PROCEEDINGS OF SCIENCE

# The modular class as a quantization invariant

**F.Bonechi**[*]
*INFN Sezione di Firenze*
*E-mail:* bonechi@fi.infn.it

**N. Ciccoli**
*Dipartimento di Matematica, Univ. di Perugia*
*E-mail:* ciccoli@dipmat.unipg.it

**M. Tarlini**
*INFN Sezione di Firenze*
*E-mail:* marco.tarlini@fi.infn.it

We show an example of quantization of a singular Poisson structure on the sphere sharing the same $C^*$–algebra quantization as the Podleś one but with different quasi invariant probability measure and the quantization of the modular automorphism.

*7th Conference Mathematical Methods in Physics - Londrina 2012,*
*16 to 20 April 2012*
*Rio de Janeiro, Brazil*

[*]Speaker.



PoS(Rio de Janeiro 2012)003

## 1. Introduction

The Karasev-Weinstein-Zakrzewski program for quantization of Poisson manifolds through the *symplectic groupoid* was, for long, slowered by the difficult problem of integrating Poisson manifolds. Since this problem was solved both in general (see [4, 5]) and in many specific cases (see [1] for the case of Poisson homogeneous spaces), interest in this quantization program was revived.

In [7] Hawkins proposed a framework in order to discuss geometric quantization of the symplectic groupoid $\mathscr{G}$; in particular he defined a notion of *multiplicative polarization* $F \subset T\mathscr{G}$ by imposing a compatibility with the groupoid structure. As usual in geometric quantization, the existence of polarizations is difficult to assess and in general requiring a smooth space of leaves puts severe constraints.

We propose here to modify the approach of [7] by allowing the real polarization $F \subset T\mathscr{G}$ to be singular. Indeed the multiplicativity condition on $F$ endows the space of lagrangian leaves $\mathscr{G}_F$ with a groupoid structure. It is enough to have a topological (rather than smooth) subgroupoid $\mathscr{G}_F^{bs}$ of *Bohr-Sommerfeld leaves* admitting a Haar system to apply Renault's theory of groupoid $C^*$-algebras (see [8]). Such requirement is strictly weaker than Hawkins'one and still produces a quantization $C^*$–algebra.

In this approach two basic invariants of the Poisson structure play a very natural role: the *modular vector* is quantized to a groupoid 1-cocycle $c$ (the *modular cocycle*) and the *Poisson tensor* to a $\mathbb{S}^1$ valued groupoid 2-cocycle $\zeta$ (*the prequantization cocycle*) of $\mathscr{G}_F^{bs}$. From the triple $(\mathscr{G}_F^{bs}, c, \zeta)$ it is possible to build up a twisted convolution $C^*$–algebra with a quasi invariant measure $\mu_c$ supported on the space of units of $\mathscr{G}_F^{bs}$. Together with a (left) Haar system this defines a measure $\nu_{\mu_c}$ on the whole $\mathscr{G}_F^{bs}$ (which turns $(\mathscr{G}_F^{bs}, \nu_{\mu_c})$ into a *measured groupoid*), a KMS state $\phi_{\mu_c}$ and a left Hilbert algebra structure on $L^2(\mathscr{G}_F^{bs}, \nu_{\mu_c}^{-1})$ with a modular operator.

In [2] we started the program of understanding this quantization procedure for Poisson Lie groups and their homogeneous spaces, in particular we want to recover the results of A.J.L. Sheu characterizing the $C^*$-algebras of the quantum spaces as groupoid $C^*$-algebras [10]. In this note we will discuss in detail the case of a specific Poisson structure on the 2–sphere (to be called a $\theta$–sphere) determined by an $\mathbb{R}^2$–action. Our aim is not to produce yet another example of the general program, but, rather, to compare the outcome with the analogous quantization of the Podleś sphere carried through in [2]. These two Poisson structures on $\mathbb{S}^2$ share the same symplectic foliation (one zero point and symplectic complement) and differ by the degree of zero of the bivector at the singular point. They have quite different behaviours from a geometrical point of view: they are not even Poisson Morita equivalent.

If quantization is carried just with an eye on the final $C^*$–algebra this difference is not visible. Indeed both Poisson manifolds are "quantized" by the same groupoid $\mathscr{G}_F^{bs}$ of Bohr-Sommerfeld leaves so that they share the same groupoid $C^*$–algebra. However, since the modular 1–cocycles are not equivalent, in the two cases we get different measured groupoids, KMS states and modular operators.

