The Simplest Neutrino Mass Matrix Revisited

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In 2004, two of us proposed a texture, the "Simplest" neutrino mass matrix, which predicted $\sin\theta_{13} = \sqrt{2\Delta m^2_{sol}/3\Delta m^2_{atm}}$ and $\delta_{CP} = \pm 90°$. Using today's measured values for neutrino mass-squared differences, this prediction gives $\sin^2 2\theta_{13} \simeq 0.086^{+0.003}_{-0.006}$, in excellent agreement with a measured value of $\sin^2 2\theta_{13} = 0.098 \pm 0.011$. Here we present a specific model based on S_4 symmetry leading to this successful texture in the context of the type-1 see-saw mechanism, assuming Majorana neutrinos. Similar predictions are obtained relating θ_{13} to the light neutrino masses, which are in accord with current experimental limits and testable at future experiments. Large *CP* asymmetries remain a generic prediction of the texture.

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1. Introduction

Leptonic mixing is characterised by two large mixing angles, $\theta_{12} \simeq 35°$ and $\theta_{23} \simeq 45°$, and one small one, θ_{13}. For several years, the data on neutrino oscillations were compatible with $\theta_{13} = 0$, and the data together were approximated by the tribimaximal (TBM) mixing scheme [1]. TBM has been used by many authors as a starting point for model building. The Daya Bay [2] and RENO [3] reactor neutrino experiments recently measured the value for the reactor mixing angle to be definitively non-zero, with a combined value of $\sin^2 2\theta_{13} = 0.098 \pm 0.010 \text{ (stat.)} \pm 0.005 \text{ (syst.)}$, a result given additional support by several other experiments [4, 5, 6].

Anticipating possible deviations from TBM, two of us proposed some generalisations [7], in which the condition $\theta_{13} = 0$ was relaxed variously. Eg. in a basis in which the charged lepton mass matrix (MM) is diagonal, a hermitian neutrino MM which has μ-τ reflection symmetry [8] and democracy (each of its rows and columns sums to a common value) [9] leads to "triχmaximal" mixing (TχM) [7]:

$$|U_{e2}|^2 = \frac{1}{3}, \quad \sin^2\theta_{23} = \frac{1}{2}, \quad \sin^2\theta_{13} = \frac{1}{3}\left(1 - \frac{1}{\sqrt{1+3k^2}}\right); \quad \delta_{CP} = \pm 90°. \tag{1.1}$$

We remark that TχM corresponds to maximal *CP* violation, for any particular value of θ_{13}.

If neutrinos were Majorana particles, the neutrino MM would be complex-symmetric. We therefore determined [10] the general complex-symmetric MM that generates TχM:

$$M_S = a\begin{pmatrix} 1 & -ik & ik \\ -ik & 1+ik & 0 \\ ik & 0 & 1-ik \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1.2}$$

with k, a, b, and c real parameters. The only free mixing parameter in TχM is the angle θ_{13} which is uniquely determined by the real parameter k in the matrix M_S.

The TχM mixing form [7], $U_{T\chi M}$, defined by Eq. (1.1), diagonalises each of the three terms of the matrix M_S of Eq. (1.2), independently. The first term of M_S alone is in fact, sufficient to generate TχM, having three distinct eigenvalues. The second and the third terms of M_S give two degenerate and three degenerate eigenvalues respectively. A symmetric matrix, M_S, can be diagonalised using the appropriate unitary matrix and its transpose to give real positive eigenvalues:

$$U_{T\chi M}^T M_S U_{T\chi M} = \text{Diag}(|a\sqrt{1+3k^2} - b + c|, |a + 2b + c|, |-a\sqrt{1+3k^2} - b + c|). \tag{1.3}$$

