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Branes, Weights and Wrapping Rules

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We show how the recent classification of half-supersymmetric branes of maximal supergravity has a simple group-theoretical characterization in terms of the longest weights of the T-duality representation to which the potentials that couple to these branes belong. We show that reducing the branes of ten-dimensional string theory leads to the half-supersymmetric branes in lower dimensions provided we impose simple wrapping rules for these branes. The origin and interpretation of these wrapping rules is discussed.

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1. Introduction

"Branes", i.e. massive objects with a number of worldvolume and transverse directions, play a crucial role in string theory and M-theory. Historically, the first example of a brane other than a string was the eleven-dimensional supermembrane [1]. An important class of branes are the Dirichlet branes or, shortly, D-branes of ten-dimensional superstring theory [2]. These branes are non-perturbative in the sense that their brane tension scales with the inverse of the string coupling constant. D-branes played a decisive role in the calculation of the entropy of a certain class of black holes [3]. Branes also play a central role in the AdS/CFT correspondence [4] and the brane-world scenario [5].

Much information about branes can be obtained by studying the low-energy approximation of string theory and/or M-theory which is a supergravity theory. For instance, the mere fact that eleven-dimensional supergravity contains a 3-form potential is already indicative of the fact that M-theory contains a membrane since 3-forms naturally couple to membranes. The fact that this membrane is actually a supermembrane which breaks half of the supersymmetry follows from the construction of a kappa-symmetric supermembrane action [1]. Kappa symmetry requires that the worldvolume action describing the dynamics of the brane contains a Nambu-Goto and a Wess-Zumino (WZ) term. The latter describes the coupling of the brane to the potentials of supergravity. A classification of branes therefore necessarily involves a classification of the supergravity potentials.

Due to their different nature it is important to distinguish between branes with more than 2 transverse directions and branes with 2 or less transverse directions. The half-supersymmetric branes with more than 2 transverse directions have been classified a long time ago. We will refer to them collectively as the "standard" branes. The classification of the remaining branes is more subtle and has only recently been obtained [6, 7, 8]. We will refer to them as the "non-standard" branes. We call the ones with 2, 1 and 0 transverse directions "defect-branes", "domain-walls" and "space-filling branes", respectively. To summarize:

$$\begin{aligned} \text{standard branes}: &\quad \text{more than 2 transverse directions}\,,\\ \text{non-standard branes}: &\quad \text{2,1 or 0 transverse directions}\,. \end{aligned}$$

One difference between the standard and non-standard branes is that the non-standard ones are not asymptotically flat. Furthermore, they are only well-defined if one considers multiple brane configurations together with an orientifold. For our purposes it will be enough to consider single brane configurations. Another difference is that the standard branes couple, via the WZ term, to potentials that describe continuous degrees of freedom. For the non-standard branes this is only the case for the defect branes which couple to the dual of the supergravity scalars. Even this case is different form the standard brane case in the sense that the number of dual potentials that fit into a U-duality representation is not equal to the number of physical scalars. From the higher-dimensional point of view the origin of this mismatch is the fact that some of the scalars originate from the higher-dimensional metric for which no dual metric can be defined. The potentials that couple to domain walls can be viewed as dual to a discrete degree of freedom such as a mass parameter or a gauge coupling constant. The space-filling branes are a bit special since they couple to potentials that do not describe any degree of freedom at all.

To verify whether a given potential couples to a half-supersymmetric brane or not we require that a gauge-invariant WZ term can be constructed. This often requires that, besides the embedding scalars, more world-volume potentials are introduced that transform under (some of) the gauge transformations of the supergravity potentials with non-trivial shifts. In this way gauge-invariance of the WZ term can always be achieved but it is not clear whether the newly introduced worldvolume potentials together with the embedding scalars fit into a worldvolume supermultiplet. This so-called "WZ-term requirement" imposes restrictions on the number of half-supersymmetric branes. Using the WZ-term requirement we have found that there is another difference between the standard and non-standard branes. Whereas for standard branes every supergravity potential (and its dual) couples to a half-supersymmetric brane, for the non-standard ones we find that there are less half-supersymmetric branes than there are potentials:

$$\text{\# half-susy standard branes} \;=\; \text{\# potentials}\,,$$

$$\text{\# half-susy non-standard branes} \;<\; \text{\# potentials}\,.$$

In the next section we will discuss in more detail the relation between branes and the WZ terms. We will review a so-called "light-cone rule" which provides a simple way, by using a light-cone basis for the T-duality indices, to specify which potentials couple to a half-supersymmetric brane and which do not. In section 3 we will show that the light-cone rule has a simple group-theoretical interpretation in terms of a "longest-weight rule" which states that the number of half-supersymmetric branes is equal to the number of longest weights of the T-duality representation to which the potentials in question belong. Having classified the half-supersymmetric branes of maximal supergravity it is natural to ask how the branes in different dimensions are related to each other via dimensional reduction. In section 4 we will show that reducing the branes of ten-dimensional string theory one obtains the half-supersymmetric branes in lower dimensions we just classified provided we impose simple wrapping rules for these branes. In the conclusions, see section 5, we will discuss the origin and interpretation of these wrapping rules.

