Calculating repetitively

*S.Gopinath Kamath** [a]

Department of Mathematics, Indian Institute of Technology Madras

Chennai 600 036, India

**e – mail: kamath@iitm.ac.in*

Abstract:

The Antonsen – Bormann idea was originally proposed by these authors for the computation of the heat kernel in curved space; it was also used by the author recently with the same objective but for the Lagrangian density for a real massive scalar field in 2 + 1 dimensional stationary curved space. Subsequently, it was reworked with advantage – but to determine the zeta function for the said model using the Schwinger operator expansion. The repetitive nature of that calculation at all higher orders($\geq$ 3) in the gravitational constant G suggests the use of the Dirac delta-function and one of its integral representations – in that it is convenient to obtain answers; in anticipation of its systematic application to all orders ≥ 3 in G and the exact evaluation of $\zeta(s)$ this paper illustrates in detail the evaluation of some integrals relevant to the third order calculation.

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1.Introduction

The integral $I_n = \int_0^{\pi/2} sin^n x \; dx$ with n a non – negative integer is a textbook[1] example of a repetitive calculation; thus, for $n > 2$ one gets

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} \frac{n-3}{n-2} I_{n-4} \tag{1}$$

Continuing in this way one arrives at $I_0 = \frac{\pi}{2}$ or $I_1 = 1$ depending on whether n is even or odd; a well – known byproduct from eq.(1) being the Wallis formula[1]

$$\frac{\pi}{2} = \lim_{m\to\infty}\left[\left\{\frac{2.4.6\ldots\ldots 2m}{1.3.5\ldots.(2m-1)}\right\}^2 \frac{1}{2m+1}\frac{I_{2m}}{I_{2m+1}}\right]$$

with $\lim_{m\to\infty}\frac{I_{2m}}{I_{2m+1}} = 1$.

We present another example of a repetitive calculation that we shall motivate later on in this paper. Much of this paper is an adjunct to an earlier version[2] in that it presents the necessary steps to complete the third order calculation of the zeta-function discussed therein; being tentative and incomplete it warranted a second look and a reader-friendly exposition is given here. Parenthetically, the method presented here is substantial and was not used in Ref.2.

2. Some integrals, their origin and evaluation

Consider the integrals

$$K_0(\vec{p}) = \left(-\frac{\lambda}{4\pi}\right)^3 (-2)^2 \int r_1 \frac{(r_2-q_2)(r_2q_1-r_1q_2)}{(\vec{r}-\vec{q})^2} e^{-xr^2} \frac{(q_2-p_2)(q_2p_1-q_1p_2)}{(\vec{q}-\vec{p})^2} e^{-z\,q^2}$$

$$K_1(\vec{p}) = \left(-\frac{\lambda}{4\pi}\right)^3 (-2)^2 \int q_1 \frac{(p_2-r_2)(p_2r_1-p_1r_2)}{(\vec{p}-\vec{r})^2} e^{-x\,r^2} \frac{(q_2-p_2)(q_2p_1-q_1p_2)}{(\vec{q}-\vec{p})^2} e^{-z\,q^2}$$

$$K_2(\vec{p}) = \left(-\frac{\lambda}{4\pi}\right)^3 (-2)^2 \int p_1 \frac{(p_2-r_2)(p_2r_1-p_1r_2)}{(\vec{p}-\vec{r})^2} e^{-x\,r^2} \frac{(r_2-q_2)(r_2q_1-r_1q_2)}{(\vec{r}-\vec{q})^2} e^{-z\,q^2}$$

$$K_3(\vec{p}) = \left(-\frac{\lambda}{4\pi}\right)^3 (-2)^3 \int e^{-x\,r^2} \frac{(p_2-r_2)(p_2r_1-p_1r_2)}{(\vec{p}-\vec{r})^2} \frac{(r_2-q_2)(r_2q_1-r_1q_2)}{(\vec{r}-\vec{q})^2} \frac{(q_2-p_2)(q_2p_1-q_1p_2)}{(\vec{q}-\vec{p})^2} e^{-z\,q^2}$$

