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Near BPS Skyrmions: Non-shell configurations and Coulomb effects

Luc Marleau[*†]
Département de Physique, de Génie Physique et d'Optique, Université Laval, Québec, Canada
E-mail: lmarleau@phy.ulaval.ca

We propose to describe nuclei as near BPS solitons emerging from a generalization of the Skyrme model in the regime where a sixth-order term and a generalized mass term dominate. The mass term is chosen such that the baryon and energy density generated by the solutions do not exhibit the usual shell configuration. Adding contributions from the rotational energy, Coulomb energy and isospin symmetry breaking, we reproduce the mass of the most abundant isotopes to rather good accuracy.

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[*]Speaker.

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1. Introduction

The idea suggested by Skyrme [1] that baryon physics could emerge as solitons from an effective Lagrangian of meson fields remains one of the most original and successful attempts for the description of the low-energy regime of the theory of strong interactions (QCD). In its original formulation, the Skyrme model succeeds in predicting the properties of the nucleon within a precision of 30% [2]. A number of generalizations of the model have been proposed to improve this concordance with baryon and nuclear physics, for example, by changing the mass term [3], by including vector mesons [4] or simply by adding of higher-order terms in derivatives of the pion fields [5]. Despite such efforts, one of the recurring problems of Skyrme-like Lagrangians is that they almost inevitably lead to large binding energy for nuclei already at the classical level. A solution may be at hand by constructing effective Lagrangians with soliton solutions that saturate the Bogomol'nyi bound, i.e. so-called Bogomol'nyi-Prasad-Sommerfield type (BPS) Skyrmions, since their classical static energy grows linearly with the baryon number A (or atomic number) much like the nuclear mass. Support for this idea comes from a recent result from Sutcliffe [6] who found that BPS-type Skyrmions seem to emerge for the original Skyrme model when a large number of vector mesons are added reaching eventually the saturation of the Bogomol'nyi bound. A different approach by Adam, Sanchez-Guillen, and Wereszczynski (ASW) [7] and by Bonenfant and L.M. [8] suggests that the Bogomol'nyi bound could be reached with a model where the mass term and the term of order six in derivatives of the pion fields dominate. However, a few issues remain to be addressed before these types of models can be considered viable. Here, we introduce a near BPS Skyrme model in an attempt to solve two of these issues: (1) obtaining non shell-like configurations for the energy and baryon densities as it is suggested by experimental data and (2) including the Coulomb energy and the isospin symmetry breaking term in the calculation of nuclear masses since they are expected to have a significant impact on the predictions binding energies. Adjusting the four parameters of our model to fit the resulting binding energies per nucleon with respect to the experimental data leads to a very good agreement [9] .

2. The near-BPS Skyrme model

The model is based on the Lagrangian density of the form

$$\mathscr{L} = -\mu^2 V(U) - \alpha \operatorname{Tr}\left[L_\mu L^\mu\right] + \beta \operatorname{Tr}\left(f_{\mu\nu} f^{\mu\nu}\right) - \frac{3}{2}\frac{\lambda^2}{16^2}\operatorname{Tr}\left(f_{\mu\nu} f^{\nu\lambda} f_\lambda{}^\mu\right) \tag{2.1}$$

where $f_{\mu\nu} = \left[L_\mu, L_\nu\right]$ with $L_\mu = U^\dagger \partial_\mu U$, the left-handed current of the meson fields represented by the $SU(2)$ matrices $U = \phi_0 + i\tau_i\phi_i$ which obey the nonlinear condition $\phi_0^2 + \phi_i^2 = 1$.

The constants μ, α, β, and λ are free parameters of the model and we shall be interested in the regime where α and β are small. The original Skyrme model corresponds to the case where $\mu = \lambda = 0$. The first term, the so-called mass term, is often added to take into account chiral symmetry breaking so that it generates a pion mass term for small fluctuations of the chiral field in $V(U)$. Finally, the term of order six in derivatives of the pion fields is equivalent to $\mathscr{L}_{J6} = -\frac{\varepsilon_6}{4}\mathscr{B}^\mu\mathscr{B}_\mu$ that was first proposed by Jackson et al. [5] to take into account ω-meson interactions. Here, $\mathscr{B}^\mu = 1/(24\pi^2)\varepsilon^{\mu\nu\rho\sigma}\operatorname{Tr}\left(L_\nu L_\rho L_\sigma\right)$ is the topological (baryon) current density, whose time

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component serves to characterize solutions by a conserved topological charge identified by baryon number or mass number in the context of nuclei, $A = \int d^3r\mathscr{B}^0$.