2. Branes and Wess-Zumino terms

It is instructive to first consider the branes of Type IIB string theory. These branes can be analysed by looking at the field content of the low-energy IIB supergravity effective action. This includes not only the propagating fields and their magnetic duals - an $\mathrm{SL}(2,\mathbb{R})$ doublet of 2-forms, corresponding to the F1 fundamental string and the D1-brane, the magnetic dual 6-forms, corresponding to the D5-brane and the NS5-brane, and a selfdual 4-form, corresponding to the D3-branes - but also forms of higher rank. These forms can be obtained by imposing the closure of the supersymmetry algebra, and they are a triplet of 8-forms and a quadruplet of 10-forms [9]. In [10] the half-supersymmetric branes associated to these latter fields were derived by looking at the corresponding brane effective action. The result of this analysis is that only two components of the triplet of 8-forms and only two components of the quadruplet of 10-forms are associated to half-supersymmetric branes. A simple explanation of this result can be given by looking at the WZ term in the effective action. Denoting with $A_{8,\alpha\beta}$ the triplet of 8-forms ($\alpha,\beta = 1,2$ are $\mathrm{SL}(2,\mathbb{R})$ doublet indices), gauge invariance implies that such a term must be of the form

$$T^{\alpha\beta}[A_{8,\alpha\beta} + A_{6,(\alpha}\mathscr{F}_{2,\beta)} + \ldots] \quad , \tag{2.1}$$

where $T^{\alpha\beta}$ is the 7-brane charge, $A_{6,\alpha}$ are the doublet of 6-forms and $\mathscr{F}_{2,\alpha} = da_{1,\alpha} + A_{2,\alpha}$ are a doublet of world-volume 2-form field-strengths ($a_{1,\alpha}$ are a doublet of world-volume vectors and $A_{2,\alpha}$ are the 2-forms). In order for the effective action to preserve one-half of the supersymmetries, we must impose that the world-volume fields fit in an 8-dimensional 16-supercharge multiplet, that is a vector multiplet (one vector and two scalars). The two scalars are the transverse scalars, while the request that eq. (2.1) contains only one world-volume vector imposes that the charge must be either T^{11} or T^{22} (the third component T^{12} would result in a WZ term containing both components of the doublet of world-volume vectors). The same analysis leads to two 9-branes in the quadruplet of 10-forms. The main lesson of this analysis is that in the IIB theory the number of standard branes is the same as the number of corresponding potentials, while the number of non-standard branes is less than the number of components of the corresponding potentials.

We now move to consider maximally supersymmetric theories in any dimension. A full classification of the potentials of these theories for all dimensions was given in [11, 12] making use of the properties of the very extended Kac-Moody algebra E_{11} [13]. Starting from this result, the study of the half-supersymmetric branes as components of the U-duality representations of the corresponding potentials, based on the analysis of the WZ terms was initiated in [14]. This analysis, completed in [6, 8], shows that as in ten dimensions the number of half-supersymmetric non-standard branes is less than the dimension of the corresponding U-duality representations. Here we are interested in the analogous analysis in terms of representations of the T-duality group. Denoting with E_{11-D} the U-duality group in D dimensions, one has

$$\mathrm{E}_{11-D} \supset \mathrm{SO}(10-D, 10-D) \times \mathbb{R}^+ \quad , \tag{2.2}$$

where $\mathrm{SO}(10-D, 10-D)$ is the T-duality group, that will be denoted from now on as $\mathrm{SO}(d,d)$, with $d = 10-D$. Decomposing the U-duality representations under T-duality allows to classify the branes according to the way their tension T scales with respect to the D-dimensional string coupling,

$$\mathrm{T} \sim g_s^{\alpha} \quad , \tag{2.3}$$

where α is related to the $\mathbb{R}^+$ weight. The value of α is always non-positive, and $\alpha = 0$ corresponds to the fundamental branes, while the other branes, with $\alpha < 0$, are non-perturbative objects in string theory.

It turns out that the classification of the potentials associated to branes as representations of $\mathrm{SO}(d,d)$ is universal for $\alpha > -4$. The fields with $\alpha = 0$ are a 1-form $B_{1,A}$ in the vector representation of $\mathrm{SO}(d,d)$ and a 2-form singlet B_2. The RR fields, with $\alpha = -1$, belong to spinor representations with alternalting chirality, and we denote them with $C_{2n+1,a}$ and $C_{2n,\dot{a}}$. The fields with $\alpha = -2$ are D_{D-4}, $D_{D-3,A}$, D_{D-2,A_1A_2}, $D_{D-1,A_1A_3A_4}$ and $D_{D,A_1A_2A_3A_4}$, where sets of indices $A_1...A_n$ are always meant to be antisymmetrised. Finally, the fields with $\alpha = -3$ are $E_{D-2,\dot{a}}$, $E_{D-1,A\dot{a}}$ and $E_{D,A_1A_2\dot{a}}$, in irreducible tensor-spinor representations.

In [15] the $\alpha = -2$, i.e. solitonic branes were classified by looking at the world-volume field content of the WZ terms. The outcome of that analysis is that the components of the T-duality representations of the $\alpha = -2$ potentials that correspond to branes are obtained from the following "light-cone rule":

We introduce light-like indices $i\pm$, $i = 1,...,d$ for $\mathrm{SO}(d,d)$. The $\alpha = -2$ fields are then denoted as $D_{D-4+n,i_1\pm\dots i_n\pm}$, with $n = 0,1,...,4$. The components associated to half-supersymmetric branes are those for which the i's are all different. The number of $(D-5+n)$-branes is therefore

$$\binom{d}{n} \times 2^n \quad , \tag{2.4}$$

which is smaller than the dimension of the representation, which is $\binom{2d}{n}$.