One may say that the non commutative $C^*$–algebra can be considered as a very rough invariant and this example points at modular properties as a possible finer quantization invariant.

## 2. The Poisson sphere $\mathbb{S}^2_\theta$

Let us consider the action of $\mathbb{R}^2$ on $\mathbb{S}^2$ defined as translation in stereographic complex coordinate $z$ and leaving the point at infinity fixed. We denote the North Pole $z = \infty$ with N and the South Pole $z = 0$ with S. Any $\theta \in \mathbb{R}$ induces a Poisson bivector $\pi_\theta$ on $\mathbb{S}^2$. In the two complex charts with coordinates $z$ and $w = 1/z$ the Poisson bivector is easily computed to be:

$$\pi_\theta\big|_{\mathbb{S}^2\setminus N} = -2\imath\theta\partial_z \wedge \partial_{\bar z}, \quad \pi_\theta\big|_{\mathbb{S}^2\setminus S} = -2\imath\theta|w|^4\partial_w \wedge \partial_{\bar w}\,. \tag{2.1}$$

We call this Poisson manifold the $\theta$ *Poisson sphere* and denote it with $\mathbb{S}^2_\theta$.

It has two distinct leaves: a singular 0–dimensional leaf corresponding to the North Pole and an open symplectic leaf $\mathbb{S}^2\setminus N$. As mentioned in the introduction the $\theta$ Poisson sphere is quite similar to the standard Podleś sphere (e.g. [2] and references therein), having the same symplectic foliation (the same topological space of leaves and symplectomorphic leaves) but a different singularity degree at the North Pole. To investigate further in its properties we will skecth Poisson cohomology computations, following the same approach as in [9]. This means that we will first compute Poisson cohomology on the space of multivectors with polynomial coefficients in the singular chart and then use a smoothing argument plus Mayer-Vietoris to end up our calculations. The use of multivectors with polynomial coefficients is justified by the fact that if we denote with $V_i$ the space of polynomials of homogeneous degree $i$ in $z$, then the Poisson cohomology complex splits as a direct sum of complexes

$$0 \to V_i \to V_{i+3}\langle\partial_w\rangle \oplus V_{i+3}\langle\partial_{\bar w}\rangle \to V_{i+6}\langle\partial_w \wedge \partial_{\bar w}\rangle \to 0\,.$$

This is evident from explicit formulas for the coboundary operator:

$$\begin{aligned} d^0_\pi(w^r\bar w^s) &= -2\imath\theta w^{r+1}\bar w^{s+1}\left(r\bar w\partial_{\bar w} - sw\partial_w\right) \\ d^1_\pi(w^r\bar w^s\partial_w + w^p\bar w^q\partial_{\bar w}) &= -2\imath\theta\left[(r-2)w^{r+1}\bar w^{s+2} + \right. \\ &\qquad \left.(q-2)w^{p+2}\bar w^{q+1}\right]\partial_w \wedge \partial_{\bar w} \end{aligned}$$

Since $d^0_\pi$ is injective for $i \geq 1$ we have $\dim \operatorname{im} d^0_\pi = i+1$. On the other hand $d^1_\pi$ is not surjective; for $i \geq 6$, its image has codimension 4. This implies immediately that $H^2_\pi(\mathbb{S}^2\setminus S)$ gets a contribution from homogeneous polynomial in every degree and thus it is infinite dimensional. Since, from a dimension calculus

$$\dim(\operatorname{Ker} d^1_\pi/\operatorname{im} d^0_\pi) = (2i+8-(i+3)-(i+1)) = 4\,,$$

also the first Poisson cohomology group gets a 4–dimensional contribution from each homogeneous subcomplex, it is infinite dimensional as well. Of course additional generators appear in between generators of degree less than 3 for vector fields and less than 6 for bivector fields, due to some degeneracy of the coboundary maps. It is an easy computation and we will leave further details to the reader. Just as in [9] a general argument immediately extends this result to smooth Poisson cohomology.

