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An anomalous hydrodynamics for chiral superfluid

Shu Lin*
RIKEN/BNL, Brookhaven National Laboratory
E-mail: slin@bnl.gov

We derived a set of hydrodynamic equations from low energy chiral Lagrangian. The effect of axial anomaly was taken into account by Wess-Zumino-Witten term. We found that the anomaly induces various effects in the presence of baryonic and axial baryonic chemical potentials. They are nonabelian analogs of chiral magnetic effect, chiral vortical effect and chiral electric effect.

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*Speaker.

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1. Introduction

Quantum anomalies violate classical conservation laws and can lead to interesting phenomena. A well-known example is the explanation of π^0 decay in terms of axial anomaly. There has been renewed interest in anomalies as they have been shown to be manifested in macroscopic hydrodynamic equations. This was first found in studies using holographic models [1]. Later, Son and Surowka derived anomaly induced terms in the framework of hydrodynamics [2]. Furthermore, they were able to show that the anomaly induced transport coefficients, being non-dissipative in nature, are almost entirely fixed by anomaly equations. The application of these results to a chirally imbalance quark gluon plasma leads to chiral magnetic effect (CME) and chiral vortical effect (CVE) [3, 4]. These effects are being actively investigated in heavy ion experiment. While its effect in deconfined phase of QCD is well understood, a natural question to ask is what happens in confined phase. The purpose of this report is to address this question.

2. Derivation of hydrodynamics from chiral Lagrangian with anomaly

The low energy degrees of freedom of QCD are pions and kaons, which are Goldstone bosons of chiral symmetry breaking. Their dynamics is governed by chiral Lagragian. At sufficient low temperature, we may ignore quantum and thermal fluctuation and consider the mean field approximation. This is a zero temperature superfluid hydrodynamics, which involves only Goldstone bosons. The effect of axial anomaly can be taken into account by including a Wess-Zumino term to the chiral Lagragian. We are interested in the low energy dynamics of QCD in the presence of external electromagnetic field, therefore we need to gauge the both the chiral Lagragian and Wess-Zumino term.

$$S = -\frac{f^2}{8}\int d^4x Tr(D_\mu U D^\mu U^+) + \Gamma_{WZ}(U,V,A) \tag{2.1}$$

where $U = e^{\frac{2i}{f}\phi}$ is the unitary matrix parameterized by the pseudoscalar ϕ and the pion decay constant f. $V = V_\mu dx^\mu$ and $A = A_\mu dx^\mu$ are one-forms of vector and axial vector flavor gauge fields. The explicit form of gauged Wess-Zumino-Witten term $\Gamma_{WZ}(U,V,A)$ has been worked out in [8]. To impose the conservation of vector currents, we need to subtract the Bardeen counter term, which is given by

$$\Gamma_c = \Gamma_{WZ}(U=1,V,A). \tag{2.2}$$

The stress energy tensor and currents can be obtained from (2.1) by variations with respect to metric and external gauge fields

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{2.3}$$

$$J_\mu = \frac{\delta S}{\delta V^\mu},\;\; J_{\mu,5} = \frac{\delta S}{\delta A^\mu}. \tag{2.4}$$

Here J_μ and $J_{\mu,5}$ are generalized non-abelian currents. We have promoted the $SU(N)$ group to $U(N)$. The addition of the $U(1)$ group allows us to introduce external baryonic and axial baryonic chemical potentials B and B_5. We will also introduce external electromagnetic field A_{em}. Our full external fields take the following form

$$V = B + A_{em}Q,\ A = B_5,$$
$$F_V = dB + dA_{em}Q,\ F_A = dB_5, \tag{2.5}$$

where Q is the charge matrix. We will put $B^\mu = (\mu, \vec{0})$ and $B_5^\mu = (\mu_5, \vec{0})$ in the end, with μ and μ_5 being baryonic and axial baryonic chemical potentials. Note that in the presence of axial anomaly, the currents J_μ and $J_{\mu,5}$ obtained from functional derivatives are "consistent"currents, which are to be distinguished from "covariant" currents commonly used in formulation of hydrodynamics. We need to add correction proportional to gradient of external gauge fields to convert "consistent" currents to "covariant" currents

$$j^\mu = J^\mu + \Delta J^\mu,\ \ j_5^\mu = J_5^\mu + \Delta J_5^\mu. \tag{2.6}$$

In terms of the "covariant" currents, the energy momentum conservation equation becomes

