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Roy–Steiner equations for πN scattering

Christoph Ditsche[*],[a] **Martin Hoferichter,**[ab] **Bastian Kubis,**[a] **and Ulf-G. Meißner**[ac]

[a] *Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, D-53115 Bonn, Germany*

[b] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, Universität Bern, CH-3012 Bern, Switzerland*

[c] *Institut für Kernphysik, Institute for Advanced Simulation, and Jülich Center for Hadron Physics, Forschungszentrum Jülich, D-52425 Jülich Germany*

E-mail: `ditsche@hiskp.uni-bonn.de`

Starting from hyperbolic dispersion relations for the invariant amplitudes of pion–nucleon scattering together with crossing symmetry and unitarity, one can derive a closed system of integral equations for the partial waves of both the s-channel ($\pi N \to \pi N$) and the t-channel ($\pi\pi \to \bar{N}N$) reaction, called Roy–Steiner equations. After giving a brief overview of the Roy–Steiner system for πN scattering, we demonstrate that the solution of the t-channel subsystem, which represents the first step in solving the full system, can be achieved by means of Muskhelishvili–Omnès techniques. In particular, we present results for the P-waves featuring in the dispersive analysis of the electromagnetic form factors of the nucleon.

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[*]Speaker.

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1. Introducing Roy–Steiner equations for πN scattering

Partial-wave dispersion relations (PWDRs) together with unitarity and crossing symmetry as well as isospin and chiral symmetry (i.e. all available symmetry constraints) have repeatedly proven to be a powerful tool for studying processes at low energies with high precision [1 – 4]. For πN scattering the (unsubtracted) hyperbolic dispersion relations (HDRs) for the usual Lorentz-invariant amplitudes read [5] (using the notation of [6], see [7] for more details)

$$
\begin{aligned}
A^+(s,t) &= \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s' \left[\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{1}{s'-a}\right] \mathrm{Im}\,A^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty}\mathrm{d}t' \,\frac{\mathrm{Im}\,A^+(s',t')}{t'-t}\,,\\
B^+(s,t) &= N^+(s,t) + \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s' \left[\frac{1}{s'-s}-\frac{1}{s'-u}\right] \mathrm{Im}\,B^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty}\mathrm{d}t' \,\frac{s-u}{s'-u'}\frac{\mathrm{Im}\,B^+(s',t')}{t'-t}\,,\\
N^+(s,t) &= g^2\left[\frac{1}{m^2-s}-\frac{1}{m^2-u}\right]\,, \qquad (s-a)(u-a)=b=(s'-a)(u'-a)\,,
\end{aligned}
\tag{1.1}
$$

and similarly for A^-, B^-, and N^-, where $N^\pm$ are the nucleon pole terms and the "external" (unprimed) and "internal" (primed) kinematics are related by real hyperbola parameters a and b (as well as via $s+t+u=2(m^2+M_\pi^2)=s'+t'+u'$), so that HDRs allow for the combination of all physical regions, which is known to be crucial for a reliable continuation into the subthreshold region and hence for an accurate determination of the πN σ-term. Furthermore, the imaginary parts are only needed in regions where the corresponding partial-wave decompositions converge and the range of convergence can be maximized by tuning the free hyperbola parameter a. While the s-channel integrals start at the threshold $s_+ = W_+^2 = (m+M_\pi)^2$, the t-channel contributes already above the pseudothreshold $t_\pi = 4M_\pi^2$ far below the threshold $t_N = 4m^2$. Depending on the asymptotic behavior of the imaginary parts, in principle it could be necessary to subtract the HDRs to ensure the convergence of the integrals, thereby parameterizing high-energy information with polynomials containing a priori unknown subtraction constants. However, (additional) subtractions may also be introduced to lessen the dependence of the low-energy solution on high-energy input; the corresponding subtraction parameters then obey respective sum rules. For πN scattering it proves particularly useful to subtract at the subthreshold point $(s=u, t=0)$, as this preserves the $s \leftrightarrow u$ crossing symmetry (which can be made explicit in terms of the crossing variable $\nu = (s-u)/(4m)$ via $D^\pm(\nu,t) = A^\pm + \nu B^\pm = \pm D^\pm(-\nu,t)$). This is especially favorable for the t-channel subproblem and facilitates matching to chiral perturbation theory [8, 9] to determine the subtraction constants, which thus can be identified with the subthreshold expansion parameters.[1] In addition to the presentation in [7], we also introduce a (partial) third subtraction, which is related to the parameters a_{10}^+ and a_{01}^- of the subthreshold expansions (with $d_{0n}^+ = a_{0n}^+$ for all $n \geq 0$)