As can be deduced from eq. (2.4), there are no solitonic branes with world-volume dimension higher than 6, because they correspond to fields with $n > d$, for which eq. (2.4) clearly gives a vanishing result. The case $n = d$, which can only occur in $D \geq 6$ and always corresponds to a 5-brane, is special because the T-duality representation with d antisymmetric indices of $\mathrm{SO}(d,d)$ splits into a selfdual and an anti-selfdual part. Correspondingly, the 2^d branes that come from eq. (2.4) split into 2^{d-1} branes supporting a vector multiplet and 2^{d-1} branes supporting a tensor multiplet. In all the other cases the branes support a world-volume vector multiplet.

A similar analysis was applied in [16] for the $\alpha = -3$ branes. In this case, as we already mentioned, the fields belong to tensor-spinor representations. The number of branes within each representation is obtained supplementing the light-cone rule above with the following rule:

For each lightlike vector index, half of the spinor components are projected out. For the $D-2$-form potential $E_{D-2,\dot{a}}$, one has

$$(D-3)-\text{branes}: \quad 2^{d-1} \quad , \tag{2.5}$$

which is equal to the dimension of the corresponding representation. For the $D-1$-form potential $E_{D-1,i\pm\dot{a}}$, one obtains

$$(D-2)-\text{branes}: \quad 2d \times 2^{d-2} \quad , \tag{2.6}$$

which is less than the dimension of the representation, $(2d-1)\times 2^{d-1}$. Finally, the D-form potentials $E_{D,i_1\pm i_2\pm\dot{a}}$ lead to

$$(D-1)-\text{branes}: \quad \binom{d}{2} \times 4 \times 2^{d-3} \quad , \tag{2.7}$$

which is less than the dimension of the representation, $d(2d-3)\times 2^{d-1}$.

The peculiarity of the branes with $\alpha > -4$ is that for each T-duality representation there is always at least one brane that comes from torus dimensional reduction from the 10-dimensional branes (either wrapped or unwrapped along some of the internal directions). There is also an $\alpha = -4$ brane in Type IIB string theory, namely the S-dual of the D9-brane. In D dimensions, this brane wraps along the T^d internal torus to give a spacefilling $(D-1)$-brane. The potential associated to this brane is the field $F^{+}_{D,A_1\dots A_d}$ in the selfdual representation of T-duality with d antisymmetric indices. According to the light-cone rule, the number of branes associated to this potential is given by eq. (2.4) with $n = d$, divided by two because of the selfduality condition. This gives 2^{d-1} branes, which is clearly less than the dimension of the representation, $\frac{1}{2}\binom{2d}{d}$.

Starting from $D = 7$, there are also half-supersymmetric branes with $\alpha \leq -4$ that belong to T-duality representations that do not contain any brane coming from 10 dimensions. These branes will not be discussed here, but their number was obtained by the WZ term analysis in [6, 8].

Before we proceed, it is important to mention that all the results discussed so far have been also obtained in [7] using a different method, namely by counting the real roots of the E_{11} Kac-Moody algebra. In the next section we will show that the light-cone rule discussed above can be replaced by an alternative group-theoretical rule, which we will denominate the "longest-weight rule". This new rule also reveals why the WZ requirement and the E_{11} method give the same result.

3. Branes and weights

The counting of branes that results from the analysis of the WZ terms in [6, 8] has a simple group-theoretical explanation: the components of the U-duality representations of the potentials that correspond to half-supersymmetric branes are those associated to the longest weights [17]. The potentials corresponding to standard branes belong to representations whose weights have all the same length, and this explains why in that case the number of half-supersymmetric branes coincides with the dimension of the representation. On the other hand, the potentials corresponding to non-standard branes belong to representations whose weights have different lengths, and therefore in this case the number of branes is less than the dimension of the representation. As an example one can consider the defect branes, associated to the $(D-2)$-form potentials, which belong to the adjoint representation. The number of such branes is equal to the dimension of the group minus the rank [18], which is the number of roots. Given that the symmetry groups of maximal supergravities are always simply laced, which means that all the roots have the same length, this implies that the roots are the longest weights of the adjoint (the other weights being the Cartan, which have zero length). The longest weights of the U-duality representation precisely correspond to the real roots of E_{11}, and therefore the observation that the branes correspond to the longest weights explains why the WZ analysis of [6, 8] and the E_{11} analysis of [7] give the same result.

We now want to give a characterisation of the length of the various weights within a representation in terms of the so-called "dominant weights". An irreducible representation is denoted in terms of the Dynkin labels of the highest weight. We recall that a weight is defined as the eigenvalue of the Cartan generators in a given representation, and the corresponding eigenvector is called a weight vector. A highest-weight vector is a weight vector annihilated by all the positive-root generators, and the non-zero (i.e. positive) Dynkin labels identify the negative-root generators that do not annihilate the highest-weight vector. As an example we can consider the group $SL(3,\mathbb{R})$, with simple roots α_1 and α_2. We first consider the fundamental representation, which is the **3**, whose Dynkin labels of the highest weight $W^{\mathbf{3}}$ are $\boxed{1\ 0}$. From this we read the weight $W^{\mathbf{3}}-\alpha_1$, with Dynkin labels $\boxed{-1\ 1}$. The lowest weight of the representation is $W^{\mathbf{3}}-\alpha_1-\alpha_2$, with Dynkin labels $\boxed{0\ -1}$. The reader can see that only the highest weight of the **3** has Dynkin labels that are all non-negative. In general one defines a dominant weight as a weight whose Dynkin labels are all non-negative. Clearly a highest weight is a dominant weight, but the opposite is not necessarily true. As we have seen, the **3** of $SL(3,\mathbb{R})$ has only one dominant weight, which is the highest weight. However, this is no longer the case if we consider instead the symmetric product $\mathbf{3}\otimes_S\mathbf{3}$, which is the **6**. This representation has highest weight $\boxed{2\ 0}$, but it also contains the weight $\boxed{0\ 1}$, which is a dominant weight.

