

Hadrons potentials and gauge/string dualities

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In 1998, J. M. Maldacena conjectured a precise duality in the low-energy (or decoupling) regime between a large N strongly-coupled $SU(N)$ super-Yang-Mills theory defined in the four-dimensional Minkowski world-volume M^4 of a stack of N coincident D3-branes and the supergravity limit of a weakly-coupled type IIB closed superstring theory on $AdS_5 \times S^5$. In this proceeding are introduced the concepts and methods used to derive, in the gauge/string correspondence framework, the interaction potentials of mesons and baryons at zero and finite temperature [1]. Especially, emphasize will be made on the linear behavior of the bound-state potentials derived in the gravity side.

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1. The area law in QCD and the heavy quark potentials in AdS/CFT

A crucial breakthrough in the attempt to deal with strongly-coupled Yang-Mills theories came with the nowadays celebrated AdS/CFT correspondence whose the bulk-to-boundary dictionary has been established essentially in [2]. Shortly afterwards, the issue of calculating expectation values of Wilson loops was considered [3], which is of significant importance since the Wilson loop, through the area law, consists of one of the most efficient tools for probing the large distance properties of confining QCD-like gauge theories [4]:

$$W[\mathcal{C}] \equiv \frac{1}{N} \langle \text{Tr} P e^{-ig \oint_{\mathcal{C}} A_{\mu}(x) dx^{\mu}} \rangle_A \Big|_{\mathcal{C} \rightarrow \infty} = e^{-\sigma_f A[\mathcal{C}]} \quad (1.1)$$

with $A[\mathcal{C}]$ the area of the minimal surface of boundary \mathcal{C} . Then, the Feynman-Kac formula gives a linear static potential for an infinitely massive quark-antiquark pair:

$$V_{Q\bar{Q}}(r) = - \lim_{T \rightarrow \infty} \frac{1}{iT} \ln W[\mathcal{C}] = \sigma_f r. \quad (1.2)$$

On the contrary, because of the underlying conformal symmetry on which AdS/CFT relies, the supergravity side computation of the infinitely heavy $Q\bar{Q}$ pair (non-dynamical external probes) potential shows instead a $1/r$ behavior for all distances r and goes in terms of the 't Hooft coupling λ as $\sqrt{\lambda}$, which reflects an intrinsically non-perturbative result [3]:

$$V_{Q\bar{Q}}(r) = - \frac{4\sqrt{2}\pi^2}{\Gamma(1/4)^4} \frac{\sqrt{\lambda}}{r}. \quad (1.3)$$

According to Maldacena's recipe, the expectation value of the *spacetime* Wilson loop (*spatial* loops - which follow the same prescription - will also be considered when deriving area law for $3d$ and $4d$ conformal field theories at finite temperature) is dual to the full partition function of the string theory which, in the low-energy supergravity limit where the stringy effects are small, reduces to the proper area of the string world-sheet with the loop as the boundary. One might sketch the AdS/CFT steps of the calculation as follows:

$$W[\mathcal{C}] \Big|_{AdS/CFT} \sim Z_{string}[\mathcal{C}] \Big|_{\text{supra}} \sim e^{-S[\mathcal{S}]} \quad (1.4)$$

where $S[\mathcal{C}]$ is the classical Euclidean action of the string world-sheet (that, in fact, does not describe the Wilson loop (1.1) but instead its supersymmetric generalization). In practice, one usually takes the simplest action which describes the dynamics of an open string, namely the Nambu-Goto action $S_{NG}[\mathcal{C}] = \frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{\det(\gamma_{ab})}$ where $\alpha' = \ell_s^2$ is related to the typical length scale ℓ_s of the string and $\gamma_{ab}(\xi)$ ($a, b = 1, 2$) is the induced metric tensor on the two-dimensional world-sheet. Furthermore, it turns out necessary to regularize the potential (by cutting off the infinite range of the holographic coordinate in the *ultraviolet*) as infinities arise: they correspond to the two infinitely stretched strings associated with the infinitely heavy Q and \bar{Q} respectively¹. Hence, the final recipe for computing Wilson loops and regularized potentials is:

$$V_{Q\bar{Q}}(r) = \lim_{\substack{T \rightarrow \infty \\ M \rightarrow \infty}} \frac{1}{T} (S_{NG} - \ell M) \quad (1.5)$$