$$
A^+(\nu,t) = \frac{g^2}{m} + d_{00}^+ + d_{01}^+ t + a_{10}^+\nu^2 + \mathscr{O}\left(\nu^4,\nu^2 t,t^2\right)\,, \qquad A^-(\nu,t) = a_{00}^-\nu + a_{01}^-\nu t + \mathscr{O}\left(\nu^3,\nu t^2\right)\,.
\tag{1.2}
$$

[1] For the PWDRs of $\pi\pi$ scattering, called Roy equations [10], an analogous matching procedure for the $\pi\pi$ scattering lengths as pertinent subtraction parameters has been conducted in [11]. In contrast to $\pi\pi$ scattering, the πN scattering lengths can be extracted with high accuracy from hadronic-atom data [12, 13] and may thus serve as additional constraints on the subtraction constants in the Roy–Steiner system.

In order to derive the partial-wave HDRs, called Roy–Steiner (RS) equations, one needs to expand the s- and t-channel imaginary parts in (1.1) into the respective partial waves and subsequently project the full expanded equations onto either s- or t-channel partial waves; the resulting sets of integral equations together with the respective partial-wave unitarity relations then form the s- and t-channel RS subsystems. According to [5], the (unsubtracted) s-channel RS equations read (based on the MacDowell symmetry $f^I_{(\ell+1)-}(W) = -f^I_{\ell+}(-W)$ for all $\ell \geq 0$ [14])

$$f^I_{\ell+}(W) = N^I_{\ell+}(W) + \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \sum_J \Big\{ G_{\ell J}(W,t')\,\mathrm{Im}\, f^J_+(t') + H_{\ell J}(W,t')\,\mathrm{Im}\, f^J_-(t') \Big\}$$
$$+ \frac{1}{\pi}\int_{W_+}^{\infty} \mathrm{d}W' \sum_{\ell'=0}^{\infty} \Big\{ K^I_{\ell\ell'}(W,W')\,\mathrm{Im}\, f^I_{\ell'+}(W') + K^I_{\ell\ell'}(W,-W')\,\mathrm{Im}\, f^I_{(\ell'+1)-}(W') \Big\}\,, \qquad (1.3)$$

where due to G-parity only even/odd J contribute for isospin $I = +/-$, respectively, and the partial-wave projections of the pole terms as well as the (lowest) kernels are analytically known, the latter including in particular the Cauchy kernel: $K^I_{\ell\ell'}(W,W') = \delta_{\ell\ell'}/(W'-W) + \dots$. The s-channel $I = \pm$ partial waves are intertwined by the usual unitarity relations, which are diagonal in the s-channel isospin basis $I_s \in \{1/2, 3/2\}$ only. Once the t-channel partial waves are known, the structure of the s-channel RS subsystem is therefore similar to the $\pi\pi$ Roy equations, cf. [1]. As shown in [7], the corresponding (unsubtracted) t-channel RS equations are given by

$$f^J_+(t) = \tilde{N}^J_+(t) + \frac{1}{\pi}\int_{W_+}^{\infty} \mathrm{d}W' \sum_{\ell=0}^{\infty} \Big\{ \tilde{G}_{J\ell}(t,W')\,\mathrm{Im}\, f^I_{\ell+}(W') + \tilde{G}_{J\ell}(t,-W')\,\mathrm{Im}\, f^I_{(\ell+1)-}(W') \Big\}$$
$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \sum_{J'} \Big\{ \tilde{K}^1_{JJ'}(t,t')\,\mathrm{Im}\, f^{J'}_+(t') + \tilde{K}^2_{JJ'}(t,t')\,\mathrm{Im}\, f^{J'}_-(t') \Big\} \qquad (1.4)$$