In general, each dominant weight in a representation identifies a set of weights which all have the same length as that dominant weight. We can consider again $SL(3,\mathbb{R})$ as an example. In the case of the **3**, all the weights have the same length, which is the length of the highest weight. In the case of the **6** instead, there are three long weights, one of them being the highest weight $\boxed{2\ 0}$, and three short weights, one of them being the second dominant weight $\boxed{0\ -1}$. This implies that the standard branes are associated to potentials belonging to representations that have only one dominant weight (which is the highest weight) while the non-standard branes are associated to potentials that are in representations with more than one dominant weight, and one can count for each representation the number of weights with the same length as each dominant weight. This was done in [17] for the U-duality representations associated to all the non-standard branes in any dimension. Here we want to apply the same analysis of [17] to the representations of the T-duality group $\mathrm{SO}(d,d)$ that are associated to branes. We will show that the longest-weight rule of [17] is the same as the light-cone rule reviewed in the previous section.

We first review what are the Dynkin labels of the highest weights of the representations of $\mathrm{SO}(d,d)$ that are relevant for our discussion. We are assuming that we are labelling the nodes of the Dynkin diagram of $\mathrm{SO}(d,d)$ in the standard way, with the last two nodes (node $d-1$ and node d) corresponding to the two spinor representations. The highest weight of the vector representation is $\boxed{1\ 0\ 0\ ...\ 0\ 0}$, while more generally the highest weight of the representation with n antisymmetric indices ($n < d-1$) has all zero Dynkin labels apart from the nth label, whose value is 1. The highest weight of the representation with $d-1$ antisymmetric indices is $\boxed{0\ 0\ 0\ ...0\ 1\ 1}$, and the ones with d antisymmetric indices are $\boxed{0\ 0\ 0\ ...0\ 2\ 0}$ (selfdual) and $\boxed{0\ 0\ 0\ ...0\ 0\ 2}$ (anti-selfdual). The spinor representations are $\boxed{0\ 0\ 0\ ...0\ 0\ 1}$ (chiral, denoted with the index a) and $\boxed{0\ 0\ 0\ ...0\ 1\ 0}$ (anti-chiral, denoted with the index $\dot{a}$).

We now discuss the various dominant weights of the T-duality representations associated to the half-supersymmetric branes discussed in the previous section for the different values of α, see eq. (2.3). The $\alpha = 0$, i.e. fundamental, branes correspond to the potentials $B_{1,A}$ (F0-branes) and B_2 (F1-brane). The vector representation clearly has only one dominant weight, which is the highest weight $\boxed{1\ 0\ 0\ ...\ 0\ 0}$. The F0-branes correspond to the lightlike directions $B_{1,i\pm}$, and their number is $2d$, which is equal to the dimension of the representation. There always is a single F1-brane (i.e. the fundamental string) associated to the T-duality singlet B_2. The $\alpha = -1$ branes, i.e. the Dp-branes, belong to the chiral (p even) and anti-chiral (p odd) representations. These representations have only one dominant weight (i.e. the highest weight) and therefore the number of branes is equal to the dimension of the representation, which is 2^{d-1}.

We then consider the $\alpha = -2$, i.e. solitonic, branes. The discussion for the $(D-5)$-branes and the $(D-4)$-branes is the same as for the fundamental branes, of which they are the magnetic dual. The $(D-3)$-branes correspond to the potentials D_{D-2,A_1A_2}. The dimension of the representation (which is the adjoint of $\mathrm{SO}(d,d)$) is $\binom{2d}{2}$. There are $\binom{d}{2} \times 4$ long weights (i.e. the roots) associated to the dominant weight $\boxed{0\ 1\ 0\ 0\ ...0\ 0\ 0}$, and d weights of zero length (the Cartan) which means that the dominant weight $\boxed{0\ 0\ 0\ ...0\ 0\ 0}$ has multiplicity d. In components, the long weights correspond to $D_{D-2,i_1\pm i_2\pm}$ with $i_1 \neq i_2$, and the short weights to $D_{D-2,i+i-}$, and given that i takes d values, this explains the degeneracy d of the short dominant weight.

The $(D-2)$-branes are associated to the potential $D_{D-1,A_1A_2A_3}$. In the representation of $\mathrm{SO}(d,d)$ with three antisymmetric indices, there are $\binom{d}{3} \times 8$ long weights, one of which being the highest weight $\boxed{0\ 0\ 1\ 0\ ...0\ 0\ 0}$, corresponding to the components $D_{D-1,i_1\pm i_2\pm i_3\pm}$ with i_1, i_2 and i_3 all different. The remaining components are $D_{D-1,i\pm j+j-}$, with $i \neq j$. These are associated to the dominant weight $\boxed{1\,0\,0\,0\,...0\,0\,0}$, which has multiplicity $(d-1)$ because there are $d-1$ possible values for j once i is fixed. The total number of short weights is $2d(d-1)$. Clearly, the sum of the long weights and the short weights is $\binom{2d}{3}$, which is the dimension of the representation.