and

$$K_4(\vec{p}) = \left(-\frac{\lambda}{4\pi}\right)^3 (-2) \int e^{-x\,r^2} \frac{(r_2-q_2)(r_2q_1-r_1q_2)}{(\vec{r}-\vec{q})^2} q_1 p_1 e^{-z\,q^2} \tag{2}$$

with $\int$ in each of the above being short hand for $\int d^2r d^2q$, $q^2 = q_1^2 + q_2^2$, $r^2 = r_1^2 + r_2^2$ and x and z being real and non-negative. Of these K_1 and K_4 are easily evaluated as

$$K_1 = -\frac{1}{\pi}\left(\frac{\lambda}{4}\right)^3 p_1 e^{-(x+z)\vec{p}^2} \frac{1}{bc^2}(-1 + e^{x\vec{p}^2})\left\{c\left(p_2^2 e^{z\vec{p}^2} - p_1^2\right) - z(p_1^2 - p_2^2)(1 - e^{z\vec{p}^2})\right\}$$

$$K_4 = \frac{1}{2\pi}\left(\frac{\lambda}{4}\right)^3 p_1 \frac{1}{xz(x+z)} \tag{3}$$

with $c = z^2\vec{p}^2$ and $b = x^2\vec{p}^2$ .The remaining integrals – especially K_3 – are tedious to evaluate thus begging an alternative; while deferring its details to the sequel it pays to briefly recall their origin here:

They are obtained from the momentum space representation of the order G^3 term in the Schwinger operator expansion[2,5,6] for $e^{-(p^2+H_I)t}$ namely,

$$(-t)^3 \int_0^1 u^2 du \int_0^1 u_1 du_1 \int_0^1 du_2 \begin{Bmatrix} e^{-t(1-u)p^2} \int d^2r d^2s \, \langle p|H_I|r\rangle e^{-tu(1-u_1)r^2} \langle r|H_I|s\rangle \\ e^{-tuu_1(1-u_2)s^2} \langle s|H_I|p\rangle e^{-tuu_1u_2p^2} \end{Bmatrix} \tag{4}$$

with the operator $H_I = -\frac{\lambda}{r^4}[(y^2-x^2)p_1 - 2xyp_2]$, $r^2 = x^2+y^2$, $\lambda = 4GJ$, G being the gravitational constant and one of the matrix elements $\langle p|H_I|r\rangle$ for example being $\langle p|H_I|r\rangle = -\frac{\lambda}{4\pi}\left(r_1 - 2\frac{(p_2-r_2)(p_2r_1-p_1r_2)}{(\vec{r}-\vec{p})^2}\right)$. Of the eight possible terms in (4) three are zero by symmetry leaving the five apparently nonzero terms given in eq.(2).The interested reader is referred to Refs.2 and 4 for details.

Returning now to evaluation of the integrals in (2) we begin with K_0 written as

$$K_0 = \left(-\frac{\lambda}{4\pi}\right)^3 (-2)^2 \int \delta(\vec{q}-\vec{s}) r_1 \frac{(r_2-s_2)(r_2s_1-r_1s_2)}{(\vec{r}-\vec{s})^2} e^{-xr^2} \frac{(q_2-p_2)(q_2p_1-q_1p_2)}{(\vec{q}-\vec{p})^2} e^{-z\,q^2} \tag{5}$$

with $\int$ now being short hand for $\int d^2r d^2q\, d^2s$. The introduction of the Dirac delta-function in (5) is a point of departure in this paper – for on using the integral representation[3]

$$\delta(\vec{q}-\vec{s}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} d\alpha d\beta \;\, e^{i\alpha(q_1-s_1)+i\beta(q_2-s_2)} \tag{6}$$

one can now do the s, q and r integration in (5) easily. Parenthetically, the Dirac delta-function was also used elsewhere[4] but with the limit representation[3]

$$\delta(x) = \lim_{\epsilon\to 0+} \frac{1}{2\sqrt{\pi\epsilon}} e^{-x^2/4\epsilon} \tag{7}$$

