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Infrared Improvement of Operator Product Expansions in non-Abelian Gauge Theory

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We present a formulation of the operator product expansion that is infrared finite to all orders in the attendant massless non-Abelian gauge theory coupling constant, which we will often-times associate with the QCD theory, the theory that we actually have as our primary objective in view of the operation of the LHC at CERN. We make contact in this way with the recently introduced IR-improved DGLAP-CS theory and point-out phenomenological implications accordingly, with an eye toward the precision QCD theory for LHC physics.

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1. Introduction

The sucessful operation of the LHC at both 7 and 8 TeV cms energies has opened the era of precision QCD, which features predictions for QCD processes at the total precision tag of 1% or better. The need for exact, amplitude-based resummation of large higher order effects is becoming more and more acute. Here, we revisit the pioneering use of Wilson operator product expansion (OPE) methods, as presented and applied by the authors in Refs. [1, 2] for short-distance limits of physical processes especially as it is realized in the DGLAP-CS [3, 4] theory. We do this from the standpoint of resummation of the attendant large infrared effects insofar as they afford application of the corresponding parton model representation to LHC precision physics. We thus make contact with the IR-improved DGLAP-CS theory in Refs. [5–7].

The infrared divergent nature of the usual formulation of Wilson's expansion in massless gauge theory is well-known [8–10] to some; for, already at one-loop, the respective leading twist operator matrix elements between fundamental particle states are in general infrared divergent and must be evaluated at off-shell (Euclidean) points in massless gauge theory as shown in Refs. [8–10]. The attendant coefficient functions in the OPE which encode the leading Q^2 dependence of the expansion are in general infrared divergent order-by-order in renormalized perturbation theory. All such infrared divergences cancel in physically observable (hadronic) matrix elements of the expansion so that, from the standpoint of such observables, the issue is one of optimizing, from the standpoint of precision, the rearrangement of the large infrared effects that remain after all infrared divergences have canceled. We choose to resum these large infrared effects so that we reformulate the OPE in such away that the respective expansion components are infrared finite. In addition, we show how the new IR-improved DGLAP-CS theory in Ref. [5–7] arises naturally in this context. For a given *exact* order in the loop expansion for the coefficient functions and respective operator matrix elements the IR-improved expansion should be closer to experiment.

The discussion proceeds as follows. In the next section, we briefly review the formulation of the OPE following the arguments used in Refs. [8–10] for the analysis of the proto-typical physical application of the method, deep-inelastic lepton-nucleon scattering [11]. In Section 3, we improve it so that its hard coefficient functions are IR finiteand we make contact with the IR-improved DGLAP-CS theory [5–7]. In Section 3, we also give our phenomenological summary remarks.

2. Recapitulation of the OPE

We use the deep inelastic electron-proton scattering problem treated by Bjorken [12] as our starting point: $e^-(\ell)+p(p_p) \to e^-(\ell')+X(p_X)$ in an obvious notation as illustrated in Fig. 1. We use $x \equiv x_{Bj} = Q^2/(2m_p\nu)$ for Bjorken's scaling variable which has the parton model interpretation as the struck parton's momentum fraction when $\nu = qp_p/m_p$ with $q = \ell - \ell'$, $Q^2 = -q^2$. In the Fig. 1, the parton momenta are $p_i(p'_i)$ before(after) the hard interaction process. Our interest is the

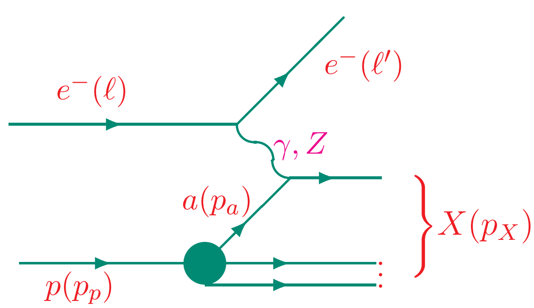

Figure 1: Deep inelastic electron-proton scattering: $q = \ell - \ell'$, $\nu = qp_p/m_p$, $x \equiv x_{Bj} = -q^2/(2m_p\nu)$, $\ell(\ell')$ is the four-momentum of the initial(final) e^-, p_A is the four-momentum of A, $A = a, p$, where a is a parton.