The last type of solitonic branes are the $(D-1)$-branes, corresponding to the potentials $D_{D,A_1\ldots A_4}$. The representation has $\binom{d}{4}\times 16$ longest weights, which are of the same length as the highest weight $\boxed{0\,0\,0\,1\,...0\,0\,0}$. They correspond to the components $D_{D,i_1\pm i_2\pm i_3\pm i_4\pm}$, with the i's all different. The next-to-longest weights correspond to the components of the form $D_{D,i_1\pm i_2\pm j+j-}$, with i_1, i_2 and j all different. The corresponding dominant weight is $\boxed{0\,1\,0\,0\,...0\,0\,0}$, and its multiplicity is $(d-2)$ because these are the possible choices that can be made for j once i_1 and i_2 are fixed. The total number of next-to-longest weights is $\binom{d}{2}\times 4\times(d-2)$. Finally, the shortest weights correspond to the components $D_{D,i+i-j+j-}$, with $i\neq j$. They all correspond to the dominant weight $\boxed{0\,0\,0\,0\,...0\,0\,0}$, with multiplicity $\binom{d}{2}$ because these are all the possible choices for i and j. This gives a total of $\binom{d}{2}$ shortest weights. The sum of the longest, next-to-longest and shortest weights gives the dimension of the representation, which is $\binom{2d}{4}$.

The discussion above is valid if $D\leq 5$. In $D=6$ the spacefilling branes split into tensor and vector branes, corresponding to the representation with 4 antisymmetric indices of SO(4,4) splitting into selfdual and anti-selfdual. For each of the two irreducible representations the number of longest, next-to-longest and shortest weights is simply half of what one would get putting $d=4$ in the analysis above. In $D=7$ the representation of SO(3,3) with 4 antisymmetric indices is dualised to the one with two antisymmetric indices. This is consistent with the fact that if one puts $d=3$ in the formulas above for the longest weights of the representation with four antisymmetric indices one obtains zero, which implies that there are no solitonic spacefilling branes in $D=7$. Similar considerations apply in $D=8$ and $D=9$.

We now consider the $\alpha=-3$ fields. The fields $E_{D-2,\dot{a}}$ are in the anti-chiral spinor representation, which has only one dominant weight. The weights all have the same length and the number of branes is 2^{d-1}. The fields $E_{D-1,A\dot{a}}$ belong to the irreducible "gravitino" representation of dimension $(2d-1)\times 2^{d-1}$. The different dominant weights, their degeneracy and the number of weights of the same length are

$$\begin{aligned}&\boxed{1\,0\,0\,0\,...0\,1\,0} \;:\; d\times 2^{d-1}\,,\\ &\boxed{0\,0\,0\,0\,...0\,0\,1}\times(d-1) \;:\; (d-1)\times 2^{d-1}\quad .\end{aligned}\tag{3.1}$$

The sum of all the weights is clearly equal to the dimension of the representation. The fields $E_{D,A_1A-2\dot{a}}$ belong to the irreducible tensor-spinor representation of dimension $d(2d-3)\times 2^{d-1}$, and the different dominant weights, their degeneracy and the number of weights of the same length are

$$\begin{aligned}&\boxed{0\,1\,0\,0\,...0\,1\,0} \;:\; \binom{d}{2}\times 2^{d-1}\,,\\ &\boxed{1\,0\,0\,0\,...0\,0\,1}\times(d-2) \;:\; d(d-2)\times 2^{d-1}\,,\\ &\boxed{0\,0\,0\,0\,...0\,1\,0}\times\binom{d}{2} \;:\; \binom{d}{2}\times 2^{d-1}\quad .\end{aligned}\tag{3.2}$$

field	dim. of repr.	dominant weights	weights of same length
D_{D-2,A_1A_2}	$\binom{2d}{2}$	$\boxed{0\,1\,0\,0\,...\,0\,0\,0}$ $d\times\boxed{0\,0\,0\,0\,...0\,0\,0}$	$\binom{d}{2}\times 4$ d
$D_{D-1,A_1A_2A_3}$	$\binom{2d}{3}$	$\boxed{0\,0\,1\,0\,...\,0\,0\,0}$ $(d-1)\times\boxed{1\,0\,0\,0\,...0\,0\,0}$	$\binom{d}{3}\times 8$ $2d(d-1)$
$D_{D,A_1A_2A_3A_4}$	$\binom{2d}{4}$	$\boxed{0\,0\,0\,1\,...\,0\,0\,0}$ $(d-2)\times\boxed{0\,1\,0\,0\,...0\,0\,0}$ $\binom{d}{2}\times\boxed{0\,0\,0\,0\,...0\,0\,0}$	$\binom{d}{4}\times 16$ $(d-2)\binom{d}{2}\times 4$ $\binom{d}{2}$
$E_{D-1,A\dot{a}}$	$(2d-1)2^{d-1}$	$\boxed{1\,0\,0\,0\,...\,0\,1\,0}$ $(d-1)\times\boxed{0\,0\,0\,0\,...0\,0\,1}$	$d\times 2^{d-1}$ $(d-1)\times 2^{d-1}$
$E_{D,A_1A_2\dot{a}}$	$d(2d-3)2^{d-1}$	$\boxed{0\,1\,0\,0\,...\,0\,1\,0}$ $(d-2)\times\boxed{1\,0\,0\,0\,...0\,0\,1}$ $\binom{d}{2}\times\boxed{0\,0\,0\,0\,...0\,1\,0}$	$\binom{d}{2}\times 2^{d-1}$ $(d-2)d\times 2^{d-1}$ $\binom{d}{2}\times 2^{d-1}$

Table 1: This table gives the dominant weights and the number of weights of same length of each dominant weight for the $\alpha=-2$ and $\alpha=-3$ non-standard branes (apart from the $\alpha=-3$ defect branes whose analysis is straightforward because the corresponding representation has only one dominant weight). The highest weights and the number of weights of the same length, which is the number of half-supersymmetric branes, are painted in red.