A special case of TχM, known as "Simplest" neutrino mixing and defined by $b = 0$ (eg. in Eqs. (1.2) and (1.3)) was proposed in 2004 [11] (having been first introduced in [7]). Simplest neutrino mixing yields an exact and very simple relation between the reactor mixing angle (see Eq. (1.1)) and the eigenvalues, e_i:

$$\sin^2\theta_{13} = \frac{2}{3}\frac{(e_2 - e_1)}{(e_3 - e_1)}. \tag{1.4}$$

In the original publications [7, 11], the Simplest texture was proposed for $M_\nu^2 := M_\nu M_\nu^\dagger$, in which case the eigenvalues are the neutrino masses-squared, resulting in the very successful prediction:

$$\sin\theta_{13} = \sqrt{\frac{2}{3}\frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2}}. \tag{1.5}$$

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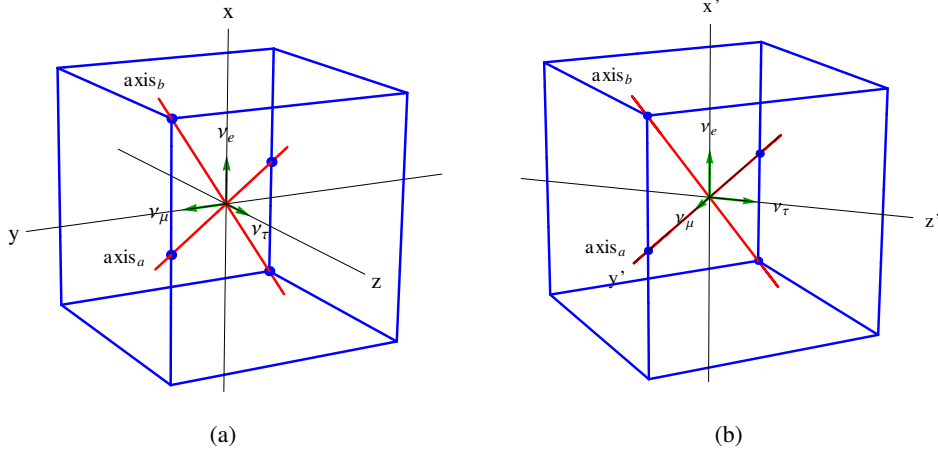
Figure 1: Octahedral symmetry in the μ-τ rotated basis (x', y', z').

i.e. $\sin^2 2\theta_{13} = 0.086^{+0.003}_{-0.006}$, predicted in 2002/2004 [7, 11], compared with $\sin^2 2\theta_{13} = 0.098 \pm 0.011$, measured in 2012 [2, 3].

In view of this very successful prediction, we decided to revisit Simplest mixing, proposing a new model for it [10], based on the group S_4. We make the following two changes with respect to the original ansatz: we adopt the Simplest form for the neutrino MM itself (rather than its hermitian square); a Majorana mass term is assumed, exploiting the type-I see-saw mechanism. Thus, we get a Majorana neutrino MM of the following Simplest complex-symmetric form:

$$M_\nu(\text{Majorana}) = a \begin{pmatrix} 1 & -ik & ik \\ -ik & 1+ik & 0 \\ ik & 0 & 1-ik \end{pmatrix} + c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.6}$$

We construct such a Majorana MM assuming symmetry under S_4, with the neutrino Dirac MM assumed to be proportional to the identity. We show that this model has a phenomenology compatible with experiment, and we use it to predict the masses of the light neutrinos.

2. The group S_4 and the μ-τ rotated basis

S_4 is the symmetry group of the cube and the octahedron. It has the abstract presentation $\langle a, b | a^2 = b^3 = (ab)^4 = e \rangle$, where a, b and ab represent the orientation-preserving rotations of a cube through angles π, $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ respectively, as shown in Fig. 1 (axis$_a$ and axis$_b$ correspond to the generators a and b respectively). The character table for the S_4 group can be found in [12].