Now we are in position to apply the Poisson Mayer-Vietoris sequence to the pair $(\mathbb{S}^2\setminus N, \mathbb{S}^2\setminus S)$. Since both $S^2\setminus N$ and the intersection $(\mathbb{S}^2\setminus N)\cap(\mathbb{S}^2\setminus S)$ are symplectic their Poisson cohomology coincide with de Rham cohomology. A rather standard application of the exact sequence allows us to conclude that $\dim H^1_\pi(\mathbb{S}^2_\theta) = \dim H^2_\pi(\mathbb{S}^2_\theta) = \infty$. As for $H^0_\pi(\mathbb{S}^2)$ it is one-dimensional since the presence of a dense symplectic leaf forbids the existence of non constant Casimir functions.

Let us consider the Poisson bivector itself. Since in the singular chart

$$d^1_\pi(w\partial_w + \bar{w}\partial_{\bar{w}}) = -2\pi_\theta$$

and since the vector field $w\partial_w + \bar{w}\partial_{\bar{w}}$ extends to a global vector field on $\mathbb{S}^2$, its 2-cohomology class is trivial. Let us fix the usual round volume form

$$V = \frac{dz \wedge d\bar{z}}{\iota(1+|z|^2)^2} \in \Omega^2\mathbb{S}^2 .$$

The corresponding modular vector field is:

$$\chi_V = \frac{4\theta}{\iota(1+|z|^2)} \left(z\partial_z - \bar{z}\partial_{\bar{z}}\right) . \tag{2.2}$$

This vector field is a Poisson cocycle for general reason and cannot be a Poisson coboundary by an easy computation (based on degree arguments); thus defines a non trivial class in Poisson cohomology and the Poisson $\theta$-sphere is not unimodular. The previous discussion can be summarized as follows.

**Proposition 2.1.** *The Poisson manifold $\mathbb{S}^2_\theta$ is exact and non unimodular. Its Poisson cohomology is infinite dimensional in degrees one and two. The Poisson $\theta$–sphere is not Poisson–Morita equivalent to the Podleś sphere.*

*Proof.* Since the Podleś sphere has finite dimensional Poisson cohomology ([9]) and since 1–Poisson cohomology is invariant under Poisson Morita equivalence ([6]) we have proven that these two Poisson structures on $\mathbb{S}^2$ are not Poisson Morita equivalent (let aside Poisson gauge equivalent or, even, Poisson diffeomorphic). ∎

## 3. Symplectic integration and prequantization

By applying the general result of [11] we know that the (source simply connected) groupoid $\mathscr{G}(\mathbb{S}^2_\theta)$ integrating $\mathbb{S}^2_\theta$ is $T^*\mathbb{S}^2$ with the canonical symplectic structure. In the symplectic chart, with complex coordinates $z$ on the base $\mathbb{S}^2 \setminus N$ and $p$ on the fiber, the source and target maps of the groupoid are

$$l(z,p) = z - \iota\frac{\theta}{2}\bar{p}; \qquad r(z,p) = z + \iota\frac{\theta}{2}\bar{p} . \tag{3.1}$$

The product of compatible pairs, *i.e.* pairs such that $z + \iota\frac{\theta}{2}\bar{p} = z' - \iota\frac{\theta}{2}\bar{p}'$ is given by:

$$(z,p)\cdot(z',p') = (z + \iota\frac{\theta}{2}\bar{p}', p + p') . \tag{3.2}$$

The inverse mapping is $(z,p)^{-1} = (z,-p)$, and finally the inclusion map is $z \hookrightarrow (z,0)$. On the fiber over the North Pole, the groupoid structure is just addition on the fibre, *i.e.*

$$l(N,p) = r(N,p) = N , \qquad (N,p)\cdot(N,p') = (N,p+p') . \tag{3.3}$$



PoS(Rio de Janeiro 2012)003

Since $\mathbb{S}^2\backslash N$ is an open symplectic leaf, then there exists a symplectic groupoid morphism between $\mathscr{G}(\mathbb{S}^2_\theta)|_{\mathbb{S}^2\backslash N}$ and the pair groupoid $\mathbb{C}\times\mathbb{C}$ given by $(z,p)\to(x,y)=(l(z,p),r(z,p))$. The canonical symplectic structure on $T^*\mathbb{S}^2$ can be rewritten either in Darboux coordinates $(z,p)$ or in pair groupoid coordinates $(x,y)$ as:

$$\Omega = dq^i\wedge dp_i = \frac{1}{2}(dz\wedge dp + d\bar{z}\wedge d\bar{p}) = -\imath\frac{1}{2\theta}(dx\wedge d\bar{x} - dy\wedge d\bar{y})\,. \tag{3.4}$$

