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Measuring the Instability of a Reducible Critical point of the Seiberg-Witten Functional

Celso Melchiades Doria[*†]
Universidade Federal de Santa Catarina (UFSC)
E-mail: c.m.doria@ufsc.br

The instability of the reducible critical points $(A,0)$ of the Seiberg-Witten functional is studied by analysing the lowest eigenvalue of the elliptic operator $L_A = \triangle_A + \frac{k_g}{4}$. A short resume about the existence and relevance of SW-monopoles is given and also how geometrical structures on a 4-manifold X plays an important role to the theory.

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1. Introduction

The Seiberg-Witten equations has been a powerful device to topologist understand the topology of smooth four dimensional manifolds. The differential phenomenons are difficult to understand because the invariants must be defined in terms of a smooth structure on the underlying topological space. The Gauge Theory developed by theoretical physicists turned out to be an important source of methods to define smooth invariants, this is so because Gauge Theory is a Vector Bundle Theory endowed with equations. In dimension 4, a smoothable topological 4-manifold may admit a infinite number of smooth structures in contrast with 3 dimension where there is only one. The Seiberg-Witten theory is claimed to be dual, in the sense of Montonen-Olivie [5] to a twisted version of $N = 2$ supersymmetric Yang-Mills Theory [9], but this is still an open question. Nevertheless, from the mathematical point of view, none is concerned with the origin of the equations as far as they are useful. Let X^4 be a closed, smooth manifold. The cohomology $H^*(X) = \oplus_{i=0}^4 H^i(X)$ turns out to be of fundamental importance to the study topological and geometrical properties of X. Let $b_i(X) = dimH^i(X)$, $1 \leq i \leq 4$, be the Betti numbers, $\chi(X) = 2-2b_1(X)+b_2(X)$ be the Euler characteristic, and $\sigma(X) = b_2^+ - b_2^-$ the signature of X, where $H^2(X) = H^2_+ \oplus H^2_-$ and $b_2^\pm = dimH^2_\pm$. In order to describe the theory on a closed 4-manifold X, let's fix a riemannian metric g and a spin^c structure on X. The choice of a riemannian metric on X reduces the structural group of the tangent bundle TX to SO_4, so the frame (vierbein) bundle FX is a principal SO_4-bundle. The space of $Spin^{\mathfrak{c}}$ structures on X is $Spin^c(X) = \{\mathfrak{s} = \alpha_{\mathfrak{s}} + \beta \in H^2(X,\mathbb{Z}) \oplus H^1(X,\mathbb{Z}_2) \mid w_2(X) = \alpha_{\mathfrak{s}} \; mod \; 2\}$. Because $Spin_4^{\mathfrak{c}} = (SU_2 \times SU_2 \times U_1)/_{\mathbb{Z}_2}$ and $U_2 = (SU_2 \times U_1)/\mathbb{Z}_2$ we get two representations $\rho_\pm : Spin_4^{\mathfrak{c}} \to U_2 = (SU_2 \times U_1)/\mathbb{Z}_2 \subset GL(2,\mathbb{C})$. In practice, a $Spin^{\mathfrak{c}}$-structure on X is given by a pair of rank 2 complex vector bundles $\mathscr{S}_{\mathfrak{s}}^\pm$ with isomorphisms $det(\mathscr{S}_{\mathfrak{s}}^+) = det(\mathscr{S}_{\mathfrak{s}}^-) = \mathscr{L}_{\mathfrak{s}}$, where $det(\mathscr{S}_{\mathfrak{s}}^\pm)$ are the determinant line bundle such that $c_1(\mathscr{L}_{\mathfrak{s}}) = \alpha_{\mathfrak{s}} \in H^2(X,\mathbb{Z})$ (denote $c_1(\mathfrak{s}) = c_1(\mathscr{L}_{\mathfrak{s}})$). Spinor bundles are powerful tools because they carry a Dirac operator $D_A : \Omega^0(\mathscr{S}_{\mathfrak{s}}^+) \to \Omega^0(\mathscr{S}_{\mathfrak{s}}^-)$. Let $\mathscr{A}_{\mathfrak{s}}$ be the space of U_1-connections on $\mathscr{L}_{\mathfrak{s}}$ and $\Omega^0(\mathscr{S}_{\mathfrak{s}}^+)$ be the space of sections of $\mathscr{S}_{\mathfrak{s}}^+$. The configuration space on X is $\mathscr{C}_{\mathfrak{s}} = \mathscr{A}_{\mathfrak{s}} \times \Omega^0(\mathscr{S}_{\mathfrak{s}}^+)$. The gauge group acting on $\mathscr{C}_{\mathfrak{s}}$ is $\mathscr{G} = Map(X,U_1)$, the space of maps from X to U_1, $g.(A,\phi) = (A + 2g^{-1}dg, g^{-1}\phi)$. The Gauge action is not free on $\mathscr{C}_{\mathfrak{s}}$, the moduli space $\mathscr{B}_{\mathfrak{s}} = \mathscr{C}_{\mathfrak{s}}/\mathscr{G}$ is singular at the points $(A,0)$; the isotropic subgroups $\mathscr{G}_{(A,0)}$ are isomorphic to U_1. By restricting to the subgroup $\mathscr{G}^* = \{g \in \mathscr{G} \mid g(x_0) = I\}$ the action becomes free and the moduli space $\mathscr{B}_{\mathfrak{s}}^* = \mathscr{C}_{\mathfrak{s}}/\mathscr{G}^*$ is an infinite dimensional manifold. Indeed, $\mathscr{C}_{\mathfrak{s}}$ is a universal principle bundle over $\mathscr{B}_{\mathfrak{s}}^*$, so the classifying space for principle $\mathscr{G}$-bundles is $B\mathscr{G}^* = \mathscr{B}_{\mathfrak{s}}^*$ whose homotopy type is the same as $\mathbb{C}P^\infty \times \mathscr{J}_X$, where $\mathscr{J}_X = H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$ is diffeomorphic to $T^{b_1}(X)$ (jacobian torus). Another way of avoiding the singular set is by blowing-up the configuration space, this new new setting will be considered in a forthcoming paper.