The Lagrangian in (2.1) represents the most general $SU(2)$ model with at most two time derivatives and as a result, the model has been studied extensively. But remarkably, this was done only for values of parameters μ, α, β, and λ close to that of the original Skyrme model and in this sector, the model fails to provide an accurate description of the binding energy of heavy nuclei. Recently, ASW [7] proposed a model (equivalent to setting $\alpha = \beta = 0$) whose solutions are BPS-type solitons and have lower binding energies. A more realistic approach was proposed in Ref. [8] to analyze the full Lagrangian (2.1) in the sector where α and β are relatively small treating these two terms as perturbations. However, nuclear matter is believed to be uniformly distributed inside a nucleus whereas the solutions of the aforementioned models display shell-like configuration for the baryon and energy densities. We propose a new mass term [9] and demonstrate that it is possible to construct an effective Lagrangian which leads to non-shell configurations and still preserves and even improves the agreement with nuclear mass data

$$V(U) = -\frac{1}{576}\mathrm{Tr}\left[\frac{\left(2I-U-U^\dagger\right)\left(2I+U+U^\dagger\right)^3}{\ln\left(\left(2I+U+U^\dagger\right)/4\right)}\right]. \tag{2.2}$$

The general static solution of (2.1) can be written in the form

$$U = e^{i\mathbf{n}\cdot\tau F} = \cos F + i\mathbf{n}\cdot\tau\sin F \tag{2.3}$$

where $\hat{\mathbf{n}} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$ is a unit vector and in general F, Θ, and Φ depend on the position variables r, θ, and ϕ. We consider the model (2.1) in the limit where α and β are small. It turns out that in the limit $\alpha = \beta = 0$, the solutions have axial symmetry and take the form $F = F(r)$, $\Theta = \theta$, and, $\Phi = n\phi$ where n is an integer that corresponds to the baryon number A. Using the change in variable $x = ar$ with $a = \left(\frac{\mu}{18n\lambda}\right)^{1/3}$, the minimization of the static energy for $\alpha = \beta = 0$ leads to the differential equation for F :

$$\frac{\sin^2 F}{288x^2}\partial_x\left(F_x\frac{\sin^2 F}{x^2}\right) - \frac{\partial V}{\partial F} = 0. \tag{2.4}$$

where $F_x = \partial F/\partial x$, from which one gets

$$\int dF\frac{\sin^2 F}{8\sqrt{V}} = \pm\left(x^3 - x_0^3\right) \tag{2.5}$$

where x_0 is an integration constant. Finally, the expression for $F(x)$ can be found analytically provided the integral on the left-hand side is an invertible function of F. Such solutions saturate the Bogomol'nyi bound [7] i.e. the corresponding static energy is proportional to the baryon number $A = n$. Moreover, the chiral angle F can only be a function of the ratio $r/n^{1/3}$. Furthermore, it is interesting to note that for such solutions, the square root of the potential $\sqrt{V(x)} = \frac{\pi}{48n}\mathscr{B}^0(x)$ where $\mathscr{B}^0(x)$ corresponds to the baryon radial density $\mathscr{B}^0(x) = \int d\Omega\,\mathscr{B}^0(\mathbf{x})$ up to a multiplicative constant. We see that in order to obtain a non-shell baryon density, it suffices to construct a potential V that does not vanish at small x. Such a potential would also imply that $F_x(0) \neq 0$. The potential (2.2) is the simplest form of potential that leads to an invertible function of F such that $F_x(0) \neq 0$.

Solving (2.5) for F one gets a very simple form

$$F(x) = \mp 2 \left| \cos^{-1}\left(e^{-x^2}\right) \right| \tag{2.6}$$

which possess the desired non-shell like baryon density configurations as observed in nature. The static energy of the soliton in the small α and β approximation can be easily calculated

$$\begin{aligned} E_{\text{stat}} = \frac{2\pi^{3/2}}{3} \Bigg[& 4\sqrt{2}\left(-3+2\sqrt{3}\right) n\lambda\mu + 3\left(4\left(\sqrt{2}-1\right)\left(n^2+1\right)+3\sqrt{2}\zeta\left(\frac{5}{2}\right)\right)\alpha a^{-1} \\ & +192\left(\left(16\sqrt{2}\left(\sqrt{3}-1\right)-15\right)n^2+1\right)\beta a\Bigg] \end{aligned} \tag{2.7}$$