The modular vector field (2.2) is integrated to the non trivial groupoid 1-cocycle $c_V\in C^\infty(T^*\mathbb{S}^2)$, the *modular cocycle*; in the pair groupoid coordinates it reads

$$c_V(x,y) = 2\log\left(\frac{1+|y|^2}{1+|x|^2}\right)\,. \tag{3.5}$$

Following [7], the groupoid structure endows the usual prequantization of $T^*\mathbb{S}^2$ as a symplectic manifold with a $\mathbb{S}^1$-valued groupoid 2-cocycle $\zeta$, the *prequantization cocycle*. Since the symplectic form is exact, the prequantization line bundle is the trivial one and $\zeta\in C^\infty(\mathscr{G}_2(\mathbb{S}^2_\theta),\mathbb{S}^1)$ (we denote with $\mathscr{G}_k$ the set of $k$ composable elements of $\mathscr{G}$). Let us choose a primitive $\Theta$ of $\Omega$, then $\zeta$ satisfies the following two conditions

*i)* $\zeta$ is multiplicative, *i.e.* for any triple of composable points $(\gamma_1,\gamma_2,\gamma_3)\in\mathscr{G}_3(\mathbb{S}^2_\theta)$ it satisfies:

$$\zeta(\gamma_1,\gamma_2\gamma_3)\zeta(\gamma_2,\gamma_3) = \zeta(\gamma_1,\gamma_2)\zeta(\gamma_1\gamma_2,\gamma_3)\,. \tag{3.6}$$

*ii)* $\zeta$ is covariantly constant, *i.e.*

$$d\zeta - \imath/\hbar\,(\partial^*\Theta)\zeta = 0\,, \tag{3.7}$$

where $\partial^*$ denotes the simplicial coboundary operator. By direct computation one shows that $\Theta$ defined as

$$\Theta = \imath/4\theta\,(\bar{x}dx - \bar{y}dy - x\,d\bar{x} + y\,d\bar{y}) \tag{3.8}$$

is a multiplicative primitive of $\Omega$, *i.e.* $d\Theta=\Omega$ and $\partial^*\Theta=0$. As a consequence, the solution to (3.7) is $\zeta=1$. The fact that this 2–cocycle is trivial is a reflection of $\pi_\theta$ being exact in Poisson cohomology.

## 4. The Bohr-Sommerfeld groupoid

We discuss in this section the polarization of $\mathscr{G}(\mathbb{S}^2_\theta)$. As mentioned in the introduction we will look for a possibly singular multiplicative lagrangian distribution such that the modular 1–cocycle descends to the leaf groupoid. It is natural to seek for a *modular multiplicative integrable system*, as we considered in [3], *i.e.* a maximal set $F$ of functions in involution, almost everywhere independent, generating the modular cocycle $c_V$ (3.5) and such that the space of level sets $\mathscr{G}_F(\mathbb{S}^2_\theta)$ inherits the groupoid structure.

Let us consider the height function $\tau=\frac{1}{1+|z|^2}\in C^\infty(\mathbb{S}^2)$ and let $(f_1,f_2)=(l^*\tau,r^*\tau)$; then $\{f_1,f_2\}=0$ and $df_1\wedge df_2\neq 0$ on a dense open subset of $T^*\mathbb{S}^2$. Therefore, the non empty level sets are, when 2–dimensional, lagrangian. It is easy to check that $\mathscr{G}_F(\mathbb{S}^2_\theta)$ inherits the groupoid structure; moreover the modular cocycle (3.5) is $c_V=2\log(l^*\tau/r^*\tau)$.

Let us describe the topology of the quotient groupoid $\mathscr{G}_F(\mathbb{S}^2_\theta)$. The level sets over $\mathbb{S}^2 \setminus N$ are described by the pair groupoid coordinates $R_1 = |x|^2$, $R_2 = |y|^2$, for $R_1, R_2 \in \mathbb{R}^+$; there is one leaf $\infty$ over the North Pole that corresponds to $f_1 = f_2 = 0$, *i.e.* $R_1 = R_2 = \infty$. In the coordinates $(w, p)$ centered at the North Pole we have

$$(x,y) = \left(\frac{1}{w} + \iota\frac{\theta}{2}\bar{w}^2\bar{p}, \frac{1}{w} - \iota\frac{\theta}{2}\bar{w}^2\bar{p}\right) \tag{4.1}$$

from which it is easily seen that when $w \to 0$ the difference $x - y$ tends to zero as well. A fundamental system of neighbourhoods for $\infty$ is given by

$$B_{N,M,\varepsilon} = \{(R_1, R_2)\,\big|\, R_1 > N\,, R_2 > M\,, |R_1 - R_2| < \varepsilon\}\,. \tag{4.2}$$