¹In a flat spacetime, the mass-squared M^2 of a string stretched between x_1 and x_2 reads in broad outline and neglecting the quantum fluctuations as $M^2 = \left(\frac{x_2 - x_1}{2\pi\alpha'}\right)^2$.

from which is derived (1.3) and where the second term on the *r.h.s.* consists precisely in subtracting the contribution of the two infinitely stretched Q and \bar{Q} strings (ℓ is the total perimeter of the loop \mathcal{C} on the boundary).

2. The static potential at finite temperature in supergravity

Following Hawking and Page's work on the thermodynamics of black holes in anti-de Sitter spacetimes, a gauge/string duality involving a gauge theory at finite temperature was proposed [5]. In this framework, the bulk accommodates a Schwarzschild black hole (BH) whose the metric in Euclidean reads as

$$ds_{BH}^2 = \alpha' \left\{ \frac{u^2}{\tilde{R}^2} (f(u) dt^2 + d\vec{x}^2) + \frac{\tilde{R}^2}{u^2} du^2 + \tilde{R}^2 d\Omega_5^2 \right\} \quad (2.1)$$

where $f(u) = 1 - \frac{u_T^4}{u^4}$ and $\tilde{R}^4 \equiv \frac{R^4}{\alpha'^2} = 4\pi g_s N = 2\lambda$ is the dimensionless *AdS* radius². u is the holographic coordinate. There is a curvature singularity at $u = 0$ hidden behind the event horizon at $u = u_T$ whose the location is given by the Bekenstein-Hawking temperature T of the black hole $u_T = \pi \tilde{R}^2 T$. In particular, at zero temperature, *i.e.* when $u_T = 0$, one recovers the line element of $AdS_5 \times S^5$. To the high (low) energy regime of the dual boundary theory corresponds $u \rightarrow \infty$ ($u \rightarrow 0$). Then, considering a *spacetime* Wilson loop (that is, a loop along one space-like dimension and one time-like dimension) and following Maldacena's recipe, the subtracted static potential is [6]:

$$V_{Q\bar{Q}} = \frac{U_0}{\pi} \int_1^{U_{max}/U_0} dv \left(\sqrt{\frac{v^4 - 1 + \varepsilon}{v^4 - 1}} - 1 \right) + \frac{U_T - U_0}{\pi} \quad (2.2)$$

where $v \equiv U/U_0$ and $\varepsilon \equiv f(U_0)$. $U(x)$ is the string coordinate along the fifth holographic dimension which is only a function of the spatial boundary coordinate x . Because the string world-sheet is symmetric under the mirror transformation $x \leftrightarrow -x$, $U(x)$ presents a minimum which then occurs at $x = 0$. By definition, $U_0 \equiv U(0)$ (and $U'(0) = 0$). U_T is the value of the string coordinate at the event horizon while U_{max} is the *ultraviolet* cutoff ($U \leq U_{max}$). Let us focus on the limiting case $U_0 \gg U_T$ ($\varepsilon \simeq 1$) where the string world-sheet is close to the boundary such that it does not feel the presence of the horizon. In fact, this configuration corresponds to the low temperature limit $rT \ll 1$. Obviously, for small temperatures, the potential behaves approximately as in the zero temperature case $V \sim -\frac{1}{r}$, Eq.(1.3). Moreover, the leading non-zero temperature correction exhibits scaling consistent with the conformal invariance of the boundary theory [6]: $V \propto -\frac{1}{r} \left(1 + a(rT)^4 \right)$ with a a positive numerical constant which does not depend on \tilde{R} . Without length scale, it is indeed meaningless to speak, at low temperature, of a large or small compactification radius of the Euclidean temporal dimension (whose the period gives the temperature). The high temperature limit $rT \gg 1$ when $U_0 \simeq U_T$ is more subtle. As shown in [6], there is a critical value of the inter-quark distance above which the potential starts to be positive. At this point, the bound-state potential (2.2) is not valid because the lowest energy configuration consists instead of two straight