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda + F_5^{\nu\lambda} j_{\lambda,5} + F_Q^{\nu\lambda} j_{\lambda,Q}, \tag{2.7}$$

with $F = dB$, $F_5 = dB_5$ and $F_Q = dA_{em}$. The currents appearing on the RHS of (2.7) are abelian baryonic, axial baryonic and electromagnetic current. The explicit form of the abelian currents written in terms of $\alpha = dUU^+$ and $\beta = U^+dU$ has been worked out as [5]

$$\begin{aligned}
J_B &= \frac{5Ci}{2} Tr(\alpha^3+\beta^3) - \frac{15C}{2}\left(dBTr(\alpha+\beta) + dA_{em}TrQ(\alpha+\beta) - A_{em}TrQ(\alpha^2-\beta^2)\right)\\
J_{B5} &= -\frac{5C}{2}\left(dB_5Tr(\alpha+\beta) + dA_{em}TrQ(\alpha-\beta)\right) + \frac{5Ci}{2}A_{em}dA_{em}(TrQ^2 - TrU^+QUQ)\\
J_Q &= \frac{5Ci}{2}TrQ(\alpha^3+\beta^3) + \frac{5C}{2}\Big[-3dBTrQ(\alpha+\beta) - dB_5TrQ(\alpha-\beta) + B_5TrQ(\alpha^2+\beta^2)\\
&-dA_{em}(2TrQ^2(\alpha+\beta) + TrQdUQU^+ - TrQUQdU^+) + A_{em}(TrQ^2(\alpha^2-\beta^2) - TrQdUQdU^+)\Big]\\
&\frac{5Ci}{2}(dB_5A_{em} + 2dA_{em}B_5)(TrQUQU^+ - TrQ^2) - \frac{5Ci}{2}B_5A_{em}Tr(QdUQU^+ + QdU^+QU)\\
\Delta J_B &= 15CidBB_5Tr1 + 15CidA_{em}B_5TrQ\\
\Delta J_{B5} &= 5CidB_5B_5Tr1\\
\Delta J_Q &= 15CidBB_5TrQ + 15CidA_{em}B_5TrQ^2.
\end{aligned} \tag{2.8}$$

We can check the "covariant" currents $j_B = J_B + \Delta J_B$, $j_B = J_{B5} + \Delta J_{B5}$ and $j_Q = J_Q + \Delta J_Q$ are gauge invariant.

2.1 Non-abelian hydrodynamics with anomaly

In the absence of anomaly and external gauge fields, the hydrodynamic equations for chiral dynamics are more conveniently formulated in the left-right basis [10]:

$$J_L = J + J_5,\ \ J_R = J - J_5. \tag{2.9}$$

The dynamical equations are given by

$$D^Q{}_\mu J_{0L}^\mu = D^Q{}_\mu J_{0R}^\mu = 0, \tag{2.10}$$

with the following constitutive equations

$$J^{\mu}_{0L} = \frac{f^2}{4} iD^{\mu}UU^{+} \tag{2.11}$$

$$J^{\mu}_{0R} = -\frac{f^2}{4} iU^{+}D^{\mu}U. \tag{2.12}$$

The index 0 denote the non-anomalous contributions and $D^Q = d - \frac{iA_{em}}{2}[Q,\ ]$. We find it more instructive to rewrite (2.10) in terms of superfluid velocity by analogy with abelian superfluid hydrodynamics. Defining the non-abelian superluid velocities xiL and ξ^R as

$$iD^{Q}{}_{\mu}UU^{+} = \xi^L_{\mu},\ \ -iU^{+}D^{Q}{}_{\mu}U = \xi^R_{\mu}, \tag{2.13}$$

we can rewrite (2.10) as follows

$$\begin{aligned}
&D^{Q}{}_{\mu}J^{\mu}_{0L} = D^{Q}{}_{\mu}J^{\mu}_{0R} = 0\\
&D^{Q}{}_{t}\xi^L_i - D^{Q}{}_{i}\xi^L_t = i[\xi^L_i, \xi^L_t] - (Q-\chi^L)\frac{E_i}{2}\\
&D^{Q}{}_{t}\xi^R_i - D^{Q}{}_{i}\xi^R_t = i[\xi^R_i, \xi^R_t] + (\chi^R - Q)\frac{E_i}{2}\\
&D^{Q}{}_{t}\chi^L = -i[\xi^L_t, \chi^L]\\
&D^{Q}{}_{t}\chi^R = -i[\xi^R_t, \chi^R].
\end{aligned} \tag{2.14}$$

Here $E_i = F^Q_{i0}$ is the external electric field and we have defined new variables $\chi^L = UQU^{+}$ and $\chi^R = U^{+}QU$ in addtional to superfluid velocities, which are emergent for non-abelian theory. The first line provides the dynamical equations for ξ^L_t and ξ^R_t. The rest of the equations provide dynamical equations for ξ^L_i, ξ^R_i, χ^L and χ^R respectively.