2. Variational Set Up

The Seiberg-Witten monopole equations on X are

$$F_A^+ = \sigma(\phi), \quad D_A^+\phi = 0, \tag{2.1}$$

where F_A^+ is the self-dual component of the curvature F_A, D_A^+ is the positive component of the Dirac operator and σ is the sel-dual 2-form

$$\sigma(v)(X,Y) = < X.Y.v,v > + \frac{1}{2} < X,Y > | v |^2, \quad | \sigma(v) |^2 = \frac{1}{4} | v |^4 .$$

The Seiberg-Witten functional $\mathscr{SW} : \mathscr{C}_{\mathfrak{s}} \to \mathbb{R}$ is defined by

$$\mathscr{SW}_{\mathfrak{s}}(A,\phi) = \int_X \{\frac{1}{4} | F_A |^2 + | \nabla^A \phi |^2 + \frac{k_g}{4} | \phi |^2 + \frac{1}{8} | \phi |^4\} dv_g + 2\pi^2 c_1^2(\mathfrak{s})[X], \tag{2.2}$$

where k_g is the scalar curvature of (X,g) and $c_1^2(\mathfrak{s}) = c_1(\mathfrak{s}) \wedge c_1(\mathfrak{s}) = \frac{1}{4\pi^2}\int_X [| F_A^+ |^2 - | F_A^- |^2] dv_g$. Because of the gauge invariance, the $\mathscr{SW}_{\mathfrak{s}}$-functional defines a function $\mathscr{SW}_{\mathfrak{s}} : \mathscr{B}_{\mathfrak{s}} \to \mathbb{R}$. The Euler-Lagrange equations are

$$d^* F_A + 4iIm(< \nabla^A \phi, \phi >) = 0, \quad \text{and} \quad \Delta_A \phi + \frac{| \phi |^2 + k_g}{4} \phi = 0, \tag{2.3}$$