² g_s is the closed string coupling constant. Using a standard convention in string theory, the world-sheet coordinates will be written with capital letters.

strings ending at the horizon. In other words, the quarks become free as screened by the effects of the temperature. Hence, the potential exhibits behavior expected for the deconfinement phase at high temperature when the meson decays into a configuration of quarks without interaction.

As a matter of fact, linear potentials can also appear in supergravity. For this, we will consider *spatial* Wilson loops (*i.e.* along two spacelike dimensions) at *fixed value* of the temperature [5][7]. Then, the heavy quark potential and the interquark distance as functions of U_0 and U_T are [7]:

$$r = \frac{2\tilde{R}^2}{U_0} \int_1^\infty \frac{dv}{\sqrt{(v^4 - 1 + \varepsilon)(v^4 - 1)}}, \quad (2.3)$$

$$V_{Q\bar{Q}} = \frac{U_0^2}{2\pi\tilde{R}^2} r + \frac{U_0}{\pi} \int_1^\infty dv \left(\sqrt{\frac{v^4 - 1}{v^4 - 1 + \varepsilon}} - 1 \right) + \frac{U_T - U_0}{\pi}. \quad (2.4)$$

In the limit $U_0 \simeq U_T$ ($\varepsilon \ll 1$) where the string world-sheet reaches the horizon, the interquark distance diverge, which thus corresponds to the large distance limit. On the other hand, when $rT \gg 1$, the circle $S^1(1/T)$ around the compactified Euclidean time direction is small and, as a result, the number of dimensions of the $4d$ gauge theory on the boundary reduces to three. By choosing appropriate boundary conditions along this circle (namely, by taking antiperiodic fermions around $S^1(1/T)$ in contrast to the periodic bosons), the supersymmetry can also be broken [5]. Moreover, as both fermions and scalars get masses related to the temperature (due to renormalization for the latter), they decouple at high enough temperature and the theory reduces to a pure non-conformal gauge theory. We are thus considering, at large distances, $3d$ non-supersymmetric Yang-Mills theory at zero temperature. On the contrary, at small distances $rT \ll 1$, the compactification radius of the circle is sizeable. We deal therefore with $4d$ supersymmetric Yang-Mills theory at zero temperature and, not surprisingly, we recover Maldacena's result (1.3). An expansion in power series of ε gives the leading and subleading terms in the static potential at large quark separation:

$$V_{Q\bar{Q}} = \frac{U_T^2}{2\pi\tilde{R}^2} r \left(1 - \frac{1}{2} e^{-\frac{2U_T}{\tilde{R}^2} r} \right). \quad (2.5)$$

As expected, the string tension is proportional to (the square of) the temperature since it is our only dimensionful parameter at hand:

$$\sigma_T = \frac{U_T^2}{2\pi\tilde{R}^2} = \frac{1}{2} \pi \tilde{R}^2 T^2 = \sqrt{\pi^3 g_s} N T^2. \quad (2.6)$$

The subleading term in the potential (2.5) is not the Lüscher term in $-1/r$ as predicted by effective string models and Lattice QCD. The result (2.5) is not so surprising in fact since the limits at work in the supergravity approach are the large N and the large 't Hooft coupling constant limits and it is known that there is no Lüscher term in the strong coupling regime on the lattice. In this respect, Ref.[7] argued that the Lüscher term could arise from quantum fluctuations of the classical world-sheet approximation. Let us also mention that the linear behavior of the potential is not spoilt by the leading stringy corrections $O(\alpha'^3)$ of the metric (2.1). The expressions (2.3) and (2.4) of r and $V_{Q\bar{Q}}$ are modified only by terms in $1/v$ which do not rule out their singular behaviors in the limit $U_0 \simeq U_T$.