Let us compute the subgroupoid $\mathscr{G}_F^{bs} \subset \mathscr{G}_F$ of Bohr-Sommerfeld (BS) leaves. We have that $(R_1, R_2)$ is a BS leaf if and only if $|z|^2 = R_i$, $i = 1, 2$, are BS leaves on $\mathbb{S}^2 \setminus N$ with respect to $\omega_S = \frac{\iota}{2\theta} dz \wedge d\bar{z}$, *i.e.*

$$\int_{|z|^2 \leq R} \omega = \frac{\pi R}{\theta} = 2\pi\hbar n\,.$$

If we fix $\hbar\theta > 0$, BS leaves are selected by the condition: $R_i = 2\theta\hbar n_i$, $i = 1, 2$ and are numbered by a pair of natural numbers $n_1, n_2$; to these leaves we have to add the leaf $\infty$ at infinity. By looking at (4.2) for $\varepsilon$ small enough, we see that around $\infty$ there is an open neighborhood $B_N = \{(R,R)\big|\, R > N\}$ so that source and target maps are homemorphisms with their image. By applying Proposition 2.8 in [8], we conclude that $\mathscr{G}_F^{bs}$ is ètale and admits as unique Haar system the counting measure on the fibres.

By construction the modular cocycle (3.5) descends to the following family of 1–cocycles of $\mathscr{G}_F^{bs}$ parametrized by $\hbar\theta$

$$c_V^{\hbar\theta}(m,n) = 2\log\left(\frac{1 + 2\hbar\theta n}{1 + 2\hbar\theta m}\right) \quad . \tag{4.3}$$

By a straightforward computation, we can check that $c_V^{\hbar\theta}$ is the modular cocycle of the following quasi invariant measure on $(\mathscr{G}_F^{bs})_0 = \bar{\mathbb{N}}$

$$\mu_{\hbar\theta}(m) = \frac{1}{(\frac{1}{2\hbar\theta} + m)^2}\,. \tag{4.4}$$

We are now ready to compare in the following Proposition this quantization output $(\mathscr{G}_F^{bs}, \zeta = 1, c_V^{\hbar\theta})$ with the same data $(\mathscr{G}_S, \zeta = 1, c_{pod}^{\hbar})$ produced by the quantization of Podleŝ sphere in [2].

**Proposition 4.1.** *The groupoid of Bohr-Sommerfeld leaves $\mathscr{G}_F^{bs}$ and Sheu's groupoid $\mathscr{G}_S$ are equivalent as topological groupoids but inequivalent as measured groupoids.*

*Proof.* Let us describe first $\mathscr{G}_S$. Let $\mathbb{Z} \times \overline{\mathbb{Z}}$ be the action groupoid where $\mathbb{Z}$ act on $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ by translation leaving $\infty$ fixed. This is a locally compact Hausdorff topological groupoid, when $\mathbb{Z}$ is endowed with the discrete topology and $\overline{\mathbb{Z}}$ is its compactification at $+\infty$. Its restriction $\mathscr{O}_1 = (\mathbb{Z} \times \overline{\mathbb{Z}})|_{\overline{\mathbb{N}}}$ is called the $(n = 1)$ Cuntz groupoid (see [8]). Inside this groupoid Sheu considered the subgroupoid

$$\mathscr{G}_S = \{(p,q) \in \mathscr{O}_1 \mid q = \infty \implies p = 0\} \subset \mathscr{O}_1\,. \tag{4.5}$$

The map $(m,n) \mapsto (m-n,n)$ gives a topological groupoid isomorphism between $\mathscr{G}_F^{bs}$ and $\mathscr{G}_S$. We have to show that the modular cocycles $c_V^{\hbar\theta}$ and $c^{\hbar}_{pod}$ are not cohomologous. The modular cocycle computed in [2] is $c^{\hbar}_{pod}(p,q) = -\hbar p$ that under the above isomorphism becomes $c^{\hbar}_{pod}(m,n) = \hbar(n-m)$. Now $c_V^{\hbar\theta} - c^{\hbar}_{pod} = \partial^*\varphi$ where $\varphi(m) = -2\log(1+2\hbar\theta m) + \hbar m$ . But such $\varphi$ does not extend continously at $\infty$. ∎

