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The Lambda parameter and strange quark mass in two-flavour QCD

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Patrick Fritzsch*
Institut für Physik,
Humboldt-Universität zu Berlin,
Newtonstr. 15, 12489 Berlin, Germany
E-mail: fritzsch@physik.hu-berlin.de

We present the final non-perturbative results for the Lambda parameter and strange quark mass as obtained by the ALPHA collaboration in $N_\mathrm{f} = 2$ lattice QCD. Starting from measurements on large volume ensembles ($m_\pi L \geq 4$), we employ two complementary strategies for the (partially quenched) chiral extrapolation of the kaon decay constant and strange quark mass. The former is used to fix the overall energy scale of QCD. Non-perturbative renormalization and a recursive finite-size scaling procedure in the continuum allows us to compute the Lambda parameter and the renormalization group invariant strange quark mass. Here renormalized perturbation theory enters well controlled at scales of O(100GeV).

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*Speaker.

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1. Introduction

Our current theoretical comprehension of the standard model of particle physics relies on some basic input parameters in order to unfold its predictive power to describe the various processes observed in nature at low and high energies. These parameters have to be determined by matching theory to experimental data. Regarding QCD there are the mass parameters $\{m_q, q = \mathrm{u,d,s,c,b,t}\}$ for the $N_\mathrm{f} = 6$ quark flavors and the strong coupling constant g_s^2. To give any meaning to the theory and its parameters one needs to apply a well-suited regularization and renormalization procedure. By doing so the input parameters become scale-dependence quantities $\overline{g}_s^2(\mu)$ and $\overline{m}_q(\mu)$. An equivalent choice of input parameters is given by the Lambda parameter

$$\Lambda = \mu\,[b_0\overline{g}_s^2(\mu)]^{-b_1/(2b_0^2)}\,\mathrm{e}^{-1/[2b_0\overline{g}_s^2(\mu)]} \times \exp\left\{-\int_0^{\overline{g}_s(\mu)} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\} \tag{1.1}$$

and the renormalization group invariant (RGI) quark mass(es)

$$M_q = \overline{m}_q(\mu)\,[2b_0\overline{g}_s^2(\mu)]^{-d_0/(2b_0)} \times \exp\left\{-\int_0^{\overline{g}_s(\mu)} \mathrm{d}g \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right\}. \tag{1.2}$$

In a massless renormalization scheme, such as $\overline{\mathrm{MS}}$, the RGI quark masses are scheme independent while Λ is still a scheme dependent constant. However, the resulting scheme dependence is trivial since two well-behaved massless schemes are exactly related by the ratio of their Λ parameters that can be calculated analytically. Hence the M_q and Λ are fundamental parameters of QCD.

Here we present results for the scale parameter Λ and the strange quark mass M_s in two-flavor QCD as the outcome of a long term project by the ALPHA Collaboration [1]. We use the lattice regularization of QCD with $N_\mathrm{f} = 2$ degenerate, non-perturbatively improved Wilson fermions and plaquette gauge action. Moreover, the Schrödinger functional (SF) is used as intermediate massless finite-volume renormalization scheme to bridge large scale ratios non-perturbatively via a recursive finite-size step-scaling procedure in the continuum. The SF provides a natural non-perturbative definition of the strong coupling $\overline{g}_\mathrm{SF}^2$ that runs with the finite extent of the box, $\mu = 1/L$. Starting from a typical low energy scale $L_\mathrm{max} \sim 0.5\mathrm{fm}$, the scale evolution of this coupling has been worked out in two-flavor QCD [2] where PT only enters at scales of $\mathrm{O}(100\mathrm{GeV})$ to connect $\overline{g}_\mathrm{SF}^2$ to its Lambda parameter, $\Lambda_\mathrm{SF}^{(2)}$. In this way the perturbative uncertainty becomes negligible. Following a similar strategy, also the connection of any renormalized quark mass $\overline{m}_q(L_\mathrm{max})$ as computed in the low-energy regime of QCD ($\mu = 1/L_\mathrm{max}$) to its RGI quark mass M_q is known [3].