For consistency, one can check that the sum of all the weights equals the dimension of the representation. The complete result of this analysis for non-standard $\alpha=-2$ and $\alpha=-3$ branes is summarised in Table 1.

To conclude this section, we consider the $\alpha=-4$ fields $F^{+}_{D,A_1\ldots A_d}$. As reviewed in the previous section, this representation contains the branes that come from the dimensional reduction of the S-dual of the D9-brane in IIB. The dimension of the representation is $\frac{1}{2}\binom{2d}{d}$. The longest weights correspond to the components $F^{+}_{D,i_1\pm i_2\pm \ldots . i_d\pm}$, with the i's all different. They are associated to the dominant (highest) weight $\boxed{0\,0\,0\,0\,...0\,2\,0}$ and their number is $\frac{1}{2}\times 2^d=2^{d-1}$ (the first factor $\frac{1}{2}$ comes from self-duality which implies that there is always an even number of minus signs). The next-to-longest weights correspond to components in which a pair of indices is of the form $j+j-$, that is $F^{+}_{D,i_1\pm i_2\pm\ldots . i_{d-2}\pm j+j-}$ with the i's and j all different. This corresponds to the dominant weight $\boxed{0\,0\,0\,0\,....1\,0\,0}$ with degeneracy 1 because there are two possible choices for j once all the i's are fixed, and a factor $\frac{1}{2}$ because of self-duality. This gives in total $\binom{d}{d-2}\times 2^{d-2}$ weights. Iterating this procedure, the kth-to-longest weights correspond to components in which k pairs of indices are of the form j_m+j_m- with $m=1,\ldots,k$ (and clearly j's all different). The fields are $F^{+}_{D,i_1\pm i_2\pm\ldots i_{d-2k}\pm j_1+j_1-\ldots j_k+j_k-}$. The dominant weight has Dynkin labels all zero apart from 1 corresponding to the node $d-2k$, and its degeneracy is $\frac{1}{2}\binom{2k}{k}$ where the factor $\frac{1}{2}$ is due to self-duality and the combinatorial factor is due to the fact that the number of j indices is equal to k and there are $2k$ possible choices once the i indices are fixed. The number of such weights is

$\frac{1}{2}\binom{2k}{k}\binom{d}{d-2k} \times 2^{d-2k}$. The shortest weights correspond to $[d/2]$ pairs of the form $j+j-$, with dominant weight $\boxed{0\,0\,0\,0\,...\,0\,0\,0}$ (d even) or $\boxed{1\,0\,0\,0\,...\,0\,0\,0}$ (d odd) of multiplicity $\frac{1}{2}\binom{2[d/2]}{[d/2]}$. The number of shortest weights is $\frac{1}{2}\binom{2[d/2]}{[d/2]}2^{d-2[d/2]}\binom{d}{d-2[d/2]}$. Summing all the weights, one arrives at the formula

$$\frac{1}{2}\binom{2d}{d} = \frac{1}{2}\sum_{k=0}^{[d/2]} 2^{d-2k}\binom{2k}{k}\binom{d}{d-2k}\,. \tag{3.3}$$

In this expression, the sum on the right-hand side is over each set of weights of the same length. The first term ($k=0$) corresponds to the longest weights, the $k=1$ term counts all the next-to-longest weights and so on.

4. Wrapping rules

Now that we know the numbers of half-supersymmetric branes, resulting from either the light-cone rule or the longest-weight rule, it is natural to ask oneself how all the branes in different dimensions are related to each other via dimensional reduction, if that can be done at all. Since the scaling of the brane tension T with respect to the D-dimensional dilaton g_s does not change under dimensional reduction it is natural to consider the reduction of branes whose tension has a given scaling. We are interested in studying the dimensional reduction of the ten-dimensional branes, whose tension scales like g_s^α with $\alpha = 0,-1,-2,-3,-4$. This means that in any dimension we are interested in the branes with these values of α. Explicitly, we refer to these branes as:

$$\begin{aligned}
\mathrm{T_F} &\sim 1: && \text{Fundamental branes}\,,\\
\mathrm{T_D} &\sim 1/g_s: && \text{Dirichlet branes (D-branes)}\,,\\
\mathrm{T_S} &\sim 1/g_s^2: && \text{Solitonic branes}\,,\\
\mathrm{T_E} &\sim 1/g_s^3: && \text{Exotic branes}\,,\\
\mathrm{T_{SF}} &\sim 1/g_s^4: && \text{Space-filling } \alpha=-4 \text{ branes}\,.
\end{aligned} \tag{4.1}$$

Note that this distinction of branes with different dilaton scaling is different from the distinction between standard and non-standard branes. For instance, one has both standard as well as non-standard D-branes.

