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Anomaly-induced effective action of gravity and stability of classical solutions

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Our goal in this work is to derive the equation for gravitational waves in gravity theory with anomaly-induced quantum corrections on the general homogeneous and isotropic background. After this we verify the stability of such background with respect to metric perturbations. This problem has several interesting applications. One is to explore the stability of the classical cosmological solutions in the theory with quantum effects. For this, we analyze the behaviour of metric perturbations during inflationary period, in the stable phase of the Modified Starobinsky inflation.

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1. Introduction

The semiclassical approach to gravity is commonly associated with the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = < T_{\mu\nu} > \tag{1.1}$$

i.e., means that the gravity itself is not quantized. The full action includes Einstein-Hilbert term S_{EH}, which is the origin of the *r.h.s.* of (1.1) with the cosmological constant term and also the higher derivative terms

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1C^2 + a_2E + a_3\Box R + a_4R^2\}\,, \tag{1.2}$$

where $C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3)\,R^2$ is the square of the Weyl tensor and $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$ is the integrand of the Gauss-Bonnet topological term.

We have the following action of vacuum (with all terms)

$$S_{vac} = S_{EH} + S_{HD}\,, \tag{1.3}$$

belong to the gravitational action.

2. Effective action induced by anomaly

The anomalous trace of the energy momentum tensor is written in the following form [1, 2],

$$< T^{\mu}_{\mu} > = -\,(wC^2 + bE + c\Box R)\,, \tag{2.1}$$

where the coefficients w, b and c depend of the number of active quantum fields of different spins (see [3]),

$$w = \frac{1}{(4\pi)^2}\left(\frac{N_0}{120} + \frac{N_{1/2}}{20} + \frac{N_1}{10}\right), \tag{2.2}$$

$$b = -\,\frac{1}{(4\pi)^2}\left(\frac{N_0}{360} + \frac{11\,N_{1/2}}{360} + \frac{31\,N_1}{180}\right), \tag{2.3}$$

$$c = \frac{1}{(4\pi)^2}\left(\frac{N_0}{180} + \frac{N_{1/2}}{30} - \frac{N_1}{10}\right). \tag{2.4}$$

It is easy to see that these coefficients are nothing else but the β-functions for the parameters $a_{1,2,3}$ in the classical action of vacuum (1.3). Due to the decoupling phenomenon [4], the number of active fields can vary from one epoch in the history of the Universe to another. As we have already mentioned in the Introduction, the present-day Universe corresponds to the particle content with $N_0 = N_{1/2} = 0$ and $N_1 = 1$.

The anomaly-induced effective action $\bar{\Gamma}_{ind}$ represents an addition to the classical action of gravity, and can be found by solving the equation

$$\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta\,\bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = < T^{\mu}_{\mu} > = (\omega C^2 + bE + c\Box R)\,. \tag{2.5}$$

The covariant and generally non-local solution can be easily found in the form

$$\bar{\Gamma} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36}\int d^4x\sqrt{-g(x)}\,R^2(x) + \int d^4x\sqrt{-g(x)}\int d^4y\sqrt{-g(y)}\,(E-\frac{2}{3}\Box R)_x\,G(x,y)\left[\frac{w}{4}C^2-\frac{b}{8}(E-\frac{2}{3}\Box R)\right]_y, \tag{2.6}$$

where $G(x,y)$ is a Green function for the operator

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu\,.$$

Finally, one can rewrite (2.6) in the local form by introducing two auxiliary fields ϕ and ψ [5] (see also [6] for an alternative albeit equivalent scheme),

$$\bar{\Gamma}_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36}\int d^4x\sqrt{-g(x)}\,R^2(x) + \int d^4x\sqrt{-g(x)}\Big\{\frac{1}{2}\varphi\Delta_4\varphi - \frac{1}{2}\psi\Delta_4\psi + \varphi\left[\frac{\sqrt{-b}}{2}(E-\frac{2}{3}\Box R) - \frac{w}{2\sqrt{-b}}C^2\right] + \frac{w}{2\sqrt{-b}}\psi C^2\Big\}. \tag{2.7}$$

The expression (2.7) is classically equivalent to (2.6), because if one uses the equations for the auxiliary fields φ and ψ, the nonlocal action (2.6) is restored. Consider now the background cosmological solution for the theory with the action including quantum corrections,

$$S_{total} = -M_P^2\int d^4x\sqrt{-g}\,R + \bar{\Gamma}\,, \tag{2.8}$$

where $M_P^2 = 1/16\pi G$ is the square of the Planck mass, and the quantum correction $\bar{\Gamma}$ is taken in the form (2.7).