To convert these results into physical units demands the determination of the lattice scale which has been done only recently [1] due to the effort of the Coordinated Lattice Simulations (CLS) [4] consortium. In order to keep control over all systematic errors that appear when QCD is regularized on a lattice, one needs to balance physical scales that typically appear in a simulation ($a \ll m_\pi^{-1}, \ldots \ll L$) against the time that is spent to produce statistically independent configurations while attaining a good signal-to-noise ratio for the observables of interest. Accordingly, the lattice community is not yet able to perform simulations at the physical point where quark mass ratios correspond to the physical situation while fulfilling $a \ll 0.1\mathrm{fm}$ *and* $m_\pi L > 4$.[1] The former is

[1] these criteria vary a bit from one collaboration to another

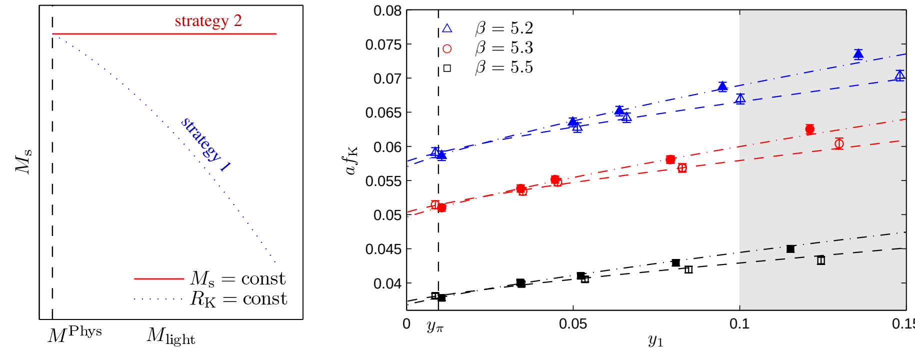

Figure 1: *Left*: Sketch of our two strategies to approach the physical point using partially quenched χPT. *Right*: Resulting chiral extrapolation of the kaon decay constant in lattice units, $F_K = af_K$, for three values of the lattice spacing corresponding to $\beta = 5.2, 5.3, 5.5$. Open symbols & dashed lines belong to strategy 1.

needed to ensure a save continuum extrapolation $a \to 0$ and the latter to get rid of finite-size effects which are exponentially suppressed in $m_\pi L$ if light meson physics is concerned. To this end CLS performed simulations with light quark masses corresponding to pseudoscalar masses $m_\pi > m_\pi^{\mathrm{phys}}$, such that beside the continuum extrapolation chiral perturbation theory (χPT) has to be invoked. For the fruitful interrelation between lattice and χPT community see for instance [5].

2. CLS ensembles and scale setting

We choose the kaon decay constant f_K as dimensionful observable to set the physical mass scale for our lattice computations. As long as all physical effects are taken into account it does not matter what quantity is used. However, since our simulations are restricted to two dynamical light quarks we have an intrinsic systematic uncertainty due to the quenched strange quark that is impossible to determine a priory. Its main advantages are: i) f_K is experimentally accessible within the envisaged precision[2], ii) at small sea quark masses χPT provides a theory for the dependence of f_K on m_π and iii) ground state properties in the meson sector can be determined without doubt.

In the following we will use two complementary strategies to approach the physical point. To this end we parametrize our observables in terms of input parameters κ_1 and κ_3 for the light and strange quark mass respectively. Since there is no perfect way to approach the physical point one has to impose a scaling condition made of physical quantities that take their experimental values at the physical point. With our choice of the scale parameter it is natural to define

$$R_K(\kappa_1,\kappa_3) = \left[m_K(\kappa_1,\kappa_3)/f_K(\kappa_1,\kappa_3)\right]^2 , \qquad R_\pi(\kappa_1,\kappa_3) = \left[m_\pi(\kappa_1)/f_K(\kappa_1,\kappa_3)\right]^2 , \tag{2.1}$$

and their values at the physical point, R_K^{phys} and R_π^{phys}, given through

$$m_\pi^{\mathrm{phys}} = 134.8\,\mathrm{MeV} , \qquad m_K^{\mathrm{phys}} = 494.2\,\mathrm{MeV} , \qquad f_K^{\mathrm{phys}} = 155\,\mathrm{MeV} . \tag{2.2}$$

Here they are given in the isospin symmetric limit with QED effects removed [6]. As the strange quarks mass on the lattice depends on the light quark mass chosen in the dynamical fermion simulations where $m_\pi(\kappa_1) > m_\pi^{\mathrm{phys}}$, κ_3 becomes a function of κ_1. Beside a theoretically well-motivated extrapolation formula for the quantity under consideration, f_K, one would like to ease the extrapolation such that $f_K(\kappa_1,\kappa_3)$ depends only little on the distance to the physical point.