The Schwarzschild black hole– AdS_5 geometry described above was required in order to deal with a $3d$ gauge theory (after compactification of the Euclidean time direction). If we are interested in studying higher-dimensional gauge theories, it is then necessary to consider the general case of a stack of N coincident D p -branes in the decoupling limit. We are therefore led to the (Euclidean) metric [8]:

$$ds^2 = \alpha' \left\{ \frac{u^{\frac{7-p}{2}}}{g_{YM}^{(p+1)} \sqrt{d_p N}} (dt^2 + d\vec{x}^2) + \frac{g_{YM}^{(p+1)} \sqrt{d_p N}}{u^{\frac{7-p}{2}}} du^2 + g_{YM}^{(p+1)} \sqrt{d_p N} u^{\frac{(p-3)}{2}} d\Omega_{8-p}^2 \right\} \quad (2.7)$$

with $d_p \equiv 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma(\frac{7-p}{2})$ and where the coupling constant $g^{(p+1)}$ of the $(p+1)$ -dimensional $SU(N)$ supersymmetric Yang-Mills theory is related to g_s as $g^{(p+1)^2} = (2\pi)^{p-2} g_s \alpha'^{\frac{(p-3)}{2}}$. The case of interest here consists of $p = 4$, for which $g^{(5)^2} = 4\pi^2 g_s \sqrt{\alpha'}$, and non-zero temperature. The solution of the equations of motion for the stack of N coincident D p -branes in the decoupling limit is then:

$$ds_{BH}^2 = \alpha' \left[\frac{u^{3/2}}{R_4^{3/2}} (g(u) dt^2 + d\vec{x}^2) + \frac{R_4^{3/2}}{u^{3/2}} \frac{du^2}{g(u)} + R_4^{3/2} \sqrt{u} d\Omega_4^2 \right] \quad (2.8)$$

where $g(u) = 1 - \frac{u_T^3}{u^3}$ and $u_T = \frac{16}{9} \pi^2 R_4^3 T^2 = \frac{4}{9} \pi g^{(5)^2} N T^2$ ³. In this case, the interquark distance and the static potential read as follows ($\varepsilon \equiv g(U_0)$):

$$r = \frac{2R_4^{3/2}}{U_0^{1/2}} \int_1^\infty \frac{dv}{\sqrt{(v^3 - 1 + \varepsilon)(v^3 - 1)}} \quad (2.9)$$

and

$$V_{Q\bar{Q}} = \frac{U_0^{3/2}}{2\pi R_4^{3/2}} r + \frac{U_0}{\pi} \int_1^\infty dv \left(\sqrt{\frac{v^3 - 1}{v^3 - 1 + \varepsilon}} - 1 \right) + \frac{U_T - U_0}{\pi}. \quad (2.10)$$

Here also, one can show that the potential presents an area law behavior (when $U_0 \simeq U_T$) with a string tension:

$$\sigma_t = \frac{U_T^{3/2}}{2\pi R_4^{3/2}} = \frac{8}{27} \pi g^{(4)^2} N T^2 \quad (2.11)$$

expressed in terms of the $4d$ gauge theory coupling $g^{(4)^2} = g^{(5)^2} T$. To summarize, we observe an area law for *spatial* Wilson loops in $4d$ and $5d$ supersymmetric Yang-Mills theories at finite temperature. This is then interpreted as the area law of *spacetime* Wilson loops (after having identified one of the spatial coordinates of the higher-dimensional theory to the non-compactified Euclidean time) in $3d$ and $4d$ non-supersymmetric Yang-Mills theories at zero temperature.