If a brane saw a standard geometry we would expect that upon dimensional reduction it would always lead to two different branes. Either one reduces along a transverse direction or along a worldvolume direction. The latter case corresponds to the wrapping of the brane, which leads to a brane with a reduced world-volume direction. We summarize this by saying that the "wrapping rules" corresponding to standard geometry are given by

$$\text{any brane} \begin{cases} \text{wrapped} & \rightarrow \quad \text{undoubled}\,,\\ \text{unwrapped} & \rightarrow \quad \text{undoubled}\,. \end{cases} \tag{4.2}$$

The use of the word 'undoubled' stresses the fact that in both cases, wrapped or un-wrapped, only a single brane is obtained. Giving these wrapping rules and given the branes of ten-dimensional string theory with a given scaling α it is non-trivial that we precisely obtain the number of half-supersymmetric brane we obtained in our earlier brane classification. Indeed, it turns out that this only happens in the case of D-branes. Given the D-branes of Type IIA or IIB string theory and applying the wrapping rules (4.2) one precisely obtains the lower-dimensional D-branes which organize themselves into spinor representations of the T-duality group.

The same strategy does not work for the fundamental branes. As we saw earlier, in each dimension we have a singlet fundamental string and fundamental 0-branes that form the components of a vector representation of the T-duality group $\mathrm{SO}(d,d)$. This means that we need for each compactified direction *two* fundamental 0-branes. Clearly only one of these two branes can come from a wrapped fundamental string. We need another source to explain the occurrence of the second 0-brane. This is provided by the T-dual of the fundamental string, which is a pp-wave which upon reduction gives rise to the second 0-brane. The extra contribution due to the pp-waves gives rise to the following effective wrapping rules for the fundamental branes:

$$\mathrm{T_F} \sim 1 \ : \quad \begin{cases} \text{wrapped} & \rightarrow \quad \text{doubled}\,, \\ \text{unwrapped} & \rightarrow \quad \text{undoubled}\,. \end{cases} \tag{4.3}$$

These wrapping rules remind the ‘doubled geometry’ proposal of [19] where each compactified direction is doubled with a T-dual direction. Note that the doubled geometry proposal is based on a perturbative symmetry and therefore only applies to the fundamental branes and not necessarily to the other type of branes. Indeed, as we saw above, the D-branes have their own wrapping rules (4.2) corresponding to standard geometry.

Things get more interesting when we consider the solitonic branes. Again we find that the wrapping rules (4.2) corresponding to standard geometry do not lead to the right number of half-supersymmetric solitonic branes in lower dimensions. In this case, however, the extra input comes from the Kaluza-Klein (KK) monopoles. In each dimension $D \geq 5$ there is a KK monopole which can be considered as the dual of the pp-wave. The KK monopole divides spacetime into *three* different directions:

$$\text{KK monopole}: \quad \begin{cases} p+1 \text{ worldvolume directions}\,, \\ 1 \text{ isometry direction}\,, \\ 3 \text{ transverse directions}\,. \end{cases} \tag{4.4}$$

A brane in lower dimensions is obtained by reducing over the single isometry direction. In each dimension we have a singlet solitonic $(D-5)$-brane which is dual to the fundamental string. This singlet follows from the worldvolume reduction of the ten-dimensional NS5-brane. We also have solitonic $(D-4)$-branes which transform as a vector of the T-duality group. This implies that for each compactified direction we need *two* $(D-4)$-branes. Consider, for instance, the doublet of solitonic 5-branes in 9D. The first 5-brane follows from a transverse reduction of the ten-dimensional NS5-brane. To obtain the second 5-brane we need the help from the 10D KK-monopole. Indeed, the ten-dimensional KK monopole has 6 worldvolume, 1 isometry and 3 transverse directions. Reducing over the isometry direction leads to the second 5-brane. We thus obtain the following effective wrapping rules for solitonic branes [20]:

$$\mathrm{T}_S \sim 1/g_s^2 \ : \quad \begin{cases} \text{wrapped} & \rightarrow \quad \text{undoubled}\,, \\ \text{unwrapped} & \rightarrow \quad \text{doubled}\,. \end{cases} \tag{4.5}$$

These rules can be viewed as dual to the fundamental wrapping rules (4.3).

An issue arises if we now also consider the non-standard solitonic $(D-3)$-, $(D-2)$- and $(D-1)$-branes which transform according to anti-symmetric tensor representations of the T-duality group, see Table 1. The precise number of such branes, which is given by the red entries in the last column, first three rows of Table 1, is reproduced if we apply the solitonic wrapping rules (4.5) also for these cases [20]. However, the KK monopole, upon reduction over the isometry directions, only gives rise to a standard brane with three transverse directions. We need to introduce something new to explain the numbers of the non-standard solitonic branes. One possibility is that one introduces 'generalized' KK monopoles which have less than three transverse directions. Such monopoles have already been considered a long time ago using T-duality arguments [21]. At the moment it is not clear how rigorously such generalized objects can be defined within string theory. Note that such objects, if they exist at all, seem to couple to mixed-symmetry tensors instead of p-form potentials. The possibility of including such mixed-symmetry tensors into a supergravity multiplet is as yet unknown. Another attitude is to say that the solitonic branes 'see' a different so-called 'dual doubled geometry' which is different from the 'doubled' geometry sensed by the fundamental branes or the standard geometry as viewed by the D-branes. In some sense, the generalized KK-monopoles represent information about this dual doubled geometry.