## 5. Kähler polarization

In this section we will show how results of the previous section may be recovered by the choice of a complex Kähler polarization in the complexified tangent bundle. Since the restriction of the symplectic groupoid to the symplectic leaf is a Kähler manifold we can take its natural Kähler polarization and extend it to the vertical polarization on $T_N^*\mathbb{S}^2$. We will therefore define $\mathscr{P}_\theta$ to be $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial \bar{y}} \rangle$ for finite $(x,y)$ and $(T_N^*\mathbb{S}^2)_{\mathbb{C}}$ over $(w=0,p_S)$. This is a smooth lagrangian multiplicative positive (if $\theta > 0$) distribution. Its associated real distributions are singular: $\mathscr{D} = \mathscr{P}_\theta \cap \bar{\mathscr{P}}_\theta$ is 0 on finite $(x,y)$ and $(T_N^*\mathbb{S}^2)_{\mathbb{C}}$ in $(w=0,p_S)$. A basis of hamiltonian vector fields is given by $\{\chi_{\bar{x}}, \chi_y\}$. By identifying $\mathscr{P}_\theta$ with $\mathscr{P}_\theta^\perp = \Omega(\mathscr{P}_\theta)$ a basis of covariantly constant sections under the Bott connection $\nabla_\chi \xi = i_\chi d\xi$ is $\underline{b} = \{d\bar{x}, dy\}$.

By using the multiplicative primitive $\Theta$ of the symplectic form $\Omega$ given in (3.8) we can easily compute the space of polarized sections to be:

$$\mathscr{S}_{\mathscr{P}} = \left\{ \sigma \otimes \sqrt{\underline{b}} \in C^\infty(T^*\mathbb{S}^2) \otimes C^\infty(\sqrt{\det \mathscr{P}_\theta}) \,\middle|\, \sigma = e^{-\frac{1}{4\theta\hbar}(|x|^2+|y|^2)} \psi(\bar{x},y) \right\} .$$

In such setting the local observables $|x|^2$ and $|y|^2$ are quantizable, since their hamiltonian fluxes preserve the polarization. From the general geometric quantization rules we get:

$$\widehat{|x|^2}\,\psi = 2\hbar\theta(\bar{x}\partial_{\bar{x}} + \frac{1}{2})\psi , \qquad \widehat{|y|^2}\,\psi = 2\hbar\theta(y\partial_y + \frac{1}{2})\psi . \tag{5.1}$$

On states $\psi_{m,n} = \bar{x}^m y^n$ the modular function $D_V = \exp(c_V)$ with $c_V$ as in (3.5) is then quantized as:

$$\widehat{D_V}\,\psi_{m,n} = \left( \frac{1+2\hbar\theta(n+1/2)}{1+2\hbar\theta(m+1/2)} \right)^2 \psi_{m,n} . \tag{5.2}$$

In order to define the convolution algebra on polarized section we need to introduce a left Haar system on $\mathscr{G}(\mathbb{S}^2_\theta)$. Any smooth Haar system on it (up to a scalar factor) is written as $d\lambda^x(x,y) = \Lambda(y)(1+|y|^2)^2 d^2y$ for $|x| < \infty$ with $\lim_{y\to\infty}\Lambda(y) = 1$ and $d\lambda^\infty = \theta^2 d^2 p_S$.

Thus, for any choice of the volume form $V_\rho = \rho(x)V$ on $\mathbb{S}^2$, where $V$ is the round volume form and $\rho \in C^\infty(\mathbb{S}^2)$ is strictly positive, there is defined the following volume form on $\mathscr{G}(\mathbb{S}^2_\theta)$:

$$\nu = \frac{(1+|y|^2)^2}{(1+|x|^2)^2} \rho(x)\Lambda(y) d^2x d^2y , \tag{5.3}$$

We obtain that $\nu = -\theta^2 \rho(x)\Lambda(y) D_V \Omega^2$ so that:

$$\nu = \nu^{-1} D_V^2 e^{\partial^* \log(\rho/\Lambda)} . \tag{5.4}$$

Remark that from this expression you get that the (groupoid) modular cocycle is $2c_V + \partial^*(\rho/\Lambda)$ .