To incorporate the effect of anomaly, we note the Wess-Zumino-Witten term does not induce correction to stress energy tensor, but does modify the currents. The correction to “covariant” currents are given by [5]

$$\begin{aligned}
&j^{\mu}_{L} = -5C\varepsilon^{\mu\nu\rho\sigma}\left(\xi^L_{\nu}\xi^L_{\rho}\xi^L_{\sigma} + \xi^L_{\nu}\xi^L_{\rho}B_{5\sigma}\right)\\
&+\frac{5Ci}{2}\varepsilon^{\mu\nu\rho\sigma}\left[\partial_{\nu}A^{em}_{\rho}(2Q+\chi^L)(\xi^L_{\sigma}+B_{5\sigma}) + (\xi^L_{\sigma}+B_{5\sigma})(2Q+\chi^L)\partial_{\nu}A^{em}_{\rho}\right]\\
&j^{\mu}_{R} = 5C\varepsilon^{\mu\nu\lambda\rho}\left(\xi^R_{\nu}\xi^R_{\rho}\xi^R_{\sigma} - \xi^R_{\nu}\xi^R_{\rho}B_{5\sigma}\right)\\
&+\frac{5Ci}{2}\varepsilon^{\mu\nu\rho\sigma}\left[\partial_{\nu}A^{em}_{\rho}(2Q+\chi^R)(-\xi^R_{\sigma}+B_{5\sigma}) + (-\xi^R_{\sigma}+B_{5\sigma})(2Q+\chi^R)\partial_{\nu}A^{em}_{\rho}\right].
\end{aligned} \tag{2.15}$$

The hydrodynamic equations are slight modified:

$$\begin{aligned}
&D^{Q}{}_{\mu}(J^{\mu}_{0L} + j^{\mu}_{L}) = 30Ci(dA_L)^2 = \frac{15Ci}{8}\varepsilon^{\mu\nu\rho\sigma}F^Q_{\mu\nu}F^Q_{\rho\sigma}Q^2\\
&D^{Q}{}_{\mu}(J^{\mu}_{0R} + j^{\mu}_{R}) = -30Ci(dA_R)^2 = -\frac{15Ci}{8}\varepsilon^{\mu\nu\rho\sigma}F^Q_{\mu\nu}F^Q_{\rho\sigma}Q^2\\
&D^{Q}{}_{t}\xi^L_i - D^{Q}{}_{i}\xi^L_t = i[\xi^L_i, \xi^L_t] - (Q-\chi^L)\frac{E_i}{2}
\end{aligned}$$

$$
\begin{aligned}
&D^Q{}_t\xi_i^R - D^Q{}_i\xi_t^R = i[\xi_i^R,\xi_t^R] + (\chi^R - Q)\frac{E_i}{2} \\
&D^Q{}_t\chi^L = -i[\xi_t^L,\chi^L] \\
&D^Q{}_t\chi^R = -i[\xi_t^R,\chi^R].
\end{aligned} \tag{2.16}
$$

It is interesting to compare anomaly induced terms in superfluid hydrodynamics and normal hydrodynamics. We will focus on baryonic and axial baryonic currents and separate temporal and spatial components of (2.15):

$$
\begin{aligned}
&J_B^0 = Tr(j_L^0 + j_R^0) = -5Ci\varepsilon^{ijk}Tr\left(D^Q{}_i\xi_j^L\xi_k^L - D^Q{}_i\xi_j^R\xi_k^R\right) + 15CiB_iTrQ(\xi_i^L - \xi_i^R) \\
&J_B^i = Tr(j_L^i + j_R^i) = -5CiTr(\Omega_i^L - \Omega_i^R) \\
&+15Ci\left[\varepsilon^{ijk}E_jTrQ(\xi_k^L - \xi_k^R) - B_iTrQ(\xi_0^L - \xi_0^R) + 2\mu_5B_iTrQ\right] \\
&J_{B5}^0 = Tr(j_L^0 - j_R^0) = 10CiB_iTrQ(\xi_i^L + \xi_i^R) \\
&J_{B5}^i = Tr(j_L^i - j_R^i) = 10Ci\left(\varepsilon^{ijk}E_jTrQ(\xi_k^L + \xi_k^R) - B_iTrQ(\xi_0^L + \xi_0^R)\right),
\end{aligned} \tag{2.17}
$$

where we have defined $\Omega_i^L = \varepsilon^{i\nu\rho\sigma}TrD_\nu\xi_\rho^L\xi_\sigma^L$ and $\Omega_i^R = \varepsilon^{i\nu\rho\sigma}TrD_\nu\xi_\rho^R\xi_\sigma^R$, which are analogs of vorticity in non-abelian superfluid, thus giving rise to the CVE. They are vanishing for abelian superfluid. E_i and B_i in (2.17) are external electric and magnetic field. The terms proportional to E_i and B_i can be interpreted as the analogs of the CEE [7] and CME. Note that CVE and CME have already been found in [6]. The identification of CEE is new. From various terms in (2.17), we observe that there is CVE in the baryonic current and CEE and CME are present in both baryonic and axial baryonic currents.

3. Conclusion

Starting with chiral Lagrangian plus Wess-Zumino-Witten term, we have derived a set of hydrodynamic equations for chiral dynamics. It is a non-abelian hydrodynamics invloving superfluid components only. We found axial anomaly induces various terms in the consitututive equations of currents, which are analogous to CME, CVE and CEE in normal hydrodynamics, albeit some of them are unique for non-abelian theory. The resulting hydrodynamic equations can be a useful tool in the study of low energy dynamics of QCD.

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