³We have defined $R_4^{3/2} \equiv g^{(5)} \sqrt{d_4 N} = g^{(5)} \sqrt{\frac{N}{4\pi}}$ such that R_4 has the dimension of a *length*^{1/3} ($g^{(5)}$ has the dimension of a *length*^{1/2}).

3. The supergravity description of baryons

In a $SU(N)$ Yang-Mills theory, a color-singlet baryon must be made of N quarks. As described in the supergravity dual, such a baryon consists of N quarks living on the boundary. On each of these quarks ends a string whose the other endpoint is attached to a D5-brane wrapped around S^5 : the so-called *baryon vertex* located at the holographic coordinate u_0 [9]. The typical radius of the baryon is denoted r . Moreover, the configuration of the N quarks on the boundary is symmetric with respect to the boundary dimensions such that the resulting force acting on the *baryon vertex* is zero along these directions. In the following, we will consider only the induced metric contribution in the Dirac-Born-Infeld action of the D5-brane [10]⁴:

$$S_{D5} = T_5 \int d^6x \sqrt{\det g_{D5}} = \frac{TN U_0}{8\pi}. \quad (3.1)$$

The total action of the baryonic system is thus:

$$\begin{aligned} S_{total} &= S_{D5} + \sum_{i=1}^N S_{string}^{(i)} \\ &= \frac{TN U_0}{8\pi} + \frac{TN}{2\pi} \int_0^r dx \sqrt{U'^2 + \frac{U^4}{R^4}} \end{aligned} \quad (3.2)$$

where the integral over the boundary spatial coordinate x runs from 0 to the typical radius r of the baryon. The stability (or no-force) condition for the *baryon vertex* along the holographic coordinate stems from variational principle and reads:

$$\delta S_{total} \Big|_{\text{term at } U_0} = 0 \Rightarrow \frac{U'_0}{\sqrt{U_0'^2 + \frac{U_0^4}{R^4}}} = \frac{1}{4}. \quad (3.3)$$

On the other hand, Maldacena's recipe gives the energy of the baryon:

$$V_B(r) = -N\alpha_B \frac{\sqrt{2\lambda}}{r} \quad (3.4)$$

with a coefficient given by

$$\alpha_B = \frac{1}{2\pi} \int_1^\infty \frac{du}{u^2 \sqrt{\beta^2 u^4 - 1}} \left\{ \frac{3}{4} - \int_1^\infty dv \left[\frac{\beta v^2}{\sqrt{\beta^2 v^4 - 1}} - 1 \right] \right\} \quad (3.5)$$

of numerical value $\alpha_B \simeq 0.036$ ($\beta = \sqrt{\frac{16}{15}}$) and whose the behavior in $1/r$ is dictated by the conformal invariance of the field theory on the boundary.

Remarkably, another string configuration has been identified [10] which allows, on the supergravity side, to account for baryons made of a smaller number of quark constituents $k < N$. In such a configuration, k strings attached to the *baryon vertex* end on the k quarks at the boundary while

⁴In general, Dp-branes carry electromagnetic fields on their $(p+1)$ -dimensional world-volume whose the dynamics is governed by the so-called Dirac-Born-Infeld action: $S_{Dp} = T_p \int d^{p+1}x \sqrt{-\det(\eta_{MN} + 2\pi\alpha' F_{MN})}$ with $T_p = \frac{2\pi}{(2\pi l_s)^{p+1} g_s}$ the brane tension and $M, N = 0, 1, \dots, p$ the spacetime indices on the (flat) world-volume of the Dp-brane. Especially, $T_5^{-1} = (2\pi)^5 \alpha'^3 g_s$.