limit of Bjorken, $\lim_{Bj}$, in which we take $Q^2 \to \infty$ with x fixed. In this limit, where for reasons of pedagogy we focus on the photon exchange in Fig. 1 [1], the standard methods can be used to represent the imaginary part of the respective current-proton forward scattering amplitude as

$$
\begin{aligned}
W^{EM}_{\alpha\beta}(p_p,q) &= \frac{1}{2\pi}\int d^4y e^{iqy} < p|[J^{EM}_\beta(y), J^{EM}_\alpha(0)]|p> \\
&= (-g_{\alpha\beta}+q_\alpha q_\beta/q^2)W_1(\nu,q^2) \\
&\quad +\frac{1}{m_p^2}(p_p-qqp_p/q^2)_\alpha(p_p-qqp_p/q^2)_\beta W_2(\nu,q^2)
\end{aligned}
\tag{2.1}
$$

Here, $J^{EM}_\alpha(y)$ is the hadronic electromagnetic current and $W_{1,2}$ are the usual deep inelastic the structure functions, which exhibit [11] Bjorken scaling already at $Q^2 \cong 1_+\mathrm{GeV}^2$, precocious scaling – we return to this point below. Henceforward we drop the superscript on J^{EM} and we always understand the average over the spin of the proton. In Bjorken's limit, we have $\lim_{Bj} m_p W_1(\nu,q^2) = F_1(x)$, $\lim_{Bj} \nu W_2(\nu,q^2) = F_2(x)$ for the scaling limits $F_{1,2}$. The QCD theory of Gross, Wilczek and Politzer [2] provides an explanation of the observed Bjorken scaling behavior via Wilson's OPE.

In Bjorken's limit, the value of the integral in (2.1) is dominated the regions which are well-known to correspond to the tip of the light-cone [13]. In this regime, we get the Wilson OPE [8–10]

$$
\begin{aligned}
J_\beta(y)J_\alpha(0) &= \frac{1}{2}g_{\beta\alpha}\left(\frac{\partial}{\partial y}\right)^2\frac{1}{y^2-i\varepsilon y_0}\sum_{n=0}^{\infty}\sum_j C^{(n)}_{j,1}(y^2-i\varepsilon y_0)O^j_{\mu_1\cdots\mu_n}(0)y^{\mu_1}\cdots y^{\mu_n} \\
&\quad +\frac{1}{y^2-i\varepsilon y_0}\sum_{n=0}^{\infty}\sum_j C^{(n)}_{j,2}(y^2-i\varepsilon y_0)O^j_{\beta\alpha\mu_1\cdots\mu_n}(0)y^{\mu_1}\cdots y^{\mu_n}+\cdots,
\end{aligned}
\tag{2.2}
$$

where we have neglected gradient terms without loss of content for our purposes here and as usual $\varepsilon\downarrow 0$. We also note that $\{O^j_{\mu_1\cdots\mu_n}(y)\}$ are traceless, symmetric of twist = dimension -spin = 2 [13]. The $\cdots$ represent operators that are suppressed by powers of q^2 to any finite order in perturbation theory. The dimensionless coefficient c-number functions $\{C^{(n)}_{j,k}\}$ can be computed in renormalized perturbation theory.