where ζ is the Riemann $\zeta-$function. The first term in (2.7) receives an equal contribution from the potential and the sixth order term of the Lagrangian. The remaining parts coming from the small perturbations have a more complex dependence, i.e. $E_2 = \alpha A^{1/3}(a_2 + b_2 A^2)$ and $E_4 = \beta A^{-1/3}(a_4 + b_4 A^2)$. Apart from the overall factor $A^{\pm 1/3}$ due to the scaling, there are additional factors of A^2 that originates from the axial symmetry of the solution. Note that it is also easy to calculate analytically the root mean square radius of the baryon density $\left\langle r^2 \right\rangle^{1/2} = (\sqrt{-1+\ln 16})/(2a)$ which is consistent with experimental observation for the charge distribution of nuclei $\left\langle r^2 \right\rangle^{1/2} = r_0 A^{1/3}$.

In order to represent physical nuclei, we must include the rotational and isorotational energies. Using the standard semiclassical quantization procedure leads to the lowest eigenvalue of the (iso)rotational Hamiltonian H_{rot} for a nucleus that is given by [8]

$$E_{\text{rot}} = \frac{1}{2}\left[\frac{j(j+1)}{V_{11}} + \frac{i(i+1)}{U_{11}} + \xi\kappa^2\right] \qquad \text{with} \qquad \xi = \frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{n^2}{V_{11}} \tag{2.8}$$

where i, j and $\kappa = \max(|k_3|)$ are the isospin, spin in the space-fixed frame and the maximum value of the third component of isospin in the body-fixed frame. The axial symmetry of the solutions requires that $\kappa = 0$ or $\frac{1}{2}$ for A even or odd respectively. Whereas the spins of the most abundant isotopes are fairly well known, this is not the case for the isospins so we shall assume that the most abundant isotopes correspond to states with lowest isorotational energy. Since $i \geq |i_3|$, the lowest value that i can take is simply $|i_3|$ where $i_3 = Z - A/2$. The (iso)rotational energy of nuclei then require the explicit calculations of only three moments of inertia V_{11}, U_{11}, and, U_{33} which can be obtained analytically [8].

We are interested in a precise calculation of the nuclei masses and an estimate of the Coulomb energy is desirable. So following Adkins et al. [2], we write the electromagnetic current $J^{\mu}_{EM} = \frac{1}{2}\mathscr{B}^{\mu} + J^{\mu 3}_{V}$, where $J^{\mu 3}_{V}$ is the vector current density. The charge density $\rho(\mathbf{r}) = J^0_{EM} = \left(\frac{1}{2}\mathscr{B}^0 + J^{03}_V\right)$ where $J^{03}_V = k_3 \frac{\mathscr{U}_{33}(\mathbf{r})}{U_{33}}$ with $U_{33}, \mathscr{U}_{33}$, the moment of inertia density for the third component of isospin and its density respectively. Finally, we obtain the Coulomb energy by computing numerically the last remaining integral

$$E_C = \frac{1}{2}\int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3r d^3r' \tag{2.9}$$

It is well known that the Coulomb energy alone leads to a neutron-proton mass difference with the wrong sign. On the other hand, isospin is not an exact symmetry, a fact that may be traced back to the up and down quark mass difference. Here we shall assume for simplicity an isospin symmetry breaking term proportional to the third component of isospin $E_I = a_I i_3$ with the parameter a_I fixed by setting the neutron-proton mass difference to its experimental value. Summarizing, the mass of a nucleus reads

$$E(A,i,j,k_3,i_3) = E_{\text{stat}}(A) + E_{\text{rot}}(A,i,j,k_3) + E_C(A,i_3) + E_I(A,i_3) \tag{2.10}$$

where the explicit dependence on the relevant quantum numbers is shown for each term. Further analysis reveals a partial correspondence between the (2.10) and the semiempirical mass formula.