We are going to define on the space of polarized sections the structure of left Hilbert algebra, *i.e.* compatible convolution and scalar product. We consider, on the space of polarized sections, the scalar product corresponding to the trivialization of $\det T^*\mathbb{S}^2$ determined by $\nu^{-1}$, hence:

$$\langle\sigma_1,\sigma_2\rangle_\nu = \int_{\mathbb{C}^2} d^2x\, d^2y\, \sqrt{\rho(y)\Lambda(x)D_V^{-1}}\; e^{-\frac{1}{2\theta\hbar}(|x|^2+|y|^2)}\; \overline{\psi_1}(\bar{x},y)\,\psi_2(\bar{x},y)\,, \tag{5.5}$$

Let $\rho(x)=\rho(|x|^2)$ and $\Lambda(y)=\Lambda(|y|^2)$. On the polynomial basis of polarized sections

$$\psi_{mn}(\bar{x},y)=\bar{x}^m y^n\,, \tag{5.6}$$

we have that $||\sigma_{mn}||_\nu=\ell(m)r(n)$, where:

$$\ell(m) \;=\; 2\pi\int_0^\infty dt\,\sqrt{\Lambda(t)}(1+t)\,t^m e^{-\frac{1}{2\hbar\theta}t} \quad , \tag{5.7}$$

$$r(n) \;=\; 2\pi\int_0^\infty dt\,\sqrt{\rho(t)}\frac{t^n}{(1+t)}e^{-\frac{1}{2\hbar\theta}t} \quad . \tag{5.8}$$

Moreover:

$$\langle\sigma_{m,n},\sigma_{m',n'}\rangle_\nu=\delta_{m,m'}\delta_{n,n'}\,\ell(m)r(n)\,. \tag{5.9}$$

By using the left Haar system on $\mathscr{G}(\mathbb{S}^2_\theta)$ we can define the following convolution product between polarized sections (and trivial 2–cocycle)

$$\sigma_1 *_\Lambda \sigma_2(x,y)=\int_{\mathbb{C}} d^2z\,\sigma_1(x,z)\sigma_2(z,y)\sqrt{\Lambda(|z|^2)}(1+|z|^2)\,, \tag{5.10}$$

and involution $\sigma^*(x,y)=\overline{\sigma(y,x)}$; we obtain, for sections $\sigma_{mn}$, the formula:

$$\sigma_{m,n} *_\Lambda \sigma_{m'n'}=\delta_{nm'}\ell(n)\sigma_{mn'}\,,\quad \sigma^*_{mn}=\sigma_{nm}\,. \tag{5.11}$$

Denoting $S(\sigma)=\sigma^*$ we get for the modular operator

$$D_{\rho\Lambda}\sigma_{mn}=S^\dagger S\sigma_{mn}=\frac{\ell(n)r(m)}{\ell(m)r(n)}\sigma_{mn}. \tag{5.12}$$

If we choose $\rho=\Lambda=1$ the integrals in $\ell(m)$ and $r(n)$, with $a=(2\hbar\theta)^{-1}$, give the following results:

$$\ell(m)=2\pi\,m!\,a^{-2-m}(1+a+m)\,,\quad r(n)=2\pi\,e^a n!\,\Gamma(-n,a)\,, \tag{5.13}$$

where $\Gamma(\alpha,z)=\int_z^\infty t^{\alpha-1}e^{-t}$. The following, then, holds true:

**Proposition 5.1.** *i) The space $\mathscr{S}_\mathscr{P}$ of polarized sections with convolution and involution (5.11), scalar product (5.9) defines a left Hilbert algebra $\mathscr{A}_\theta^\mathscr{P}$.*

*ii) $\mathscr{A}_\theta^\mathscr{P}$ is isomorphic to the left Hilbert algebra $L^2(\mathscr{G}_F^{bs},\nu^{-1}_{\mu_{\rho\Lambda}})$ defined by the quasi invariant measure*

$$\mu_{\rho\Lambda}(m)=\frac{r(m)}{\ell(m)}\,.$$

*iii) The measures $\mu_{\rho\Lambda}$ and $\mu_{\hbar\theta}$ defined in* (4.4) *are equivalent.*

*Proof.* By direct computation one shows on basis elements $\sigma_{mn}$ of $\mathscr{S}_{\mathscr{P}}$ that the left regular representation is involutive and bounded so that its completion with respect to the scalar product is a left Hilbert algebra.