If we define

$$
< p|O^j_{\mu_1\cdots\mu_n}(0)|p>|_{\text{spin averaged}} = i^n\frac{1}{m_p}p_{p_{\mu_1}}\cdots p_{p_{\mu_n}}M^n_j+\cdots,
\tag{2.3}
$$

where the second $\cdots$ denotes trace-terms, we get [8–10, 14]

$$
\begin{aligned}
\int_0^1 dx x^n F_1(x,q^2) &= \sum_j \overline{C}^{(n+1)}_{j,1}(q^2)M^{n+1}_j, \\
\int_0^1 dx x^n F_2(x,q^2) &= \sum_j \overline{C}^{(n)}_{j,2}(q^2)M^{n+2}_j,
\end{aligned}
\tag{2.4}
$$

where [8]

$$
\overline{C}^{(n)}_{j,k}(q^2) = \frac{1}{2}i(q^2)^{n+1}\left(-\frac{\partial}{\partial q^2}\right)^n\int d^4y e^{iqy}\frac{C^{(n)}_{j,k}(y^2)}{y^2-i\varepsilon y_0}.
\tag{2.5}
$$

The $\overline{C}^{(n)}$ satisfy the Callan-Symanzik equation [4]:

$$
\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right)\delta_{ij}-\gamma^{(n)}_{ij}(g)\right]C^{(n)}_{j,k}=0
\tag{2.6}
$$

[1]As it is well-known, adding in the effects of the Z exchange is straightforward and does not require any essentially new methods that are not already exhibited by what we do for the photon exchange case.

where μ denotes the renormalization scale, $\beta(g) = \mu \frac{\partial g}{\partial \mu}$ for the renormalized coupling g, and the anomalous dimension matrix $\gamma_{ij}^{(n)}(g)$ is given by

$$\gamma_{ij}^{(n)}(g) = \left(Z_O^{-1} \mu \frac{\partial}{\partial \mu} Z_O \right)_{ij} |_{g_0,\text{regularization fixed}} \tag{2.7}$$

where the operators $O_j^{(n)} \equiv O^j_{\mu_1 \cdots \mu_n}$ are renormalized via [8,9]

$$O_i^{(n)} \equiv O_{i,R}^{(n)} = \sum_j O_{j,bare}^{(n)} \left(Z_O^{-1} \right)_{ji} \tag{2.8}$$

in a standard notation. The implied behavior from the solution of (2.6) for the RHS of (2.4) in Bjorken's limit in the asymptotically free theory QCD in Ref. [2] agrees with experiment [11]. Here, we discuss the IR-improvement of the $C_{j,k}^{(n)}$, M_j^n.

3. IR-Improved OPE

To facilitate isolation of the infrared aspects of the $C_{j,k}^{(n)}$ we focus on the parton level version [15–20] of hadronic tensor $W_{\alpha\beta}$ which for definiteness we associate with a fermion F in the underlying asymptotically free theory(QCD):

$$\begin{aligned} W_{\alpha\beta}^F(p_F, q) &= \frac{1}{2\pi} \int d^4y e^{iqy} < p_F | [J_\beta(y), J_\alpha(0)] | p_F > \\ &= (2\pi)^3 \sum_X \delta(q + p_F - p_X) < p_F | J_\beta(0) | p_X >< p_X | J_\alpha(0) | p_F >, \end{aligned} \tag{3.1}$$

where we use the fact that $q^0 > 0$ to drop the remaining term in the commutator and we always average over the spin of the fermion F, as we do for the proton$p$. The RHS of (3.1) and that of (2.1) involve the same OPE.

We first isolate [5,6] the dominant incoming line virtual IR divergences in the matrix element $\mathscr{M}_{X,\alpha} \equiv < p_X | J_\alpha(0) | p_F >$ via the formula

$$\mathscr{M}_{X,\alpha} = e^{\alpha_s B_{QCD}} < p_X | J_\alpha(0) | p_F >_{IRI-virt}, \tag{3.2}$$

where B_{QCD} is given in Refs. [5,6]. One computes $< p_X | J_\alpha(0) | p_F >_{IRI-virt}$ from $< p_X | J_\alpha(0) | p_F >$ by comparing the coefficients of the powers of $\alpha_s \equiv g^2/(4\pi)$ on both sides of (3.2) iteratively.