We define a right-handed neutrino triplet $\nu_R = (\nu_e, \nu_\mu, \nu_\tau)$. Many authors work in a basis with the coordinate axes normal to the faces of the cube as shown in Fig. 1(a) and with the elements of ν_R defined parallel to these coordinate axes. Then the ν_e, ν_μ and ν_τ are simply the invariant eigenstates (eigenstates with eigenvalue equal to $+1$) of the $\frac{\pi}{2}$ symmetry rotations of the cube. For our model of the "Simplest" texture, we define instead the ν_μ and ν_τ basis states rotated by an angle $\frac{\pi}{4}$ in the $y - z$ plane relative to the cube, as shown in Fig. 1(b). For covenience, we also rotate the y and z coordinate axes to align with the new ν_μ and ν_τ flavour basis states respectively, denoting

the rotated coordinate system, (x', y', z') as shown in Fig. 1(b). The state ν_e is unchanged and still corresponds to the $\frac{\pi}{2}$-rotation symmetry of the cube about the x-axis. However, ν_μ and ν_τ are no longer invariant eigenstates of $\frac{\pi}{2}$-rotations, but rather are invariant under π rotations of the cube, as may be seen in Fig. 1(b). We call this new basis, the μ-τ rotated basis.

The three-dimensional representation of S_4 corresponding to the rotational symmetries of the cube is denoted by $\mathbf{3'}$ here. Thus the neutrino triplet $(\nu_e, \nu_\mu, \nu_\tau)$ belongs to the $\mathbf{3'}$ representation. A Majorana mass term contains two neutrino fields, and thus it is of interest to consider the tensor product decomposition of two $\mathbf{3'}$s. This decomposition is as follows:

$$\mathbf{3'} \times \mathbf{3'} = \mathbf{1} + \mathbf{2} + \mathbf{3} + \mathbf{3'}, \tag{2.1}$$

where $\mathbf{1}$ is the trivial representation, and in the μ-τ rotated basis we have:

$$\chi_1 = \frac{1}{\sqrt{3}}(\nu_e.\nu_e + \nu_\mu.\nu_\mu + \nu_\tau.\nu_\tau), \qquad \chi_2 = \begin{pmatrix} -\sqrt{\frac{2}{3}}\nu_e.\nu_e + \frac{1}{\sqrt{6}}\nu_\mu.\nu_\mu + \frac{1}{\sqrt{6}}\nu_\tau.\nu_\tau \\ \frac{1}{\sqrt{2}}(\nu_\mu.\nu_\tau + \nu_\tau.\nu_\mu) \end{pmatrix},$$

$$\chi_3 = \begin{pmatrix} \frac{1}{\sqrt{2}}(\nu_\mu.\nu_\mu - \nu_\tau.\nu_\tau) \\ \frac{1}{\sqrt{2}}(\nu_\tau.\nu_e + \nu_e.\nu_\tau) \\ \frac{1}{\sqrt{2}}(\nu_e.\nu_\mu + \nu_\mu.\nu_e) \end{pmatrix}, \qquad \chi_3' = \begin{pmatrix} \frac{1}{\sqrt{2}}(\nu_\mu.\nu_\tau - \nu_\tau.\nu_\mu) \\ \frac{1}{\sqrt{2}}(\nu_\tau.\nu_e - \nu_e.\nu_\tau) \\ \frac{1}{\sqrt{2}}(\nu_e.\nu_\mu - \nu_\mu.\nu_e) \end{pmatrix} = 0, \tag{2.2}$$

where the bi-linears χ_1, χ_2, χ_3, χ_3' transform as $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$ respectively. The product $\nu_i.\nu_j$ is the Lorentz-invariant product of the right-handed neutrino Weyl spinors.