[2] considering the CKM matrix element V_{us} known

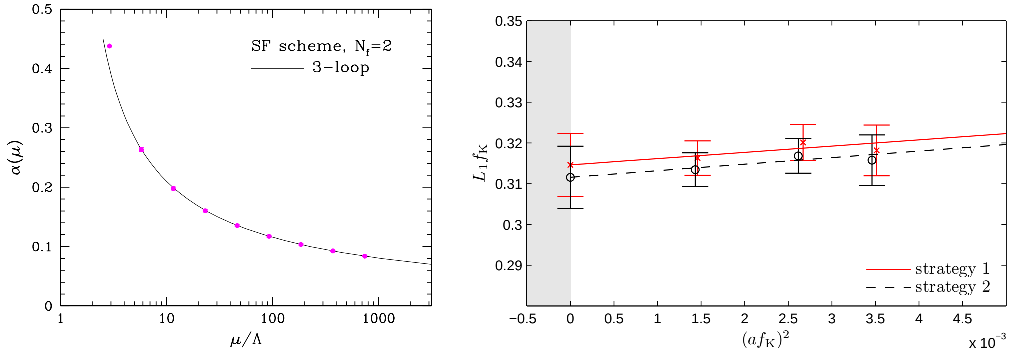

Figure 2: *Left*: Non-perturbative running of the strong coupling in the SF [2]. *Right*: Continuum extrapolation of $L_1 f_{\rm K}$ as function of $F_{\rm K}^2 = (af_{\rm K})^2$ after scale setting using large volume (CLS) ensembles.

1st strategy: We employ NLO partially quenched SU(3) χPT as in [7] and impose the condition

$$R_{\rm K}\big(\kappa_1, h(\kappa_1)\big) = R_{\rm K}^{\rm phys}\,. \tag{2.3}$$

This defines a trajectory where $f_{\rm K}$ varies only little in χPT according to

$$\begin{aligned} f_{\rm K}\big(\kappa_1, h(\kappa_1)\big) &= f_{\rm K}^{\rm phys}\,\big[1 + \overline{L}_{\rm K}(y_1,y_{\rm K}) + (\alpha_4 - \tfrac{1}{4})\,(y_1 - y_\pi) + {\rm O}(y^2)\big]\,,\\ \overline{L}_{\rm K}(y_1,y_{\rm K}) &= L_{\rm K}(y_1,y_{\rm K}) - L_{\rm K}(y_\pi,y_{\rm K})\,,\\ L_{\rm K}(y_1,y_{\rm K}) &= -\tfrac{1}{2} y_1 \log(y_1) - \tfrac{1}{8} y_1 \log(2y_{\rm K}/y_1 - 1)\,, \end{aligned} \tag{2.4}$$

where the parameters y_i take their typical form in a chiral expansion:

$$y_1 = \tfrac{1}{2}\big[m_\pi(\kappa_1)/2\pi f_{\rm K}\big(\kappa_1,h(\kappa_1)\big)\big]^2, \quad y_\pi = \tfrac{1}{2}\big[m_\pi^{\rm phys}/2\pi f_{\rm K}^{\rm phys}\big]^2, \quad y_{\rm K} = \tfrac{1}{2}\big[m_{\rm K}^{\rm phys}/2\pi f_{\rm K}^{\rm phys}\big]^2.$$

Here a big advantage is that $m_{\rm K} \approx m_{\rm K}^{\rm phys}$ or correspondingly $M_{\rm s} + M_{\rm light} \approx$ const, such that $m_{\rm K}(\kappa_1,\kappa_3)$ is not larger than its physical value which increases the chance of being inside the domain of applicability of the chiral expansion. The left panel of Fig. 1 shows a sketch of this trajectory in the $(M_{\rm light}, M_{\rm s})$-plane and the corresponding chiral extrapolation, eq. (2.4), in the right panel.