Eq.(7) however is ineffectual and is therefore given up in favour of (5) here; on doing the r integration first one gets

$$\frac{\pi}{2} e^{-xs^2} \int_0^\infty d\mu \; \frac{e^{\frac{a}{h}}}{h^3} \; \left(hs_2^2 + \mu(s_1^2 - s_2^2)\right) \tag{8}$$

with $a = x^2s^2$, $h = \mu + x$; returning to (5) the s integration can now be completed to get $\frac{\pi e^{-\frac{c}{b}}}{4b^3}[h(2b-\beta^2) + \mu(\beta^2-\alpha^2)]$ the answer after the two integrations being

$$\left(\frac{\pi}{2}\right)^2 \int_0^\infty d\mu \, \frac{e^{-\frac{c}{b}}}{2b^3h^3}[h(2b-\beta^2)+\mu(\beta^2-\alpha^2)] = \left(\frac{\pi}{2}\right)^2 \frac{1}{2c^2x^3} e^{-\frac{c}{x}} \left\{\frac{x}{2}(\alpha^2-\beta^2) - c\alpha^2\right\} \tag{9}$$

with $hb = \mu x$, $4c \equiv \alpha^2+\beta^2$. It is prudent to complete the α,β integration now prior to the q integration to get

$$T \equiv \frac{\pi x}{8}(q_2^2 - q_1^2) \int_0^\infty \nu \, d\nu \frac{e^{-\frac{q^2}{4k}}}{k^3} - \frac{\pi}{4} \int_0^\infty d\nu \frac{e^{-\frac{q^2}{4k}}}{k^3}(2k - q_1^2)$$

$$= 8\pi\left\{\frac{x}{q^2}(q_2^2-q_1^2)+\frac{2(q_2^2-q_1^2)}{q^4}\left(-1+e^{-xq^2}\right)-\frac{2xq_1^2}{q^2}e^{-xq^2}\right\}\equiv 8\pi F \qquad (10)$$

Therefore
$$K_0=-\frac{1}{\pi^2}\left(\frac{\lambda}{4x}\right)^3\int F\,\frac{(q_2-p_2)(q_2p_1-q_1p_2)}{(\vec{q}-\vec{p})^2}e^{-z\,q^2} \qquad (11)$$

with only the q integration left in (11); term-wise one gets (with $d\equiv x+z$, $b\equiv z+\mu$ and, $c\equiv d+\mu$):

1. $x\int_0^\infty d\mu d\nu\; e^{-(\mu+\nu+z)q^2+2\nu\vec{q}\cdot\vec{p}-\nu p^2}(q_2-p_2)\,(q_2^2-q_1^2)(q_2p_1-q_1p_2)$

$$=\frac{\pi x p_1}{2}e^{-zp^2}\int_0^\infty d\mu e^{-\mu p^2}\left\{\frac{(p_1^2-p_2^2)}{b^2p^2}+\frac{(p_1^2-3p_2^2)}{b^3p^4}\left(2+e^{\frac{a}{b}}\right)+3\,\frac{(p_1^2-3p_2^2)}{b^4p^6}\left(1-e^{\frac{a}{b}}\right)\right\}$$

$$=\frac{\pi x p_1}{2}\left\{\frac{e^{-zp^2}}{2z}\left[1+(p_1^2-3p_2^2)\left(\frac{1}{zp^4}+\frac{2}{z^2p^6}\right)\right]+\frac{(p_1^2-3p_2^2)\left(\frac{1}{2}-zp^2\right)}{z^2p^4}-\frac{p^2}{2}\Gamma(0,zp^2)\right\}\equiv\frac{\pi p_1}{2}s_1 \qquad (12a)$$