Introducing (3.2) into (3.1), we get

$$\begin{aligned} W_{\alpha\beta}^F(p_F, q) = (2\pi)^3 \sum_X \delta(q + p_F - p_X) e^{2\alpha_s \Re B_{QCD}} {}_{IRI-virt} < p_F | J_\beta(0) | p_X > \\ < p_X | J_\alpha(0) | p_F >_{IRI-virt} . \end{aligned} \tag{3.3}$$

To isolate the leading soft, spin independent incoming line real emission infrared function we separate $\{X\}$ into its multiple gluon subspaces via

$$\{X\} = \{X : X = X' \otimes \{G_1 \otimes \ldots \otimes G_n\}, \text{for some } n \geq 0, X' \text{ is non-gluonic}\}. \tag{3.4}$$

This allows us to write

$$\begin{aligned} & e^{2\alpha_s \Re B_{QCD}} {}_{IRI-virt} < p_F | J_\beta(0) | p_X > < p_X | J_\alpha(0) | p_F >_{IRI-virt} \\ & = e^{2\alpha_s \Re B_{QCD}} \Big[\tilde{S}_{QCD}(k_1) \cdots \tilde{S}_{QCD}(k_n) \ {}_{IRI-virt} < p_F | J_\beta(0) | p_{X'} > \\ & < p_{X'} | J_\alpha(0) | p_F >_{IRI-virt} + \cdots + {}_{IRI-virt\&real} < p_F | J_\beta(0) | p_{X'}, k_1, \cdots, k_n > \\ & < p_{X'}, k_1, \cdots, k_n | J_\alpha(0) | p_F >_{IRI-virt\&real} \Big], \end{aligned} \tag{3.5}$$

where the real infrared function $\tilde{S}_{QCD}(k)$ is given in Refs. [5, 6]. The IR-improved quantities $_{IRI-virt\&real} < p_F|J_\beta(0)|p_X > < p_X|J_\alpha(0)|p_F >_{IRI-virt\&real}$ are defined iteratively from (3.2),(3.5) to all orders in α_s and they no longer contain the infrared singularities associated to B_{QCD} and to $\tilde{S}_{QCD}$, although, because of the non-Abelian infrared algebra of the theory, they do contain other IR singularities which cancel in the structure functions by the KNL theorem for massless and massive [21] fundamental fermions.

Using (3.5) in (3.1) we get

$$
\begin{aligned}
W^F_{\beta\alpha}(p_F,q) &= (2\pi)^3\sum_X \delta(q+p_F-p_X)e^{2\alpha_s\Re B_{QCD}}\Big[\tilde{S}_{QCD}(k_1)\cdots\tilde{S}_{QCD}(k_n)\\
&\qquad _{IRI-virt} < p_F|J_\beta(0)|p_{X'} > < p_{X'}|J_\alpha(0)|p_F >_{IRI-virt} +\cdots\\
&\qquad + {}_{IRI-virt\&real} < p_F|J_\beta(0)|p_{X'},k_1,\cdots,k_n >\\
&\qquad < p_{X'},k_1,\cdots,k_n|J_\alpha(0)|p_F >_{IRI-virt\&real}\Big]\\
&= \frac{1}{2\pi}\int d^4y\sum_{X'}\sum_n\int \Pi^n_{j=1}\frac{d^3k_j}{k^0_j}e^{SUM_{IR}(QCD)}e^{i(q+p_F-p_{X'}-\sum_j k_j)+D_{QCD}}\\
&\qquad _{IRI-virt\&real} < p_F|J_\beta(0)|p_{X'},k_1,\cdots,k_n >\\
&\qquad < p_{X'},k_1,\cdots,k_n|J_\alpha(0)|p_F >_{IRI-virt\&real}\\
&= \frac{1}{2\pi}\int d^4y e^{iqy}e^{SUM_{IR}(QCD)+D_{QCD}}\\
&\qquad _{IRI-virt\&real} < p_F|[J_\beta(y),J_\alpha(0)]|p_F >_{IRI-virt\&real},
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
\mathrm{SUM_{IR}(QCD)} &= 2\alpha_s\Re B_{QCD} + 2\alpha_s\tilde{B}_{QCD}(K_{\max}),\\
2\alpha_s\tilde{B}_{QCD}(K_{\max}) &= \int\frac{d^3k}{k^0}\tilde{S}_{\mathrm{QCD}}(k)\theta(K_{\max}-k),\\
D_{\mathrm{QCD}} &= \int\frac{d^3k}{k}\tilde{S}_{\mathrm{QCD}}(k)\left[e^{-iy\cdot k}-\theta(K_{\max}-k)\right],
\end{aligned}
\tag{3.7}
$$