and similarly for the f^J_- except for the fact that these do not receive contributions from the f^J_+. Here, only even or odd J' couple to even or odd J (corresponding to t-channel isospin $I_t = 0$ or $I_t = 1$), respectively, and $\tilde{K}^1_{JJ'}$ (as well as the analogous $\tilde{K}^3_{JJ'}$ for the f^J_-) contains the Cauchy kernel. Moreover, it turns out that only higher t-channel partial waves contribute to lower ones. Assuming Mandelstam analyticity, the equations (1.4) are valid for $\sqrt{t} \in [2M_\pi, 2.00\,\mathrm{GeV}]$ using $a = -2.71 M_\pi^2$, whereas (1.3) holds for $W \in [m + M_\pi, 1.38\,\mathrm{GeV}]$ using $a = -23.19 M_\pi^2$. The t-channel unitarity relations are diagonal in I_t and only linear in the $f^J_\pm$ (below the first inelastic threshold t_{inel})

$$\mathrm{Im}\, f^J_\pm(t) = \sigma^\pi_t \big(t^{I_t}_J(t)\big)^* f^J_\pm(t)\,\theta\big(t - t_\pi\big)\,, \qquad \sigma^\pi_t\, t^{I_t}_J(t) = \sin\delta^{I_t}_J(t)\, e^{i\delta^{I_t}_J(t)}\,, \qquad \sigma^\pi_t(t) = \sqrt{1 - t_\pi/t}\,,$$

from which one can infer Watson‘s final state interaction theorem [15] stating that (in the “elastic” region) the phase of $f^J_\pm$ is given by the phase $\delta^{I_t}_J$ of the respective $\pi\pi$ scattering partial wave $t^{I_t}_J$.

Due to the simpler recoupling scheme for the $f^J_\pm$, the t-channel RS subsystem can be recast as a (single-channel) Muskhelishvili–Omnès (MO) problem [16] with a finite matching point t_{m} [3] for f^0_+, f^J_-, and the linear combinations $\Gamma^J(t) = m\sqrt{J/(J+1)}\, f^J_-(t) - f^J_+(t)$ with $\Gamma^J(t_N) = 0$ for all $J \geq 1$ of the generic form (the details are given in [7])

$$f(t) = \Delta(t) + \frac{1}{\pi}\int_{t_\pi}^{t_{\mathrm{m}}} \mathrm{d}t' \frac{\sin\delta(t')\, e^{-i\delta(t')}\, f(t')}{t'-t} + \frac{1}{\pi}\int_{t_{\mathrm{m}}}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\, f(t')}{t'-t} \equiv \big|f(t)\big| e^{i\delta(t)} \quad \text{for } t \leq t_{\mathrm{m}} < t_{\mathrm{inel}}\,,$$

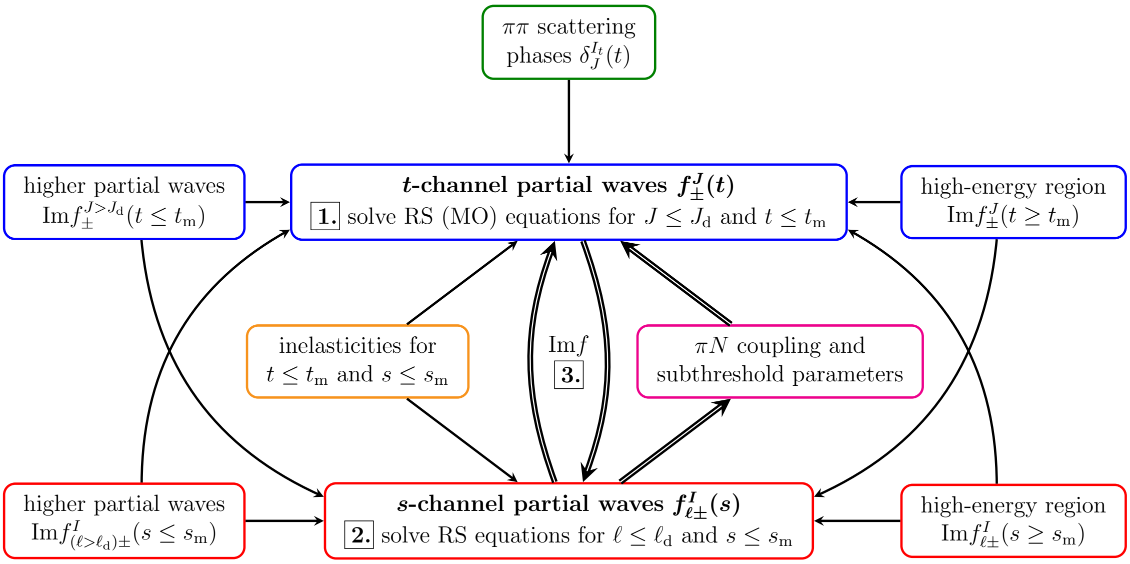

Figure 1: Flowchart of the solution strategy for the Roy–Steiner system for πN scattering. The third step consist in the self-consistent iteration (denoted by thick arrows) of the preceding steps until convergence.