2. $-2x\int_0^\infty d\mu d\nu\; e^{-(\mu+\nu+x+z)q^2+2\nu\vec{q}\cdot\vec{p}-\nu p^2}(q_2-p_2)\,q_1^2(q_2p_1-q_1p_2)$

$$=-\frac{\pi x p_1}{2}e^{-(x+z)p^2}\int_0^\infty d\mu\, e^{-\mu p^2}\left\{-\frac{2p_1^2}{c^2p^2}+\frac{1}{c^3}\left[\frac{4(p_2^2-p_1^2)}{p^4}+\frac{e^{\frac{b}{c}}}{p^4}(p^2+4p_2^2)\right]+\frac{3(p_1^2-3p_2^2)}{c^4p^6}\left(-1+e^{\frac{b}{c}}\right)\right\}$$

$$=\frac{\pi x p_1}{4d}\left[e^{-dp^2}\left\{1+\frac{p_2^2-3p_1^2}{dp^4}+\frac{2(p_1^2-3p_2^2)}{d^2p^6}\right\}-dp^2\Gamma(0,dp^2)+\frac{2}{dp^2}\left\{1+\frac{4p_2^2}{p^2}+\frac{2}{dp^4}(p_1^2-3p_2^2)\right\}\right]$$

$$\equiv\frac{\pi p_1}{2}s_2 \qquad (12b)$$

3. $-2\int_0^\infty \mu\, d\mu d\nu\; e^{-(\mu+\nu+z)q^2+2\nu\vec{q}\cdot\vec{p}-\nu p^2}(q_2-p_2)\,(q_2^2-q_1^2)(q_2p_1-q_1p_2)$

$$=-2\frac{\pi p_1}{2}e^{-zp^2}\int_0^\infty \mu\, d\mu\, e^{-\mu p^2}\left\{\frac{(p_1^2-p_2^2)}{b^2p^2}+\frac{(p_1^2-3p_2^2)}{b^3p^4}\left(2+e^{\frac{a}{b}}\right)+3\,\frac{(p_1^2-3p_2^2)}{b^4p^6}\left(1-e^{\frac{a}{b}}\right)\right\}$$

$$=\frac{\pi p_1}{2}\left\{e^{-zp^2}\left[1-\frac{(p_1^2-3p_2^2)}{z^2p^6}\right]-(1+zp^2)\Gamma(0,zp^2)-\frac{(p_1^2-3p_2^2)}{z^2p^6}(1+zp^2)\right\}\equiv\frac{\pi p_1}{2}s_3 \qquad (12c)$$

4. $2\int_0^\infty \mu\, d\mu d\nu\; e^{-(\mu+\nu+x+z)q^2+2\nu\vec{q}\cdot\vec{p}-\nu p^2}(q_2-p_2)\,(q_2^2-q_1^2)(q_2p_1-q_1p_2)$

$$=2\frac{\pi p_1}{2}e^{-(x+z)p^2}\int_0^\infty \mu\, d\mu\, e^{-\mu p^2}\left\{\frac{(p_1^2-p_2^2)}{b^2p^2}+\frac{(p_1^2-3p_2^2)}{b^3p^4}\left(2+e^{\frac{a}{b}}\right)+3\,\frac{(p_1^2-3p_2^2)}{b^4p^6}\left(1-e^{\frac{a}{b}}\right)\right\}$$

$$=\frac{\pi p_1}{2}\left\{e^{-dp^2}\left[-1+\frac{(p_1^2-3p_2^2)}{d^2p^6}\right]+(1+dp^2)\Gamma(0,dp^2)+\frac{(p_1^2-3p_2^2)(1+dp^2)}{d^2p^6}\right\}\equiv\frac{\pi p_1}{2}s_4 \qquad (12d)$$