and we note that (3.6) does not depend on K_{max}. Using the standard partonic view, where

$$
W_{\beta\alpha} = \sum_a\int_0^1\frac{dx}{x}\mathscr{F}_a(x)W^a_{\beta\alpha} \tag{3.8}
$$

for appropriately defined parton distribution functions $\{\mathscr{F}_a\}$, we introduce (2.2) into (3.6) and use (2.1) to get the IR-improved results

$$
\begin{aligned}
\int_0^1 dx x^n F_1(x,q^2) &= \sum_j \tilde{C}^{n+1}_{j,1}(q^2)\tilde{M}^{n+1}_j,\\
\int_0^1 dx x^n F_2(x,q^2) &= \sum_j \tilde{C}^{n}_{j,2}(q^2)\tilde{M}^{n+2}_j,
\end{aligned}
\tag{3.9}
$$

where [8]

$$
\tilde{C}^{(n)}_{j,k}(q^2) = \frac{1}{2}i(q^2)^{n+1}\left(-\frac{\partial}{\partial q^2}\right)^n\int d^4y e^{iqy}e^{SUM_{IR}(QCD)+D_{QCD}}\frac{\tilde{C}^{(n)}_{j,k}(y^2)}{y^2-i\varepsilon y_0} \tag{3.10}
$$

and now

$$\begin{aligned} &< p|\tilde{O}^{j}_{\mu_1\cdots\mu_n}(0)|p> |_{\text{spin averaged}} \equiv \\ {}_{IRI-virt\&real} &< p|O^{j}_{\mu_1\cdots\mu_n}(0)|p>_{IRI-virt\&real} |_{\text{spin averaged}} = i^n \frac{1}{m_p} p_{p_{\mu_1}} \cdots p_{p_{\mu_n}} \tilde{M}^n_j \\ &+ \cdots, \end{aligned} \tag{3.11}$$

where the second $\cdots$ again denotes trace-terms and the $\{\tilde{C}^{(n)}_{j,k}\}$ are the respective (new) IR-improved OPE coefficient functions. These latter functions satisfy the analogous Callan-Symanzik equation [4] to (2.6) with a new anomalous dimension matrix $\tilde{\gamma}^{(n)}_{ij}(g)$ determined by the renormalization properties of the IR-improved matrix elements in (3.11). In writing (3.10) we work to one-loop order in the various coefficients in this paper.

