. [4])

The results in our formalism are independent of phase information in the inelastic region and leads without any assumptions to a rather large domain where complex zeros are excluded.

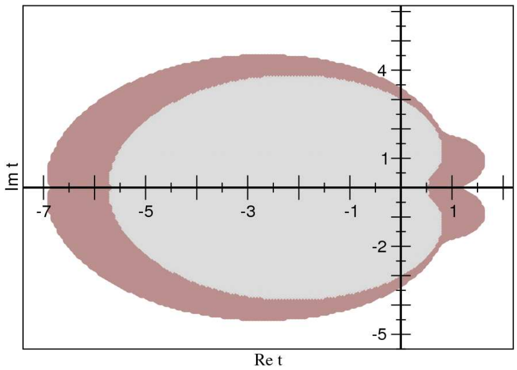

Figure 5: Domain without zeros for the scalar form factor: the small domain is obtained without including phase and modulus in the elastic region, the bigger one using phase, modulus and CT constraint.

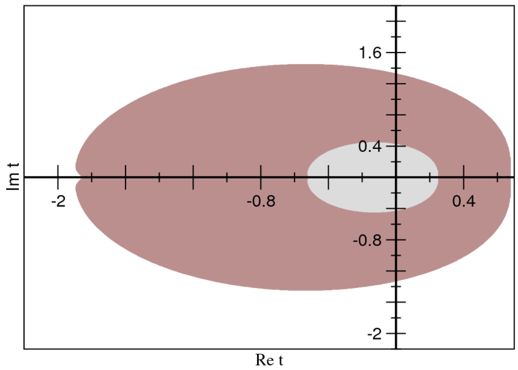

Figure 6: Domain without zeros for the vector form factor: the small domain is obtained without including phase and modulus in the elastic region, the bigger one using phase and modulus.

5. Conclusion

We have derived constraints on the shape parameters of the form factors of K_{l3} decay which is the best source for the extraction of CKM matrix element V_{us}. The results are especially stringent in the case of the scalar form factor. The most recent results from NA48 [3] are consistent with our prediction for the slope of scalar form factor and restrict the range of the slope to $\sim 0.01-0.02$. We have also excluded zeros in a rather large domain at low energies both for the scalar and vector form factor. The Callan-Treiman input provides an additional constraint in the case of the scalar form factor and as a result excludes a larger domain of the energy plane where zeros can exist. Thus, this work represents a powerful application of the theory of unitarity bounds, which relies not so much on experimental information, but on theoretical inputs from perturbative QCD, low energy theorems and lattice calculations. It provides a powerful consistency check on determinations of shape parameters from experimental analyses.

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