the $N - k$ remaining strings stretch out up to the brane at $u = 0$. The total action governing the dynamics of the baryon is then:

$$\begin{aligned} S_{total} &= S_{D5} + \sum_{i=1}^k S_{string}^{(i)} + \sum_{j=1}^{N-k} S_{string}^{(j)} \\ &= \frac{TN U_0}{8\pi} + \frac{kT}{2\pi} \int_0^r dx \sqrt{U'^2 + \frac{U^4}{\tilde{R}^4}} + \frac{T(N-k)U_0}{2\pi} \end{aligned} \quad (3.6)$$

with the following stability condition for the *baryon vertex* along the holographic coordinate:

$$\delta S_{total} \Big|_{\text{surface term at } U_0} = 0 \Rightarrow \frac{U'_0}{\sqrt{U_0'^2 + \frac{U_0^4}{\tilde{R}^4}}} = \frac{5N - 4k}{4k} \equiv A. \quad (3.7)$$

If $k = N$, then $A = \frac{1}{4}$ and we recover (3.3). The upper bound for A (which corresponds to the lower bound for k) is obtained for radial straight k -type strings ending on the *baryon vertex* such that $U'_0 \rightarrow \infty$. Then, $A = 1$ or $k = \frac{5N}{8}$. Finally, the condition for having a stable string-brane system into the bulk demands $\frac{5N}{8} \leq k \leq N$. As for the potential, if $k = \frac{5N}{8}$ then $V_B(U_0) = 0$ independently of the location U_0 of the *baryon vertex* along the holographic coordinate (actually, the size $r(U_0)$ of the baryon vanishes). If $\frac{5N}{8} < k \leq N$ then the energy $V_B(r) = -\alpha U_0(r)$ can be written as the product of a negative constant $-\alpha$ with U_0 expressed in terms of the baryon radius r [10].

To conclude this section, let us consider a *spatial* string/brane configuration in the Schwarzschild black hole– AdS_5 background (2.1). Following the same techniques as described above, the total action as well as the stability condition of the system can be derived. We are interested in the large distance regime where the typical radius of the baryons is large and where the D5-brane reaches the horizon ($U_0 \simeq U_T$). A linear potential then arises with a string tension equals to N times the mesonic string tension (2.6):

$$V_B(r) = N \left(\frac{1}{2} \pi \tilde{R}^2 T^2 \right) r. \quad (3.8)$$

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References

- [1] F. Jugeau, *Annals Phys.* **325** (2010) 1739-1789 [arXiv:hep-ph/0812.4903].
- [2] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [arXiv:hep-th/9711200]; E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253 [arXiv:hep-th/9802150]; S. S. Gubser, I. R. Klebanov, A. M. Polyakov, *Phys. Lett.* **B428** (1998) 105 [arXiv:hep-th/9802109].
- [3] J. M. Maldacena, *Phys. Rev. Lett.* **80** (1998) 4859 [arXiv:hep-th/9803002].
- [4] K. G. Wilson, *Phys. Rev.* **D10** (1974) 2445.
- [5] E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 505 [arXiv:hep-th/9803131].

- [6] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Yankielowicz, Phys. Lett. **B434** (1998) 36 [arXiv:hep-th/9803137]; S.-J. Rey, S. Theisen, J.-T. Yee, Nucl. Phys. **B527** (1998) 171 [arXiv:hep-th/9803135].
- [7] J. Greensite, P. Olesen, JHEP **9808** (1998) 009 [arXiv:hep-th/9806235], JHEP **9904** (1999) 001 [arXiv:hep-th/9901057]; A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Yankielowicz, JHEP **9806** (1998) 001 [arXiv:hep-th/9803263].
- [8] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, S. Yankielowicz, Phys. Rev. **D58** (1998) 046004 [arXiv:hep-th/9802042].
- [9] D. J. Gross, H. Ooguri, Phys. Rev. **D58** (1998) 106002 [arXiv:hep-th/9805129]; E. Witten, JHEP **9807** (1998) 006 [arXiv:hep-th/9805112].
- [10] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Yankielowicz, JHEP **9807** (1998) 020 [arXiv:hep-th/9806158].