The solutions (A,ϕ) of the monopole equations (2.1) are stable critical points of the $\mathscr{SW}$-functional. Moreover, they also satisfy the equations (2.3);

$$d^*(F_A) = 2d^* F_A^+ - = d^*[\sigma(\phi)] = -4iIm\left(< D^+\phi, X.\phi > + < \nabla_X \phi, \phi >\right)$$
$$D^+\phi = 0 \;\Rightarrow\; 0 = D^- D^+ \phi = \triangle_A + \frac{k_g}{4}\phi + \frac{F_A^+}{2}.\phi = \triangle_A \phi + \frac{k_g}{4}\phi + \frac{| \phi |^2}{\phi}$$

However, it is not always true that a stable critical point satisfies equation (2.1). In [9], Witten proved that at most a finite number of classes in $Spin^{\mathfrak{c}}(X)$ admit solutions for the monopole equations (2.1). $(A,\phi) \in \mathscr{C}_{\mathfrak{s}}$ is called a $\mathscr{SW}_{\mathfrak{s}}$-monopole if satisfies eqs. (2.1) and is called a $\mathscr{SW}_{\mathfrak{s}}$-critical point if satisfies the equations (2.3). There are two kinds of critical points, the irreducibles when $\phi \neq 0$ and the reducibles $(A,0)$. The difference between these categories is measured by the isotropic subgroup $\mathscr{G}_{(A,\phi)} = \{g \in \mathscr{G} \mid g.(A,\phi) = (A,\phi)\}$; $G_{(A,\phi)} = \{I\}$ if $\phi \neq 0$ and $\mathscr{G}_{(A,0)} = U_1$. The reducible $\mathscr{SW}_{\mathfrak{s}}$-monopoles satisfy $F_A^+ = 0$, they are abelian instantons. The set of reducible $\mathscr{SW}_{\mathfrak{s}}$ critical point satisfying $d^* F_A = 0$ is exactly the Jacobian torus $\mathscr{J}_X$. In the $\mathscr{SW}$-functional formula the scalar curvature k_g plays a important role by noticing that if it is non-negative, then $\mathscr{J}_X$ is a stable critical submanifold of $\mathscr{C}_{\mathfrak{s}}$ because $\mathscr{SW}_{\mathfrak{s}}(A,0) < \mathscr{SW}_{\mathfrak{s}}(A,\phi)$ for all $\phi \neq 0$. Due to Hodge theory the space $\mathscr{J}_X$ is never empty because for all $(A,0) \in \mathscr{J}_X$ the curvature F_A is a harmonic 2-form. When $\pi_1(X) = 0$, by considering $[\Theta]$ the class of the trivial connection, $\mathscr{J}_X = [\Theta]$ is just a point. By measuring the instability at each point in $\mathscr{J}_X$ it might be possible to learn about the existence of a $\mathscr{SW}_{\mathfrak{s}}$-monopole. It would be a big achievement to find a sufficient condition on a smooth manifold to guarantee the existence of an irreducible $\mathscr{SW}_{\mathfrak{s}}$-monopole, though it is a hard question to me answered by now. The classes $\mathfrak{s} \in Spin^{\mathfrak{c}}(X)$ admitting a irreducible $\mathscr{SW}_{\mathfrak{s}}$-monopole are named *Basic Classes*.

3. Existence x Non-Existence

The Basics Classes play a central role to the applications in differential topology. Let $\mathscr{M}_{\mathfrak{s}} \subset \mathscr{B}_{\mathfrak{s}}$ be the moduli space of $\mathscr{SW}$-monopoles. If $b_2^+(X) \geq 2$, then $\mathscr{M}_{\mathfrak{s}}$ is either empty or a smooth, compact and orientable manifold whose dimension is giving by the formula (ref. [4])

$$d(\mathfrak{s}) = \frac{1}{4}\left\{\alpha_{\mathfrak{s}}^2[X] - [2\chi(X) + 3\sigma(X)]\right\}.$$

The abelian nature of the gauge group in the Seiberg-Witten theory is an essential ingredient to turn it into a simpler theory than the Donaldson theory. A 4-manifold X is simple type if for all $\mathfrak{s} \in Spin^c(X)$ either $\mathscr{M}_{\mathfrak{s}} = \varnothing$ or $d(\mathfrak{s}) = 0$. In the last case, $\mathscr{M}_{\mathfrak{s}}$ must be a finite set of points $\{\alpha_1, \ldots, \alpha_n\}$, $\alpha_i = (A_i, \phi_i)$, each one carrying attached a sign $n_i = \pm 1$ according with orientation. The Seiberg-Witten invariant associated to a class $\mathfrak{s} \in Spin^c(X)$ is

$$\mathscr{SW}(\mathfrak{s}) = \begin{cases} \sum_{i=1}^n n_i, \ \mathscr{M}_{\mathfrak{s}} \neq \varnothing, \\ 0, \ \mathscr{M}_{\mathfrak{s}} = \varnothing \end{cases} \tag{3.1}$$