2nd strategy: Here we intend to use NLO SU(2) χPT [8] for the chiral extrapolation which amounts to an extrapolation in the light quark mass while keeping the strange quark mass constant. To do so we tune κ_3 such that the PCAC mass am_{34}, with $\kappa_4 = \kappa_3$, has a prescribed value μ, i.e. $am_{34}(\kappa_1,\kappa_3) = \mu$ implicitly defines a function $\kappa_3 = s(\kappa_1,\mu)$ at which am_{34} becomes independent of κ_1. The value of μ at the physical point needs to be known, thus we first extrapolate $M_{\rm K}^2 = (am_{\rm K})^2$ and $F_{\rm K} = af_{\rm K}$ at fixed μ according to

$$\begin{aligned} F_{\rm K}\big(\kappa_1, s(\kappa_1,\mu)\big) &= p(\mu)\,\big[1 - \tfrac{3}{8}[y_1\log(y_1) - y_\pi\log(y_\pi)] + \alpha_{\rm f}(\mu)\,(y_1 - y_\pi) + {\rm O}(y_1^2)\big]\,,\\ M_{\rm K}^2\big(\kappa_1, s(\kappa_1,\mu)\big) &= q(\mu)\,\big[1 + \alpha_{\rm m}(\mu)\,(y_1 - y_\pi) + {\rm O}(y_1^2)\big]\,, \end{aligned} \tag{2.5}$$

to $y_1 = y_\pi$. One obtains the $q(\mu), p(\mu)$ as polynomials in the neighborhood of $\mu = \mu_{\rm s}$ where $q(\mu)/p(\mu)^2 = R_{\rm K}$ attains its physical value. Solving numerically for μ_s gives $a = p(\mu_{\rm s})/f_{\rm K}^{\rm phys}$.

Results: With these two strategies we finally arrive at the three lattice spacings in physical units

$$a/{\rm fm} = 0.0755(9)(7)\,,\; 0.0658(7)(7)\,,\; 0.0486(4)(5)\,, \tag{2.6}$$

corresponding to values of the bare gauge couplings $\beta = 5.2, 5.3, 5.5$, respectively. We have checked that higher order terms are small and include only data in our final analysis that fulfills $y_1 < 0.1$. The second error is systematic and given by the difference of the two strategies.

3. Computation of the Lambda parameter

After the discussion in the previous section we know the scale in our large volume lattice simulations and thus are able to perform continuum extrapolation of other quantities such as the Lambda parameter where the master formula of the ALPHA collaboration for $N_\mathrm{f} = 2$ reads

$$\frac{\Lambda^{(2)}_{\overline{\mathrm{MS}}}}{f_\mathrm{K}} = \frac{1}{L_1 f_\mathrm{K}} \times \Lambda^{(2)}_\mathrm{SF} L_1 \times \frac{\Lambda^{(2)}_{\overline{\mathrm{MS}}}}{\Lambda^{(2)}_\mathrm{SF}} \,. \tag{3.1}$$

Accordingly $\Lambda^{(2)}_{\overline{\mathrm{MS}}}$ is given in units of the scale setting parameter f_K. It is traditionally decomposed into different parts that are all determined independently in the continuum limit of QCD. The ratio $\Lambda^{(2)}_{\overline{\mathrm{MS}}}/\Lambda^{(2)}_\mathrm{SF} = 2.382035(3)$ [9] connects the $\overline{\mathrm{MS}}$ scheme to the Schrödinger functional scheme, since in the latter the RG running of the strong coupling is non-perturbatively known [2]. The left panel of figure 2 shows the corresponding scale dependence of the strong coupling in the SF plotted against the inverse of $L\Lambda^{(2)}_\mathrm{SF}$. For additional details we have to refer the reader to [2].

In the SF the renormalization scale is given by the inverse box length, $\mu = 1/L$, which is implicitly fixed through a prescribed value of the non-perturbatively renormalized strong coupling $\bar{g}^2_\mathrm{SF}(L)$ at that scale. In our case it has been kept fix at $\bar{g}^2_\mathrm{SF} = 4.484$, corresponding to a typical hadronic length scale $L = L_1 \sim 0.4\,\mathrm{fm}$ which allows to make contact to physical units in eq. (3.1). Re-evaluating the data from ref. [2] at this specific value of the SF coupling leads to $\Lambda^{(2)}_\mathrm{SF} L_1 = 0.264(15)$ in the continuum.