We now assume three types of flavons, ϕ_1, ϕ_2 and ϕ_3, which transform as $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ respectively, allowing us to write an invariant mass term:

$$\text{Inv} = c_1\chi_1\phi_1 + c_2\chi_2^T\phi_2 + c_3\chi_3^T\phi_3, \tag{2.3}$$

where c_1, c_2 and c_3 are constants. Once the flavons acquire specific forms of vacuum expectation values (vevs), the required MM can be obtained from the invariant mass term given in Eq. (2.3). Suppose the flavons get vevs

$$\langle\phi_1\rangle = 1, \langle\phi_2\rangle = \left(-\frac{1}{2} \quad \frac{\sqrt{3}}{2}\right), \langle\phi_3\rangle = (1,1,-1), \tag{2.4}$$

the MM obtained will be

$$M = c_1 I + \frac{c_2\sqrt{3}}{2\sqrt{2}}\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 1 \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} + \frac{c_3}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \tag{2.5}$$

which is in the same form as Eq. (1.6), assuming c_1 and c_2 to be real and c_3 to be imaginary (where here the trace is removed from the second term and is absorbed into the first). The details of how to obtain the flavon vevs by minimisation of flavon potentials are given in the appendix to Ref. [10].

3. The model

The model is constructed in a Standard Model framework with the addition of heavy right-handed neutrinos. Light Majorana neutrinos are produced via a type-1 seesaw mechanism. The

fermion and flavon content of the model is given in Table 1. The Standard Model Higgs field is assigned to the trivial representation of S_4.

	e_R	μ_R	τ_R	L	ν_R	ϕ_1^-	ϕ_1	ϕ_2	ϕ_3	ϕ_{3e}'	$\phi_{3\mu}'$	$\phi_{3\tau}'$
S_4	**1**	**1**	**1**	**3′**	**3′**	**1**	**1**	**2**	**3**	**3′**	**3′**	**3′**
C_{2e}	-1	1	1	1	1	1	1	1	1	-1	1	1
$C_{2\mu}$	1	-1	1	1	1	1	1	1	1	1	-1	1
$C_{2\tau}$	1	1	-1	1	1	1	1	1	1	1	1	-1
C_{2D}	-1	-1	-1	-1	1	-1	1	1	1	1	1	1

Table 1: The flavour structure of the model. L are the three left-handed lepton weak isospin doublets and ν_R are the three right-handed heavy neutrinos. The **3′** representations are in the μ-τ rotated basis and the **2** and the **3** representations are in the basis given by the tensor product expansion in the text.

The C_2 flavour symmetries C_{2e}, $C_{2\mu}$ and $C_{2\tau}$ ensure that the flavons ϕ_{3e}', $\phi_{3\mu}'$ and $\phi_{3\tau}'$ couple only to e_R, μ_R and τ_R respectively. We thus obtain a diagonal MM for the charged leptons from a mass term of the form

$$\left(y_e L^\dagger e_R \phi_{3e}' + y_\mu L^\dagger \mu_R \phi_{3\mu}' + y_\tau L^\dagger \tau_R \phi_{3\tau}' \right) (H/\Lambda) + H.C. \tag{3.1}$$

where H is the standard model Higgs, Λ is the cut-off scale and the y_i are coupling constants. After the weak gauge symmetry and the flavour symmetry are spontaneously broken, with the flavons ϕ_{3e}', $\phi_{3\mu}'$ and $\phi_{3\tau}'$, getting vevs of $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$ respectively, we get the required masses m_e, m_μ and m_τ for the charged leptons. We postulate a singlet flavon ϕ_1^- and introduce another C_2 symmetry C_{2D} to allow only this singlet flavon to enter the Dirac mass term:

$$y_w L^\dagger \nu_R \tilde{H} (\phi_1^-/\Lambda) + H.C., \tag{3.2}$$

where $\tilde{H}$ is the conjugate Higgs and y_w, a coupling. Thus the neutrino Dirac MM is proportional to the identity. The Majorana mass term containing the right-handed neutrinos is as follows:

$$\left(y_1 \chi_1 \phi_1 + y_2 \chi_2^T \phi_2 + i y_3 \chi_3^T \phi_3 \right) /\Lambda, \tag{3.3}$$