3. Gravitational waves

Let us derive the equation for the tensor modes of metric perturbations. First we rewrite the action in a more appropriate way and then derive linear perturbations for the tensor mode.

It proves useful to present the action (2.8) of a more useful way. After performing some integrations by parts, it can be cast into the form

$$S = \int d^4x\sqrt{-g}\left[f_0 R + f_1 R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} + f_2 R^{\alpha\beta}R_{\alpha\beta} + f_3 R^2 + f_4\,\varphi\Box R + f_5\,\varphi\Delta\varphi\right], \tag{3.1}$$

where the f-terms are defined as

$$f_0 = -\frac{{M_P}^2}{16\pi};\quad f_1 = a_1 + a_2 - \frac{b+\omega}{2\sqrt{-b}}\varphi + \frac{\omega}{2\sqrt{-b}}\psi;$$
$$f_2 = -2a_1 - 4a_2 + \frac{\omega+2b}{\sqrt{-b}}\varphi - \frac{\omega}{\sqrt{-b}}\psi;\quad f_3 = \frac{a_1}{3} + a_2 - \frac{3c+2b}{36} - \frac{3b+\omega}{6\sqrt{-b}}\varphi + \frac{\omega}{6\sqrt{-b}}\psi;$$
$$f_4 = -\frac{4\pi\sqrt{-b}}{3};\quad f_5 = \frac{1}{2},$$

and the coefficients $a_{1,2}$ are the same defined in (1.3).

3.1 Perturbation equations

Using the conditions (with $\mu = 0,1,2,3$ and $i = 1,2,3$),

$$\partial_i h^{ij} = 0 \text{ and } h_{kk} = 0\,, \tag{3.2}$$

together with the synchronous coordinate condition $h_{\mu 0} = 0$, we introduce metric perturbation in the action above. After all, the equation for tensor mode can be cast into the form

$$\begin{aligned}
&\Big(2f_1+\frac{f_2}{2}\Big)\ \ddddot{h}+\Big[3H\Big(4f_1+f_2\Big)+4\dot{f}_1+\dot{f}_2\Big]\ \dddot{h}+\Big[3H^2\Big(6f_1+\frac{f_2}{2}-4f_3\Big)\\
&+\ H\Big(16\dot{f}_1+\frac{9}{2}\dot{f}_2\Big)+6\dot{H}\Big(f_1-f_3\Big)+2\ddot{f}_1+\frac{1}{2}(\ddot{f}_2+f_0+f_4\ddot{\varphi})+\frac{3}{2}f_4H\dot{\varphi}-\frac{2}{3}f_5\dot{\varphi}^2\Big]\ddot{h}\\
&-\ \Big(4f_1+f_2\Big)\frac{\nabla^2\ddot{h}}{a^2}+\Big[\dot{H}\Big(4\dot{f}_1-6\dot{f}_3\Big)-21H\dot{H}\Big(\frac{1}{2}f_2+2f_3\Big)-\ddot{H}\Big(\frac{3}{2}f_2+6f_3\Big)\\
&+\ 3H^2\Big(4\dot{f}_1+\frac{1}{2}\dot{f}_2-4\dot{f}_3\Big)-9H^3(f_2+4f_3)+H\Big(4\ddot{f}_1+\frac{3}{2}\ddot{f}_2\Big)+\frac{3}{2}f_4\dot{\varphi}\Big(3H^2+\dot{H}\Big)\\
&+\ H(3f_4\ddot{\varphi}+\frac{3}{2}f_0-2f_5\dot{\varphi}^2)+\frac{1}{2}f_4\ \dddot{\varphi}\ -\frac{4}{3}f_5\dot{\varphi}\ddot{\varphi}\Big]\dot{h}-\Big[H\Big(4f_1+f_2\Big)+4\dot{f}_1+\dot{f}_2\Big]\frac{\nabla^2\dot{h}}{a^2}\\
&+\ \Big[5f_4H\ \dddot{\varphi}\ +f_4\ \ddddot{\varphi}\ -(36\dot{H}H^2+18\dot{H}^2+24H\ddot{H}+4\ \dddot{H}\ )\Big(f_1+f_2+3f_3\Big)\\
&-\ H\dot{H}\Big(32\dot{f}_1+36\dot{f}_2+120\dot{f}_3\Big)-8\ddot{H}\Big(\dot{f}_1+\dot{f}_2+3\dot{f}_3\Big)-H^2\Big(4\ddot{f}_1+6\ddot{f}_2+24\ddot{f}_3\Big)\\
&-\ 4\dot{H}\Big(\ddot{f}_1+\ddot{f}_2+3\ddot{f}_3\Big)-9f_4\dot{\varphi}\Big(H^3+H\dot{H}\Big)+f_4\ddot{\varphi}\Big(3H^2+5\dot{H}\Big)-H^3\Big(8\dot{f}_1+12\dot{f}_2+48\dot{f}_3\Big)\\
&+\ f_5\dot{\varphi}^2\Big(\frac{1}{2}H^2+\frac{1}{3}\dot{H}\Big)+\frac{2}{3}f_5H\dot{\varphi}\ddot{\varphi}-\frac{1}{6}f_5\ddot{\varphi}^2+\frac{1}{3}f_5\dot{\varphi}\ \dddot{\varphi}\ \Big]h+f_0\Big[2\dot{H}+3H^2\Big]h\\
&+\ \Big[H^2\Big(4f_1+4f_2+12f_3\Big)+H\Big(2\dot{f}_1+\frac{1}{2}\dot{f}_2\Big)+2\dot{H}\Big(f_1+f_2+3f_3\Big)\\
&-\ \frac{1}{2}(\ddot{f}_2+f_4\ddot{\varphi}+f_0+3f_4H\dot{\varphi})-\frac{1}{3}f_5\dot{\varphi}^2\Big]\frac{\nabla^2 h}{a^2}+\Big[2f_1+\frac{1}{2}f_2\Big]\frac{\nabla^4 h}{a^4}=0\,.
\end{aligned} \tag{3.3}$$