Due to the results reported in the previous section it recently has become possible to also evaluate the last missing and most expensive piece in the computation of the Lambda parameter. Computing $L_1 f_\mathrm{K}$ actually means to perform the continuum extrapolation of $[L_1/a] \times [af_\mathrm{K}]$ where L_1/a is the volume in lattice units – implicitly fixed by $\bar{g}^2_\mathrm{SF}(L_1) = 4.484$ – and af_K to be taken from lattice QCD simulations at the physical pion mass. The set of CLS ensembles that have been used in the scale setting are roughly described by $Lm_\pi \geq 4$, $a < 0.08\,\mathrm{fm}$, $m_\pi \lesssim 500\,\mathrm{MeV}$. Again using the results of $F_\mathrm{K} = af_\mathrm{K}$ from the two complementary strategies described before we obtain in the continuum limit

$$L_1 f_\mathrm{K} = 0.315(8)(2)\,. \tag{3.2}$$

Its very flat continuum extrapolation in F^2_K is shown in the right panel of figure 2. The first error is statistical while the second error is due two the systematic uncertainty from the scale setting; here the difference of $L_1 f_\mathrm{K}$ between strategy 1 and strategy 2. Combining all pieces in the computation of the Lambda parameter finally leads to

$$\Lambda^{(2)}_{\overline{\mathrm{MS}}} = 310(20)\,\mathrm{MeV}\,. \tag{3.3}$$

4. Computation of the strange quark mass

To compute the strange quark's mass in physical units in two-flavor QCD we express the RGI quark mass in terms of the scale setting parameter and apply a similar decomposition as in the case of the Lambda parameter:

$$\frac{M_\mathrm{s}}{f_\mathrm{K}} = \frac{M}{\overline{m}(L_1)} \times \frac{\overline{m}_\mathrm{s}(L_1)}{f_\mathrm{K}} \qquad \Rightarrow \qquad \frac{\overline{m}^{\overline{\mathrm{MS}}}_\mathrm{s}(\mu)}{f_\mathrm{K}} = \frac{\overline{m}^{\overline{\mathrm{MS}}}(\mu)}{M} \times \frac{M_\mathrm{s}}{f_\mathrm{K}} \,. \tag{4.1}$$

The first universal continuum factor represents the non-perturbative RG running of the quark mass in the SF scheme from the same scale as before. The running is already known [3] and shown in the top panel of figure 3. Re-evaluating the data at this scale gives $\frac{M}{\overline{m}(L_1)} = 1.308(13)$. Computing the second factor in large volume is most natural using strategy 2. Its continuum extrapolation is plotted in figure 3 together with the one from strategy 1. We observe rather large cutoff effects and obtain $\frac{\overline{m}_s(L_1)}{f_K} = 0.678(12)(5)$. Combining both numbers and converting to physical units via $f_K^{\rm phys}$ from eq. (2.2) we obtain the RGI strange quark mass:

$$M_s = 138(3)(1)\,\mathrm{MeV}\,. \tag{4.2}$$

The conversion of our result to the $\overline{\rm MS}$ scheme is the only part of the computation where we have to take recourse to perturbation theory. Using our value for the Lambda parameter, equation (3.3), and the method described in [3] that employs the 4-loop beta function and mass anomalous dimension in the perturbative running we obtain the factor $\frac{\overline{m}^{\overline{\rm MS}}(\mu{=}2\mathrm{GeV})}{M} = 0.740(12)$ that connects M_s to

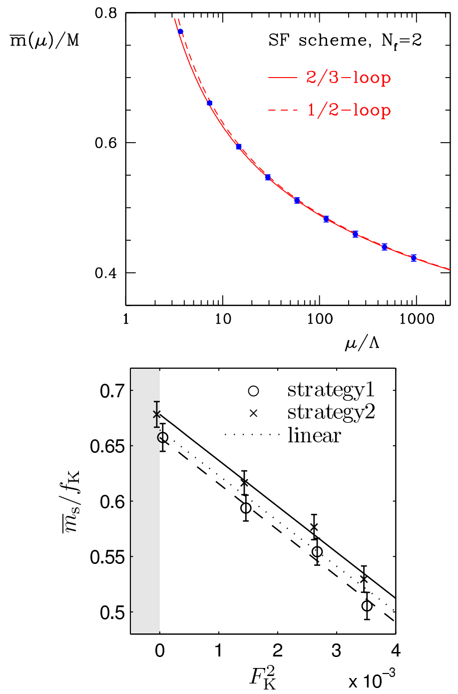

Figure 3: *Top*: Non-perturbative running of the quark mass in the SF [3]. *Bottom*: Continuum extrapolation of the strange quark mass.

$$\overline{m}_s^{\overline{\rm MS}}(2\,\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}\,. \tag{4.3}$$

This result also includes the statistical uncertainty of $\Lambda^{(2)}_{\overline{\rm MS}}$. For all the additional details that could not be covered here we refer the interested reader to [1].

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