where the inhomogeneities $\Delta(t)$ subsume the nucleon pole terms, all s-channel integrals, and the higher t-channel partial waves. For $t_\pi \le t \le t_\mathrm{m}$, solving for $|f(t)|$ only according to Watson's theorem requires $\delta(t)$ for $t_\pi \le t \le t_\mathrm{m}$ and $\mathrm{Im}\, f(t)$ for $t \ge t_\mathrm{m}$. Introducing $n \ge 1$ subtractions does not change the general structure of the RS/MO system, e.g. the P-waves are given by

$$\Gamma^1(t) = \Delta^1_\Gamma(t)\Big|^{n\text{-sub}} + \frac{t^{n-1}(t-t_N)}{\pi}\int_{t_\pi}^{\infty}\frac{\mathrm{d}t'\,\mathrm{Im}\,\Gamma^1(t')}{t'^{n-1}(t'-t_N)(t'-t)}\,, \qquad f^1_-(t) = \Delta^1_-(t)\Big|^{n\text{-sub}} + \frac{t^n}{\pi}\int_{t_\pi}^{\infty}\frac{\mathrm{d}t'\,\mathrm{Im}\,f^1_-(t')}{t'^n(t'-t)}\,,$$

demonstrating that Γ^J and hence f^J_+ is effectively subtracted by one power less than f^J_-, which motivates the additional (partial) third subtraction in $A^\pm$, cf. (1.2), that affects solely the f^J_+.

The solution strategy for the full RS system in the low-energy (or even subthreshold/pseudo-physical) regions, where only the lowest partial waves are relevant and inelastic contributions may be (approximately) neglected, is shown in Fig. 1; see [7] for more details.

2. The t-channel Muskhelishvili–Omnès problem: P-wave solutions

As the first step in the numerical solution of the full RS system, we check the consistency of our t-channel MO solutions with the results of the KH80 analysis [17], which are still used nowadays although no thorough error estimates are given (and despite the availability of more modern experimental data). Here, we present results for the P-waves in the (elastic) single-channel approximation of the MO problem, which is well justified for the P- and higher partial waves, whereas the S-wave requires a two-channel description including $\bar{K}K$ intermediate states as described in [7]. To produce the results (that will also serve as input for the solution of the s-channel RS subsystem, cf. Fig. 1) partly shown in Fig. 2, we have used as input $\pi\pi$ phase shifts from [18], s-channel partial waves ($l \le 4$) from SAID [19] for $W \le 2.5\,$GeV, and above the Regge model of [20]. To facilitate

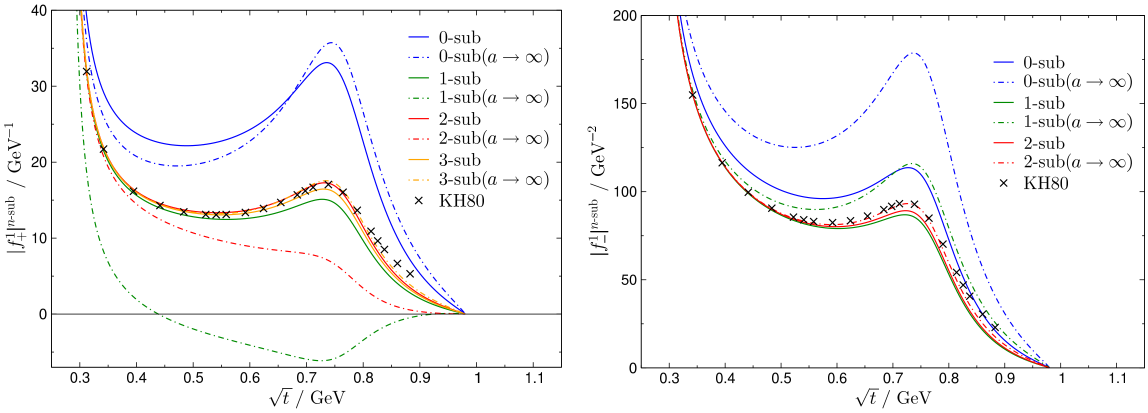

Figure 2: *n*-subtracted MO solutions for the *P*-wave moduli.