Though naively defined as the sum (3.1), $\mathscr{SW}(\mathfrak{s})$ has a cohomological interpretation and defines a smooth invariant of X (see ([4]). Thus the existence of irreducible $\mathscr{SW}_{\mathfrak{s}}$-monopoles is essential to apply the SW-theory. In contrast to the instanton equation, there are no finite energy $\mathscr{SW}_{\mathfrak{s}}$-monopole on $\mathbb{R}^4$. The deepest result concerning the existence of monopoles is Taubes' theorem.

3.1 Existence Theorems

The main theorems concerning the existence of $\mathscr{SW}$-monopoles are enunciated next. In [9] Witten proved the following theorem for Kähler 4-manifolds (ref [4], thm 7.3.1);

Theorem Let X be a Kähler surface of general type and minimal endowed with the Kähler metric. (i) If $c_1^2(\mathscr{L}_{\mathfrak{s}})[X] < 0$, then there is no irreducible $\mathscr{SW}$-monopoles, the only critical points are reducible. (ii) If $c_1^2(\mathscr{L}_{\mathfrak{s}})[X] > 0$, then $\mathscr{SW}(\kappa^*) = \pm 1$. ($\kappa^*$ is the canonical class)

Shortly after, Taubes proved [8] that the symplectic structure implies existence;

Theorem (Taubes) Let X be a simply connected 4-manifold with $b_2^+(X) \geq 2$. If X admits a symplectic structure ω ($\omega \wedge \omega > 0$), then $\pm\kappa^* = \pm c_1(J_\omega)$ are basic classes, and $\mathscr{SW}(\pm K^*) = \pm 1$.

A deep theorem (ref. [7]) due to Taubes relates the existence of $\mathscr{SW}_{\mathfrak{s}}$-monopoles with the existence of pseudo J-holomorphic curves (J=almost complex struc.). Indeed, he proved $\mathscr{SW}_{\mathfrak{s}}(\alpha) = \#\{\Sigma \subset X \mid [\Sigma] = \alpha \in H_2(X;\mathbb{Z}),\ \Sigma \text{ is a } J\text{-holomorphic curve}\}$. There is no known sufficient condition on a smooth manifold to guarantee the existence of Basic Class. The differential topologist managed to produce examples of non-symplectic 4-manifod with non-trivial SW-invariant, a remarkable manner is by using the Knot Surgery developed by Fintushel-Stern ([3]). They discovered a wide amount of non-symplectic homotopic $K3$ surfaces, almost as much as the number of isotopic classes of knots $K \subset S^3$. So far, there is no way of proving the existence of monopoles on any X without using Taubes' thm 3.1. Thus, it rises the following question "is every 4-manifold homemorphic to a symplectic one ?"

4. Instability of Critical points

The existence of the critical manifold $\mathscr{J}_X$ is guarantee by the topology of X. Starting from them, it could be argued if their instability could provide us information about the existence of $\mathscr{SW}_{\mathfrak{s}}$-monopoles. The instability of $\mathscr{J}_X$ is established by performing the analysis of the 2nd variation $\frac{\delta^2 \mathscr{SW}}{\delta\alpha\delta\beta}$ of the $\mathscr{SW}$-functional. In order to do so, a short review on the tangent space of $\mathscr{B}_{\mathfrak{s}}^*$

is listed next. The tangent space to the orbit $\mathscr{O}_{(A,\phi)} = \{g.(A,\phi) \mid g \in \mathscr{G}\}$ is $T_{(A,\phi)}\mathscr{O}_{(A,\phi)}\} = Imag(T)$, where

$$\begin{aligned} T &: \Omega^0(X,i\mathbb{R}) \to \Omega^1(X,i\mathbb{R}) \oplus \Omega^0(\mathscr{S}^+_{\mathfrak{s}}), \\ T_\phi(\lambda) &= (d\lambda, -\lambda.\phi). \end{aligned}$$