3. Results and discussion

The values of the parameters μ, α, β and λ remain to be fixed. We consider three parametrizations: In Set I (pure BPS Skyrmion), we fix $\alpha = \beta = 0$ and set the remaining coefficients using the mass of the nucleon and the Helium-4 nuclei as input. Set II optimizes the four parameters μ, α, β and λ to fit the masses of the nuclei of the most abundant 144 isotopes. Set III is obtained similarly but the fit is performed with respect to the binding energy per nucleon, B/A. A summary of the results is presented in Fig. 1 which displays the general behavior of B/A as a function of the baryon number for Set I ($\alpha = \beta = 0$), Set II (best fit for nuclear masses), Set III (best fit for B/A), and experimental values (black circles).

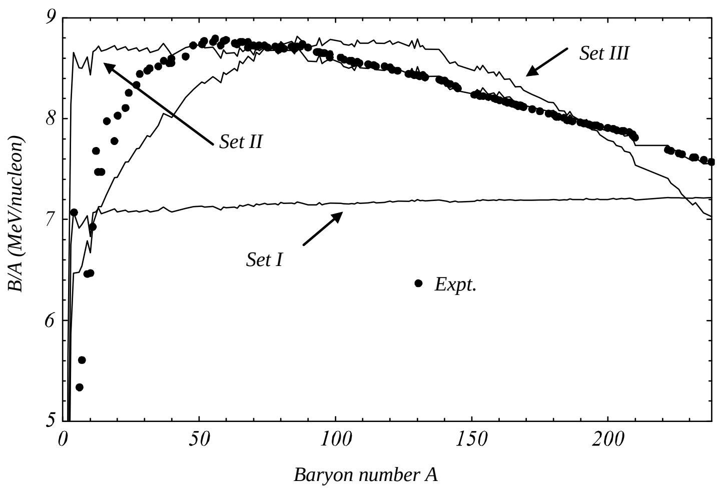

Figure 1: Binding energy per nucleon B/A as a function of the baryon number A.

Clearly, the model in (2.1) (in the regime where α and β are small) provides a significant improvement with regard to the properties of the nuclei of nuclear masses. Even for Set I, the least accurate with respect to the experimental data (black circles), the agreement remains within 0.4% of the experimental masses which is much better than with the original Skyrme model. Moreover, since the ratio B/A is quite sensitive to small variation of the nuclear masses, the results for B/A may be considered as rather good. The second fit (Set II), which is optimized for nuclear masses, overestimates the binding energies of the lightest nuclei while it reproduces almost exactly the remaining experimental values ($A \gtrsim 40$). Finally, the least square fit based on B/A (Set III) is the best fit overall but in order to provide a better representation for light nuclei, it abdicates some of the accuracy found in Set II for $A \gtrsim 40$.

	Set I	Set II	Set III
μ (10^4 MeV^2)	1.490 80	1.505 71	1.729 55
α (10^{-3} MeV^2)	0	5.881 18	22.0821
β (10^{-6} MeV^0)	0	-1.69031	-5.80989
λ (10^{-3} MeV^{-1})	6.413 62	6.339 73	5.536 91

We find that the sets of parameters II and III are very close to Set I (see Table above). The potential and the sixth order terms remain very large compared to the nonlinear σ and Skyrme terms. This provides support for our assumption that they can be treated as perturbation. But despite the very accurate predictions for the nuclear masses, there remain open questions: (1) Comparing Set II and Set III to the original Skyrme model with a pion mass term, we can identify $F_\pi = 4\sqrt{\alpha}$, $e^2 = 1/(32\beta)$ and $m_\pi = 1.0621\mu/\sqrt{\alpha}$. These quantities, F_π, e^2 and m_π take values which are orders of magnitude away from the experimental values. Although this is expected since we have assumed from the start that α and β are relatively small, the link established with soft-pion physics in the original Skyrme model by obtaining realistic values for F_π and m_π and baryon masses seems to be lost or at least more obscure here. (2) The predicted value of $\langle r^2 \rangle^{1/2}$ is about twice the experimental value. Also nucleon-delta mass difference is too small due to the rather small (iso)rotational energy. This is a direct consequence of the choice of potential which leads in this case to baryon or energy densities spread to a larger volume. Further investigations involving different potentials are now underway. (3) From a more fundamental point of view, the role of the ω -meson in the two BPS Skyrmion approaches [6] and [7, 8, 9] seem to be quite different. In the former approach, the addition of the vector meson moves the energy away from BPS saturation whereas here the term of order six suggested by ω interactions must be dominant to achieve the near BPS regime. Note that the two approaches converge to BPS solutions that are quite different.

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