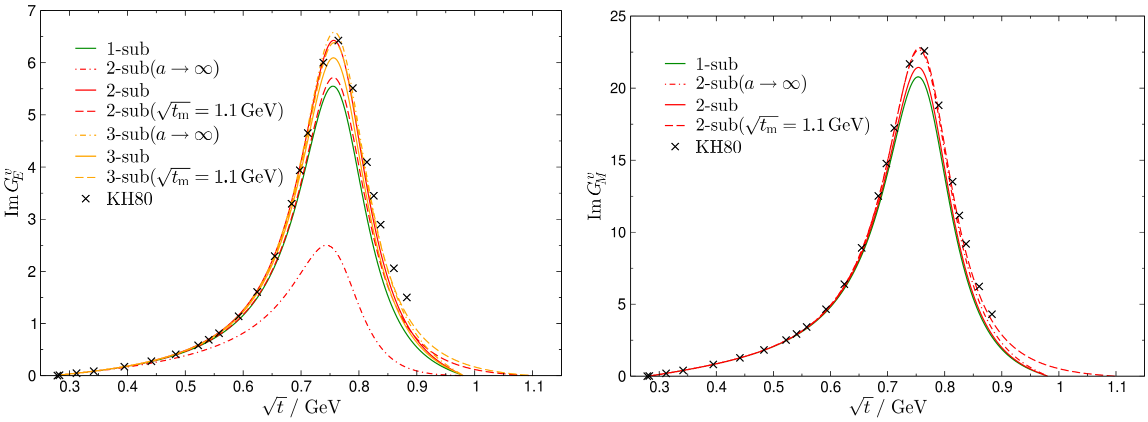

Figure 3: Two-pion-continuum contribution to $\operatorname{Im} G_E^v$ and $\operatorname{Im} G_M^v$.

comparison with the results of KH80, we use the respective subthreshold parameter values and a πN coupling of $g^2/(4\pi) = 14.28$ [6, 17] (as starting point, the final values will result from the iteration procedure, cf. Fig. 1).[2] Moreover, KH80 uses different types of dispersion relations, in particular so-called fixed-t ones, which can be emulated (up to the t-channel contributions that are not present at all in the fixed-t case) by taking the "fixed-t limit" $|a| \to \infty$. As argued in [7], all t-channel input above $\sqrt{t_m} = 0.98\,\text{GeV}$ is set to zero, which forces the MO solutions to match zero at $t = t_m$. While Fig. 2 displays the results for $|a| \to \infty$, investigating the effect of using a different (i.e. higher) matching point leads to the same conclusion: with increasing number of subtractions, thus lowering the dependence on the high-energy input by introducing more subthreshold parameter contributions as subtraction polynomials, the solutions show a nice convergence pattern both in general (proving the internal consistency and numerical stability of our RS/MO framework) and in particular towards the KH80 results (being consistent with relying on KH80 values for g and the subtraction parameters). The *P*-waves feature prominently in the dispersive analysis of the nucleon electromagnetic form factors, see e.g. [21] and references therein, and in Fig. 3 we illustrate the effects on the spectral functions (by approximating the vector pion form factor F_π^V via a

[2] Modern analyses yield significant smaller values for the πN coupling, cf. e.g. $g^2/(4\pi) = 13.7 \pm 0.2$ of [13].

twice-subtracted Omnès representation, cf. [7])

$$\operatorname{Im} G_E^v(t) = \frac{t(\sigma_t^\pi)^3}{8m}\big(F_\pi^V(t)\big)^* f_+^1(t)\,\theta\big(t-t_\pi\big)\,, \qquad \operatorname{Im} G_M^v(t) = \frac{t(\sigma_t^\pi)^3}{8\sqrt{2}}\big(F_\pi^V(t)\big)^* f_-^1(t)\,\theta\big(t-t_\pi\big)\,.$$

We are confident that a self-consistent iteration procedure between the solutions for the s- and t-channel eventually will yield a consistent and precise description (including error estimates) of the low-energy πN scattering amplitude in all kinematical channels.

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