Since T has closed range, a local slice for the space $\mathscr{A}_{\mathfrak{s}} \times_{\mathscr{G}} \Gamma(S^+_{\mathfrak{s}})$ at (A,ϕ) is given by $Ker(T^*)$, where

$$\begin{aligned} T^*_\phi &: \Omega^1(X,i\mathbb{R}) \oplus \Omega^0(\mathscr{S}^+_{\mathfrak{s}}) \to \Omega^0(X,i\mathbb{R}), \\ T^*_\phi(\theta,V) &= d^*\theta - < V,\phi > . \end{aligned}$$

So, $ker(T^*_\phi) = ker(d^*) \oplus \phi^\perp$. Because $(d^*)^2 = 0$, it can be further decomposed into $ker(d^*) = imag(d^*) \oplus \mathscr{H}_1$, where $\mathscr{H}_1 = \{\theta \in \Omega^1(X,i\mathbb{R}) \mid d\theta = d^*\theta = 0\}$ is the subspace of harmonic 1-forms and also the tangent space to the Jacobian torus $\mathscr{J}_X$ at $(A,0)$.

The tangent space of $\mathscr{C}_{\mathfrak{s}}$ at (A,ϕ) is $T_{(A,\phi)}\mathscr{C}_{\mathfrak{s}} = \Omega^1(X;i\mathbb{R}) \oplus \Omega^0(\mathscr{S}^+_{\mathfrak{s}})$, so $\frac{\delta^2\mathscr{SW}}{\delta\alpha\delta\beta}$ defines a symmetrical bilinear form $H^{\mathscr{SW}}_{(A,\phi)}((\theta_1,V_1),(\theta_2,V_2)) =< (\theta_1,V_1), H(\theta_2,V_2) >$, where the operator $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ has entries given by

$$\begin{aligned} &\frac{\delta^2\mathscr{SW}_{\mathfrak{s}}}{\delta\Lambda\delta\theta} \mid_{(A,\phi)} .(\theta,\Lambda) =< \theta, (d^*d\Lambda + 4 < \Lambda(\phi),\phi >) =< \theta, h_{11}(\Lambda) >, \\ &\frac{\delta^2\mathscr{SW}_{\mathfrak{s}}}{\delta W\delta\theta} \mid_{(A,\phi)} .(\theta,W) = 2\left(< \nabla^A\phi, \theta(W) > + < \nabla^A W, \theta(\phi) >\right) =< \theta, h_{12}(W) >, \ (h_{21} = h_{12}) \\ &\frac{\delta^2\mathscr{SW}_{\mathfrak{s}}}{\delta W\delta V} \mid_{(A,\phi)} .(V,W) =< V, \triangle_A W + \frac{k_g + \mid \phi \mid^2}{4} W + \frac{1}{4} < \phi, W > \phi >=< V, h_{22}(W) > . \end{aligned}$$

The induced 2^{nd}-variation on $\mathscr{B}^*_{\mathfrak{s}}$, at (A,ϕ), is defined by just restricting it to the subspace $ker(T^*_\phi) = ker(d^*) \oplus \phi^\perp$. Therefore, $H : ker(T^*_\phi) \to ker(T^*_\phi)$ is an elliptic operator because the leading terms $d^*d = \triangle$ and $\triangle_A$ are laplacians whose symbol are isomorphisms.