Finally, if we take $H = constant$ in (3.3), we found the result which perfectly fits the one of [7].

4. Numerical and Analytical analysis

By performing analytical (analyzing the signs of the eigenvalues) and numerical (through the software Mathematica [8]) analysis in Eq. (3.3) for the three cases of our interest, namely for exponential expansion, radiation and matter epochs, we found:

4.1 Exponential expansion

When we choose $a_1 < 0$ we found all eigenvalues real and negative. So, we have stability in this case, exactly as we could expect from comparison to the inflationary case [9].

However, if we choose $a_1 > 0$, we found three negative and one positive eigenvalue. So, we can observe the instability in this case.

Let us remark that the sign of a_1 defines whether the massless tensor mode in the classical theory is a graviton or a ghost [10]. From this perspective our result means that the stability property of the theory with higher derivative classical term (1.2) and quantum correction (2.6) is completely defined by classical part (1.2) and, quite unexpectedly, does not depend on the quantum term (2.6). Is it a general feature or just a peculiarity of the de Sitter background solution? Let us consider other cases to figure this out.

4.2 Radiation

When we choose $a_1 < 0$ we found two real eigenvalues, which are both negative and also two complex eigenvalues with negative real parts. So, we have stability in this case. But if we take $a_1 > 0$ we found two negative eigenvalues and two positive ones. So, we have instability in this case. Again, the stability of the classical solution is completely dependent on the classical term (1.2).

4.3 Matter

With $a_1 < 0$ we found two real negative eigenvalues and also two complex eigenvalues with negative real parts. So, we have the stability for $a_1 < 0$. However, if we choose $a_1 > 0$, we have instability again.

5. Conclusion

As a result of our consideration we can conclude that there is a stability for Eq. (3.3), if and only if a_1 is negative. Taking into account the mentioned feature of classical higher derivative gravity, we see that the linear (in)stability of tensor mode in the classical higher derivative theory (1.2) completely defines a linear (in)stability in the theory with quantum correction (2.6). The qualitative explanation for this output is quite clear. The quantum terms (2.6) consist of two types of terms. The simplest one is the local R^2-term, which contributes to the propagator of gravitational perturbations on flat background, but not to the one of the tensor mode (see, e.g., [11] for detailed explanations and original references). The more complicated non-local terms are at least third order in curvature, and hence do not contribute at all to the propagator of gravitational perturbations on flat background. Indeed, we are interested in the perturbations on curved cosmological background and not on the flat one. However, the typical length scale related to the expansion of the Universe is defined by the Hubble radius and are much greater than the length scale of the linear perturbations we are interested in here. Therefore, the stability of theory under such perturbations, in a given approximation, is the same as for the flat background and for the algebraic reasons explained above, there is no essential role of the anomaly-induced quantum terms (2.6) here.

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