where the χ_i are given by the expressions in Eqs. (2.2) and the y_i ($i = 1$, 2, 3) are couplings leading to very heavy right-handed Majorana masses. This leads to a Majorana MM of the form given by Eq. (2.5) as explained in the previous section. The flavons ϕ_1, ϕ_2 and ϕ_3 get the vevs given in Eqs. (2.4) and also with $\langle \phi_1^- \rangle = 1$, we get the following 6×6 mass matrix, M, for the neutrinos:

$$\nu_\alpha^T M \nu^\alpha; \qquad M = \begin{pmatrix} 0 & M_{\text{Dir}} \\ M_{\text{Dir}} & M_{\text{Maj}} \end{pmatrix}, \qquad \nu = \begin{pmatrix} \nu_L^* \\ \nu_R \end{pmatrix} \tag{3.4}$$

with ν_L the left-handed neutrino triplet and where $M_{\text{Dir}} = \frac{y_w}{2\Lambda}I$, and M_{Maj} is of the required form given in Eqs. (1.6) and (2.5). Here, M_{Maj} is at a very high mass scale and M_{Dir} is of order the weak scale, so that the seesaw mechanism comes into play. The effective mass matrix, M_{ss}, thus generated for the left-handed light neutrinos is in the form [13]

$$M_{\text{ss}} = -M_{\text{Dir}} M_{\text{Maj}}^{-1} M_{\text{Dir}}. \tag{3.5}$$

A matrix of the form $U_{T\chi M}$ diagonalises M_{ss} giving light neutrino masses proportional to $(1/e_1)$, $(1/e_2)$, $(1/e_3)$, where e_1, e_2 and e_3 are the eigenvalues of M_{Maj}.

4. Fitting the model with experimental data

The squared differences of the light neutrinos masses are known experimentally, $m_2^2 - m_1^2 = 75.9 \pm 2.1$ meV^2, $|m_3^2 - m_2^2| = 2430 \pm 130$ meV^2. The eigenvalues of the Majorana mass matrix, M_{Maj} from Eq. (1.6), are

$$e_1 = c + a\sqrt{1 + 3k^2}, \quad e_2 = c + a, \quad e_3 = c - a\sqrt{1 + 3k^2} \tag{4.1}$$

(a, c and k are real in our model, given our earlier assumptions). Since the light neutrino masses are inversely proportional to these eigenvalues, we get

$$\frac{m_2^2 - m_1^2}{m_3^2 - m_2^2} = \pm\frac{75.9 \pm 2.1}{2430 \pm 130} = \frac{(1/e_2^2) - (1/e_1^2)}{(1/e_3^2) - (1/e_2^2)} = -\frac{(r-s)^2(1-s)(1+2r+s)}{(r+s)^2(1+s)(1+2r-s)} \tag{4.2}$$