The spectrum of $\mathscr{H} : ker(T^*_\phi) \to ker(T^*_\phi)$ is a discrete set such that each eigenvalue has finite multiplicity and no accumulation points, besides, there are but a finite number of eigenvalues below any given number. All of these are consequences from the fact that the spectral analysis depends on the leading terms of $\mathscr{H}$ which in this case are laplacian operators. At each point $(A,0)$, the hessian operator is $H = \begin{pmatrix} d^*d & 0 \\ 0 & L_A \end{pmatrix}$, where $L_A : \Omega^0(\mathscr{S}^+_{\mathfrak{s}}) \to \Omega^0(\mathscr{S}^+_{\mathfrak{s}})$ is the elliptic self-adjoint operator $L_A(V) = \triangle_A V + \frac{k_g}{4}V$. L_A can be diagonalized and the eingenspace $\mathscr{V}_\lambda \subset T_{(A,0)}\mathscr{B}_{\mathfrak{s}}$ associated to the eigenvalue λ is finite dimension for all λ. Furthermore, the spectrum of $\mathscr{H}$ is bounded below. Thus, $ker(\mathscr{H}) = T_{(A,0)}\mathscr{J}_X \oplus ker(L_A)$, where $ker(L_A) = \mathscr{V}_0$ is a finite dimensional space. By assuming $ker(L_A) = \{0\}$, the Morse-Bott index of the critical submanifold $\mathscr{J}_X$ is equal to the dimension of the largest negative eingenspace of L_A. The lower eingenvalue $\lambda^{\mathfrak{s}}_m(g,A)$ of L_A can be estimate by the Rayleigh quotient

$$\lambda_m^{\mathfrak{s}}(g,A) = \inf_{V\in\Omega^0(\mathscr{S}_{\mathfrak{s}}^+)} \left\{ \frac{\int_X \left[\mid \nabla^A V \mid^2 + \frac{k_g}{4} \mid V \mid^2 \right] dv_g}{\int_X \mid V \mid^2 dv_g} \right\}$$

The purpose is to compare $\lambda_m^{\mathfrak{s}}(g,A)$ with the lowest eigenvalue of the linear, elliptic and self-adjoint operator $L(u) = \triangle_g u + \frac{k_g}{4} u$ defined on functions $u : X \to \mathbb{R}$, ($\triangle_g = -d^*d$ is the Laplace-Beltrami operator on (X,g)). The Kato's inequality $\mid d \mid V \mid^2 \mid \leq \mid \nabla^A V \mid^2$ turns out to be useful tool; fix an orthonormal frame $\beta = \{e_i \mid i = 1,2,3,4\}$ on X, then

$$\big(d \mid V \mid^2\big) = \sum_i (\partial_i \mid V \mid)^2 \quad \text{and} \quad \mid \nabla^A V \mid^2 = \sum_{i=1}^4 \mid \nabla_i^A V \mid^2 . \qquad (\partial_i = \frac{\partial}{\partial_i}).$$

Besides, by taking the identities (i) $\frac{1}{2}\partial_i \mid V \mid^2 = \mid V \mid .\partial_i \mid V \mid$ and (ii) $\frac{1}{2}\partial_i \mid V \mid^2 = < \nabla_i^A V, V >$, and applying Cauchy-Schwartz inequality, we get $\mid V \mid . \mid \partial_i(\mid V \mid) \mid \leq \mid V \mid . \mid \nabla_i^A V \mid$. So, if $V \neq 0$, then

$$\mid d \mid V \mid^2 \mid = \sum_i (\partial_i \mid V \mid)^2 \leq \sum_i \mid \nabla_i^A V \mid^2 = \mid \nabla^A V \mid^2 .$$

The equality happens if, and only if, there exist complex functions $\alpha_i : X \to \mathbb{C}$, $i = 1,2,3,4$ such that $\nabla_i^A V = \alpha_i V$. If there exists $V \in \mathscr{S}_{\mathfrak{s}}$ and functions $\alpha_i \in C^\infty(X)$ such that $\nabla_i^A V = \alpha_i V$ and the 1-form $\omega = \sum_i \alpha_i dx^i$ is closed, then X is Kähler. This relies on the relationship among the existence of parallel spinors and a Kähler structure on X, namely, the section $f.V$ is parallel ($\nabla^A fV = 0$), $f : X \to \mathbb{R}$, if the system $\partial_i f + \alpha_i f = 0$, $i = 1,2,3,4$, admits solution. Consider the 1-form $\omega = \sum_i \alpha_i dx^i$ and assume it is closed. If the system admits a solution, then $\partial_i \partial_j f = \partial_j \partial_i f$ implies $\partial_j \alpha_i = \partial_i \alpha_j$, so ω is closed. Now, let's assume ω is closed, so the identity $\partial_j \alpha_i = \partial_i \alpha_j$ allow us to define the function

$$f(x_1,\ldots,x_n) = e^{-\int_0^{x_1} \alpha_1(t,x_2,\ldots,x_n)dt}.$$

Of course, $\partial_1 f = -\alpha_1 f$. The ω closeness guarantee that

$$\partial_i f = \left(-\int_0^{x_1} \alpha_1(t,x_2,\ldots,x_n)dt \right) f, \; i = 1,2,3,4.$$

Indeed, it follows that $\omega = d(ln(f))$, hence ω is an exact 1-form.