Moreover, the map $e : \mathscr{A}_\theta^{\mathscr{P}} \to L^2(\mathscr{G}_F^{bs}, \nu^{-1}_{\mu_{\rho\Lambda}})$ defined by

$$e(\sigma_{m,n}) = e_{m,n}\sqrt{\ell(m)\ell(n)}\ , \tag{5.14}$$

where $e_{m,n}$ is a basis for $C_c(\mathscr{G}_F^{bs})$, is an algebra isomorphism of the convolution algebras $\mathscr{S}_{\mathscr{P}}$ and $C_c(\mathscr{G}_F^{bs})$; moreover it is an isometry if $C_c(\mathscr{G}_F^{bs})$ is equipped with the scalar product corresponding to the GNS state $\phi_{\mu_{\rho\Lambda}}$ defined by

$$\phi_{\mu_{\rho\Lambda}}(f) = \int \mu_{\rho\Lambda} f\ .$$

In fact, we have that

$$\phi_{\mu_{\rho\Lambda}}(e(\sigma_{m,n})^* * e(\sigma_{m,n})) = \phi_{\mu_{\rho\Lambda}}(e_{n,0})\ell(m)\ell(n) = \mu_{\rho\Lambda}(n)\ell(m)\ell(n) = ||\sigma_{m,n}||^2_\nu .$$

In order to prove (*iii*) let us define

$$\varphi(m) = \log\frac{\mu_{\hbar\theta}(m)}{\mu_{\rho\Lambda}(m)} = \log\left(\frac{\ell(m)}{r(m)}\right) - 2\log\left(m + \frac{1}{2\hbar\theta}\right) \quad .$$

Let us show the result in the case $\rho = \Lambda = 1$ so that one can make use of (5.13); by using the asymptotic limit $\Gamma(-m,a) \approx e^{-a}a^{-m}/m$ we get that $\lim_{m\to\infty}\varphi(m) = 2\log 2\hbar\theta$. ∎

## References


[1] Bonechi F., Ciccoli N., Staffolani N. and Tarlini M., *On the integration of Poisson homogeneous spaces*, Journ. Geom. Phys. **58** (2008) 1519–1529 [arXiv:math.SG/0711.0361v2].

[2] Bonechi F., Ciccoli N., Staffolani N. and Tarlini M., *The quantization of the symplectic groupoid of the standard Podleś sphere*, Journ. Geom. Phys. **62** (2012) 1851–1865 [arXiv:math.SG/1004.3163].

[3] Bonechi F., Ciccoli N. and Tarlini M., *Quantization of the symplectic groupoid* , PoS(Corfu2011) 060, Proceed. Corfu Summer Instute 2011 on Elementary Particle Physics, September 4–18 2011, Corfu Greece.

[4] Cattaneo A.S. and Felder G., *Poisson sigma models and symplectic groupoids*. In "Quantization of Singular Symplectic Quotients", (ed. N. P. Landsman, M. Pflaum, M. Schlichenmeier), Progress in Mathematics **198** (2001) 41–73 [arXiv:math.SG/0003023].

[5] Crainic M. and Fernandes R. L., *Integrability of Lie brackets*. Ann. of Math. **157** (2003) 575–620 [arXiv:math.DG/0105033].

[6] Ginzburg V., *Momentum mappings and Poisson cohomology,* Int. J. Math. **7** (1996) 329–358.

[7] Hawkins E., *A groupoid structure of quantization*, Journal of Symplectic Geometry **6** (2008) 61–125 [arXiv:math.SG/0612363].

[8] Renault J., *A groupoid approach to $C^*$-algebras*. Lectures Notes in Mathematics 793, Springer Verlag (1980).

[9] Roytenberg D., *Poisson cohomology of $SU(2)$–covariant "necklace" Poisson structures on $\mathbb{S}^2$*, Journ. Nonlin. Math. Phys. **9** (2002) 347–356 [arXiv:math.QA/0302036].

[10] Sheu A.J.–L., *Compact Quantum Groups and Groupoid $C^*$-Algebras* , Journal of Functional Analysis **144** 2 (1997) 371–393.

[11] Xu P., *Poisson manifolds associated with group actions and classical triangular r–matrices*, Journal of Functional Analysis **112** (1993) 218–240.

[12] Weinstein A. and Xu P., *Extensions of symplectic groupoids and quantization* Journ. für die Reine und Angew. Math. **417** (1991) 159–189.