The new matrix $\tilde{\gamma}^{(n)}_{ij}(g)$ can be obtained from the pioneering analysis of the authors in Ref. [19, 20]. Working from (3.8, the authors in Ref. [19] make contact with the unimproved matrix $\gamma^{(n)}_{ij}(g)$ as follows. For the non-singlet operator [8] ${}^{N}O^{F,b}(x) = \frac{1}{2} i^{N-1} S\bar{\psi}(y)\gamma_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_N}\lambda^b\psi(y) -$ trace terms, where $\nabla_\mu = \partial_\mu + ig\tau^a A^a_\mu$ is the covariant derivative, λ^b is a flavor group generator and S denotes symmetrization with respect to the indices $\mu_1\cdots\mu_n$, the authors in Ref. [19] show that the following relation holds:

$$\begin{aligned} -\gamma^{(N)}(\alpha_s) &= 2\int_{-1}^{1} dx x^{N-1}[P_{qq}(x,\alpha_s)\theta(x) - P_{q\bar{q}}(-x,\alpha_s)\theta(-x)] \\ &= 2[P_{qq}(N,\alpha_s) + (-1)^N P_{q\bar{q}}(N,\alpha_s)] \end{aligned} \tag{3.12}$$

where

$$F(N) = \int_0^1 dx x^{N-1} F(x)$$

and the P_{BA} are the usual DGLAP-CS [3, 4] splitting kernels defined in the convention of Ref. [19] and $\gamma^{(N)}(\alpha_s)$ is the respective anomalous dimension of the operator ${}^{N}O^{F,b}$.

In Ref. [22], we show that one can apply our IR-improvement calculus as illustrated above to the arguments in Refs. [19] to get the respective IR-improved anomalous dimension as

$$-\tilde{\gamma}^{(N)}(\alpha_s) = 2\frac{\alpha_s}{2\pi}[P^{exp}_{qq}(N,\alpha_s) + (-1)^N P^{exp}_{q\bar{q}}(N,\alpha_s)] \tag{3.13}$$

where the P^{exp}_{qq}, $P^{exp}_{q\bar{q}}$ are the respective IR-improved kernels as introduced in Refs. [5, 6], where we advise that the notation of Ref. [19] differs from that in Refs. [5, 6] by whether or not one includes the factor $\alpha_s/(2\pi)$ on the RHS of (3.12) in the definition of the kernels. This yields at IR-improved one-loop level the identifications

$$-\tilde{\gamma}^{(N)}(\alpha_s)_{ij} = 2\frac{\alpha_s}{2\pi} P^{exp}_{ij}(N) \tag{3.14}$$

where the labels i, j span the usual values for the one-loop anomalous dimension matrix for the evolution of the parton distributions as given in Refs. [3, 4, 8–10] for example. This establishes in a rigorous way the connection between the IR-improved DGLAP-CS theory in Ref. [5, 6] and the OPE methods of Wilson as used by Refs. [8–10] in the study of deep inelastic lepton-nucleon scattering.

Evidently, this connection may be manifested in the analysis of other physical processes as well. We refer the reader to Refs. [6, 7] wherein the new precision-baseline MC Herwiri1.031 which realizes the IR-improved DGLAP-CS kernels has been introduced and compared to the Tevatron data [23, 24] on single Z production. Its application to the various physical processes at LHC is in progress [25], where we need to stress that Herwiri1.031 can be applied to *any process* to which Herwig6.5 [26] can be applied and that it interfaces to MC@NLO [27] the *same way* that does Herwig6.5. As we have shown in Refs. [6, 7], we have an improved agreement between the IR-improved MC's shower and the Tevatron data with no need of an abnormally large intrinsic transverse momentum parameter, PTRMS$\sim$ 2GeV in the notation of Herwig [26], as it is required for similar agreement with Herwig6.5 [28]. We point-out that, consistent with the precociousness of Bjorken scaling, the IR-improved MC Herwiri.031 gives us a paradigm for reaching a precision QCD MC description of the LHC data, on an event-by-event basis with realistic hadronization from the Herwig6.5 environment, that does not involve an ad hoc hard scale parameter, where we define "hard" relative to the observed precociousness of Bjorken scaling. The discussion above shows that this paradigm has a rigorous basis in quantum field theory. In closing, we thank Prof. Ignatios Antoniadis for the support and kind hospitality of the CERN TH Unit while part of this work was completed.

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