4.1 Estimating the Lowest Eigenvalue

Whenever the lowest eigenvalue of operator L_A is negative, there is a change of existing a $\mathscr{S}\mathscr{W}_{\mathfrak{s}}$-monopole, so let's investigate this possibility by comparing with the lowest eigenvalue of the operator L_g. Let $\lambda_m(g)$ be lowest eigenvalue of L_g and define $\bar{\lambda}(g) = \lambda_m(g).[vol(X,g)]^{1/2}$.
Let $\mathcal{M}(X)$ be the space of riemannian metrics on X and $[g] = \{\zeta.g \mid \zeta : X \to (0,\infty)\}$ the conform class of g. The Yamabe constant of $[g]$ is defined by

$$Y_{[g]} = \inf_{\hat{g}\in[g]} \frac{\int_X k_{\hat{g}} dv_{\hat{g}}}{[vol(X,\hat{g})]^{1/2}}. \tag{4.1}$$

The condition $Y_{[g]} \leq 0$ implies the existence of unique metric realizing the the Yamabe constant ([1]). The smooth Yamabe invariant is defined as $Y(X) = \sup_{[g]\subset \mathcal{M}} Y_{[g]}$. In [1] they prove, under the condition $Y(X,g) < 0$, the relation $Y(X) = \sup_{g\in[g]} \bar{\lambda}(X,g) < 0$. By analogy, associated to the operator L_A we define $\bar{\lambda}^{\mathfrak{s}}(g,A) = \lambda_m^{\mathfrak{s}}(g,A).[vol(X,g)]^{1/2}$ and $Y^{\mathfrak{s}}(X,A) = \sup_{g\in[g]} \bar{\lambda}^{\mathfrak{s}}(g,A)$. From the Euler-Lagrange equations we get

$$\int_X \left[\mid \nabla^A \phi \mid^2 + \frac{k_g}{4} \mid \phi \mid^2 \right] dv_g = -\frac{1}{4} \int_X \mid \phi \mid^4 dv_g, \quad \text{and so,}$$

$$\lambda_m^{\mathfrak{s}}(g,A). \int_X \mid \phi \mid^2 dv_g \ < \ -\frac{1}{4} \int_X \mid \phi \mid^4 dv_g.$$

The Cauchy-Schwartz inequality for integrals implies that $\int_X \mid \phi \mid^2 dv_g \leq [vol(X,g)]^{1/2}. \left[\int_X \mid \phi \mid^4 dv_g\right]^{1/2}$ Therefore, the lowest eigenvalue is negative whenever there exist an irreducible solution for the Euler-Lagrange eqs. If there exists an irreducible $\mathscr{SW}_{\mathfrak{s}}$-monopole than the lowest eigenvalue is upper bounded by a topological number because the equation $F_A^+ = \sigma(\phi)$ implies $\mid F_A^+ \mid^2 = \frac{1}{4} \mid \phi \mid^4$ and $c_1^2(\mathscr{L}_{\mathfrak{s}}) = \frac{1}{(2\pi)^2} \int_X [\mid F_A^+ \mid^2 - \mid F_A^- \mid^2] dv_g$, hence

$$\bar{\lambda}^{\mathfrak{s}}(g,A) \leq -\frac{1}{4} \left[\int_X \mid \phi \mid^4 \right]^{1/2} \leq - \left[\int_X \mid F_A^+ \mid^2 dv_g \right]^{1/2} \leq -2\pi \sqrt{c_1^2(\mathscr{L}_{\mathfrak{s}})[X]}$$

If $\bar{\lambda}(g) = \bar{\lambda}^{\mathfrak{s}}(g,A)$, then from last section we conclude X is Kähler; in this case the Yamabe invariant guarantee the negativeness.

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