where $r = c/a$ and $s = \sqrt{1 + 3k^2}$. Using the observed reactor mixing angle, θ_{13}, in Eq. (1.1), we can find the value of k and use this in Eq. (4.2) to obtain the parameter r, and thus predict the values of the light neutrino masses. Eq. (4.2) is cubic in r, giving three real solutions for normal hierarchy and one for inverted hierarchy. One of the normal hierarchy solutions gives the wrong sign for the solar mass-squared difference, leaving three solutions compatible with data. Thus the predictions for the light neutrino masses fall into three sets, as shown in Fig. 2, where the best fit values are used. The predicted ranges for the mass of the neutrino eigenstate ν_1, allowing for
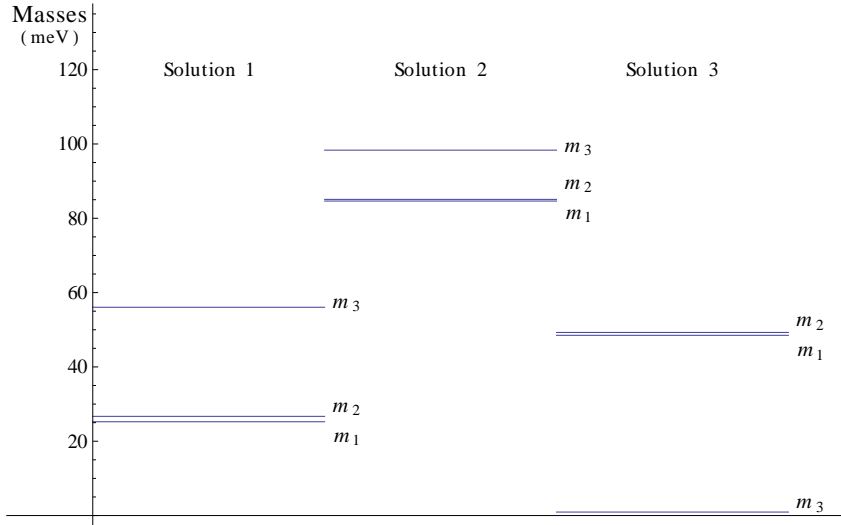
Figure 2: The predicted values of the neutrino masses corresponding to the best fit ($m_2^2 - m_1^2 = 75.9$ meV^2, $|m_3^2 - m_2^2| = 2430$ meV^2, $\sin^2 2\theta_{13} = 0.098$). Case 1 ($r = 0.4101$) and case 2 ($r = 14.452$) are in normal hierarchy. Case 3 ($r = -1.0405$) is in inverted hierachy.

experimental uncertainties in the observed neutrino masss-squared differences are shown in Fig. 3. For solution 2 we do not consider a mass above 100 meV in order to keep our prediction compatible with the cosmological upper limit of the masses of the neutrinos [14]. It should be emphasised that TχM, as obtained here implies maximal *CP* violation ($|\delta_{CP}| = 90°$). The positive and negative signs of the parameter k correspond to the *CP*-violating phase $\delta_{CP} = \pm 90°$.
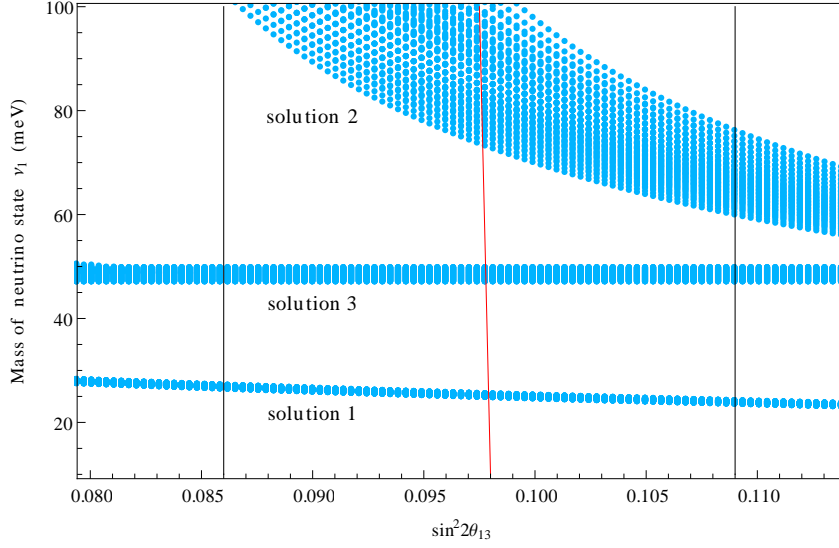
Figure 3: The predicted value of m_1 (the mass of the ν_1 eigenstate) vs the measured value of $\sin^2 2\theta_{13}$. The finite thickness of the bands is due to the errors in the measurement of the neutrino mass-squared differences. The red and the black lines indicate the best fit value and the errors on $\sin^2 2\theta_{13}$ respectively.

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