Thus
$$K_0=-\frac{1}{2\pi}\left(\frac{\lambda}{4x}\right)^3 p_1(s_1+s_2+s_3+s_4) \qquad (13)$$

with the s_i defined by eqs.(12a – d). Repeating the above exercise for $K_2(\vec{p})$ now written as

$$K_2(\vec{p})=\left(-\frac{\lambda}{4\pi}\right)^3(-2)^2\int p_1\,\delta(\vec{r}-\vec{s})\frac{(p_2-r_2)(p_2r_1-p_1r_2)}{(\vec{p}-\vec{r})^2}e^{-x\,r^2}\frac{(s_2-q_2)(s_2q_1-s_1q_2)}{(\vec{s}-\vec{q})^2}e^{-z\,q^2}$$

yields for the s integration $\frac{\pi}{2c^2}e^{-i(\alpha q_1+\beta q_2)}\left[\frac{q_1}{4}(\alpha^2-\beta^2)+\frac{\alpha\beta}{2}q_2\right]$ with $4c=\alpha^2+\beta^2$;

the q integration now yields: $\left(\frac{\pi}{2}\right)^2 \left(-\frac{i\alpha}{z^2 c} e^{-\frac{c}{z}}\right)$; and the α,β integration $\left(\frac{\pi}{2}\right)^3 \frac{16 r_1}{r^2 z^2}\left(1-e^{-zr^2}\right)$ leaving only the r – integration the relevant integral for which is

$$16\left(\frac{\pi}{2}\right)^3 \frac{1}{z^2} \int r_1 \frac{(p_2-r_2)(p_2 r_1 - p_1 r_2)}{r^2\,(\vec{p}-\vec{r})^2} e^{-x\,r^2}\left(1-e^{-zr^2}\right)$$

One finally gets with $d \equiv (x+z)$:

$$K_2 = -\frac{z}{\pi}\left(\frac{\lambda}{4z}\right)^3 p_1 \left[\left\{-\frac{e^{-xp^2}}{2x}\left(1+\frac{(p_1^2-p_2^2)}{xp^4}\right)+\frac{1}{xp^2}\left[p_2^2+\frac{(p_1^2-p_2^2)}{2xp^4}\right]+\frac{p^2}{2}\Gamma(0,zp^2)\right\}-\left\{-\frac{e^{-dp^2}}{2d}\left(1+\frac{(p_1^2-p_2^2)}{dp^4}\right)+\frac{1}{dp^2}\left[p_2^2+\frac{(p_1^2-p_2^2)}{2dp^4}\right]+\frac{p^2}{2}\Gamma(0,dp^2)\right\}\right] \quad (14)$$

The remaining integral K_3 is too cumbersome to work out below; therefore its calculation will only be sketched here. By writing it as

$$K_3 = \left(\frac{\lambda}{2\pi}\right)^3 \int \delta(\vec{r}-\vec{s})\delta(\vec{t}-\vec{q}) e^{-x\,r^2} \frac{(p_2-r_2)(p_2 r_1 - p_1 r_2)}{(\vec{p}-\vec{r})^2} \frac{(s_2-t_2)(s_2 t_1 - s_1 t_2)}{(\vec{s}-\vec{t})^2} \frac{(q_2-p_2)(q_2 p_1 - q_1 p_2)}{(\vec{q}-\vec{p})^2} e^{-z\,q^2} \quad (15)$$

with $(\vec{r}-\vec{s}) = (\frac{1}{2\pi})^2 \iint_{-\infty}^{\infty} d\alpha d\beta\ e^{i\alpha(r_1-s_1)+i\beta(r_2-s_2)}$, $\delta(\vec{t}-\vec{q}) = (\frac{1}{2\pi})^2 \iint_{-\infty}^{\infty} d\nu d\theta\ e^{i\theta(t_1-q_1)+i\nu(t_2-q_2)}$

the s integration leads to $\frac{\pi}{2c^2} e^{-i(\alpha t_1+\beta t_2)}\left(\frac{1}{4}(\alpha^2-\beta^2)t_1+\frac{1}{2}\alpha\beta t_2\right)$, $4c \equiv \alpha^2+\beta^2$ .Integrating over α,β gives

$$\frac{2\pi^2}{h^2}\{h_2(r_2 t_1 - r_1 t_2) - h_1(h_1 t_1 + h_2 t_2)\} \quad (16)$$

with $\vec{h} = \vec{r}-\vec{t}$; and the t integration yields $\frac{\pi^3}{2g^2} e^{i(\theta r_1+\nu r_2)}[(\theta^2-\nu^2)r_1 + 2\theta\nu r_2]$, $4g = \theta^2+\nu^2$; the θ,ν integration can now be done to get with $\vec{b} = \vec{r}-\vec{q}$: $\frac{8\pi^4}{b^2}\{b_2[r_2 q_1 - r_1 q_2] - r_1[r_1 b_1 + r_2 b_2]\}$

Only the r and q integration now remain; to take up the latter first we have with $\vec{m} = \alpha\vec{r}+\beta\vec{p}$, $j \equiv \alpha + l$, $l = \beta + z$, $k = \alpha + z$, $g \equiv [\alpha(\vec{r}-\vec{p}) - z\vec{p}]^2$and $n = [\beta(\vec{p}-\vec{r}) - z\vec{r}]^2$:

$$8\pi^4 \int d^2q \frac{(q_2-p_2)(q_2 p_1 - q_1 p_2)}{b^2(\vec{q}-\vec{p})^2} e^{-z\,q^2}\{b_2[r_2 q_1 - r_1 q_2] - r_1[r_1 b_1 + r_2 b_2]\}$$

$$= \pi^5 e^{-z\,r^2} \int_0^\infty d\alpha d\beta \left\{ e^{-\beta(\vec{p}-\vec{r})^2} \frac{e^{\frac{n}{j}}}{j^7}\{-8\alpha\beta j^2 m_2^2(p_1 r_2 - p_2 r_1)^2 + 4j^3[r_2 p_2(m_1^2+m_2^2) + 3m_2(\beta p_1 - \alpha r_1)(r_1 p_2 - r_2 p_1) + 2\alpha\beta m_2(r_2+p_2)(p_1 r_2 - p_2 r_1)^2] + 2j^4[(r_2 p_2 + 3 r_1 p_1) + 2(r_2+p_2)(\beta p_1 - \alpha r_1)(p_1 r_2 - p_2 r_1) - 2m_2(r_2+p_2)(r_2 p_2 + r_1 p_1) - 4\alpha\beta r_2 p_2(p_1 r_2 - p_2 r_1)^2] + 4j^5 r_2 p_2(r_2 p_2 + r_1 p_1)\} - 2r_1 e^{-\alpha(\vec{p}-\vec{r})^2} \frac{e^{\frac{g}{j}}}{j^6}\{4\alpha j^2 m_2[(r_1 p_2 - r_2 p_1)(m_1 r_1 + m_2 r_2)] + 2j^3[(r_1 p_2 - r_2 p_1)(m_2 + \alpha r_2) - p_1(m_1 r_1 + m_2 r_2) - 2\alpha(r_1 p_2 - r_2 p_1)\{m_2 r^2 + p_2(m_1 r_1 + m_2 r_2)\}] + 2j^4\left[r^2\left(p_1 + 2\alpha p_2(r_1 p_2 - \right.\right.\right.$$

$$r_2 p_1)) - p_2(r_1 p_2 - r_2 p_1)]\}\Big\} \tag{17}$$

There now remains the integration over r and α(or β) on each term in (17) ; this will be presented elsewhere.

3.Summary

The calculation of both K_0 and K_2 in the preceding section has been tedious but has been catalyzed using the Dirac δ-function and its integral representation; its workout in detail was motivated by the simplicity of the method and also because to the best of our knowledge this has not been used elsewhere.

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