The pattern that arises is that each type of brane, depending on the scaling of the brane tension with the string coupling constant, sees a different geometry. For instance, the ten-dimensional Type IIB string theory has only one brane with a tension that scales with $\mathrm{T} \sim 1/g_s^3$. This is the S-dual of the D7-brane. Clearly, this type of brane is highly non-perturbative and therefore difficult to study with the usual string theory techniques that one can use for the Dirichlet branes. Nevertheless, IIB supergravity suggests that this type of 'exotic' branes do exist. We find that, in order to explain the number of 'exotic' $\alpha = -3$ branes in lower dimensions that follows from our classification (see the red entries in the last column, forth and fifth row of Table 1) we need to impose the following new wrapping rule:

$$\mathrm{T_E} \sim 1/g_s^3 : \quad \begin{cases} \text{wrapped} & \rightarrow \quad \text{doubled}\,, \\ \text{unwrapped} & \rightarrow \quad \text{doubled}\,. \end{cases} \tag{4.6}$$

We call the new geometry defined by these wrapping rules 'exotic geometry'. Like in the previous cases the realization of this wrapping rule requires the input of new objects. How to precisely define these new objects within string theory is not clear but one could think about them as 'generalized' KK monopoles with less than three transverse and/or more than one isometry direction.

The only other type of brane that exists within ten-dimensional string theory is a space-filling brane whose tension scales as $\mathrm{T} \sim 1/g_s^4$. It is the S-dual of the D9-brane. Space-filling branes are a bit special in the sense that they can only wrap to give a space-filling brane in lower dimensions. As we reviewed in the previous sections, the field that contains the $(D-1)$-brane that comes from the wrapping of this brane is the D-form $F^{+}_{D,A_1\ldots A_d}$, and from the light-cone rule (or the longest-weight rule) one obtains 2^{d-1} branes in D dimensions. To explain this number from the wrapping of the S-dual D9-brane we need to impose the following wrapping rule:

$$\mathrm{T_{SF}} \sim 1/g_s^4 : \quad \text{wrapped} \quad \rightarrow \quad \text{doubled}\,. \tag{4.7}$$

Since these branes can only wrap, one cannot tell whether they see a doubled geometry or an exotic geometry.

Based upon the above wrapping rules we conclude that the different branes of ten-dimensional string theory see the following kind of geometries:

fundamental branes :	doubled geometry
Dirichlet branes :	standard geometry
solitonic branes :	dual doubled geometry
exotic branes :	exotic geometry .

This is not yet the end of the story. Starting from $D = 7$, there are additional $\alpha = -4$ branes apart from those associated to the potential $F^{+}_{D,A_1\ldots A_d}$. More generally, maximal supergravity in lower dimensions suggests the existence of branes with $\alpha < -4$. Clearly, all such branes can never result from the reduction of any brane in ten dimensions. They should either follow from the reduction of new objects within string theory or result as the effect of a new kind of geometry. Clearly, the last word has not been said on this issue.

5. Conclusions

We have classified the half-supersymmetric branes of maximal supergravity by investigating the worldvolume WZ term that describes the coupling of the supergravity potentials to these branes. By requiring that a gauge-invariant WZ term could be constructed involving only worldvolume fields that fit into a half-maximally supersymmetric matter multiplet we were able to classify the branes. The worldvolume content of these branes is either a vector multiplet or a 6-dimensional (self-dual) tensor multiplet. We call such branes vector branes and tensor 5-branes, respectively. Note that for branes with a low-dimensional world-volume, such as membranes and strings, the vector multiplet becomes equivalent to a scalar multiplet. [1] The dynamics of vector branes is governed by a Dirac-Born-Infeld/Volkov-Akulov (DBI-VA) action. Such vector branes have recently been considered in discussions on the quantum properties of 4D supergravity theories [22].

The investigation of the WZ term led to two simple, equivalent, rules that specify the number of half-supersymmetric branes. The first, so-called 'light-cone rule', is based on decomposing the SO(d,d) indices into its light-cone directions. The second, so-called 'longest-weight rule', states that the light-cone rule is equivalent to the group-theoretical rule that the half-supersymmetric branes correspond to the longest weights of the T-duality representation in which the supergravity potentials transform. We have not commented on the role of the next-to-longest weights etc. They are related to bound states of half-supersymmetric branes. These states, unlike bound states of standard branes, can be 1/2-supersymmetric threshold bound states [23, 17].

Having classified the branes we went on to investigate the way in which the branes in different dimensions are related to each other by dimensional reduction. This led us to consider the introduction of several wrapping rules, one set of rules for each brane with a given brane tension scaling α for $\alpha = 0, -1, -2, -3, -4$. These wrapping rules can be found in eqs. (4.3), (4.2), (4.5), (4.6) and

[1] In 3D a vector is dual to a scalar, whereas in 2D a vector is equivalent to an integration constant.

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(4.7), respectively. In some cases, the origin of the doubling in the wrapping rules is understood. They come from pp-waves and KK monopoles that upon reduction give rise to additional branes. But this is not enough. In order to explain the doubling in all cases something new is needed. Here there are two different points of view. Either one introduces new objects in string theory. We called them 'generalized KK monopoles' but the precise status of these monopoles in string theory is not clear. They seem to be related to the issue whether mixed-symmetry tensors can be introduced in supergravity. Another point of view is to say that the extra branes result from a new geometry that is described by the brane wrapping rules. This is more in line with the doubled geometry proposal that can be used to explain the wrapping rules of the fundamental branes.

We did not discuss several other interesting brane properties that follow from our methods. For instance, our techniques allow to determine the BPS conditions of the branes and their relation to the central charges in the supersymmetry algebra [2]. Again, we find here an important distinction between the standard and non-standard branes. Whereas for the standard branes each brane has its own BPS condition, in the case of non-standard branes the same BPS condition can be satisfied by several branes. We have calculated the degeneracies of each BPS condition [17]. Apart from this, one may also study brane orbits and multi-charge configurations [8, 17] [3].

It remains to be seen what the precise role is of the non-standard branes we discussed. Recently, is has been argued that in particular the defect branes play a role in describing the microscopic degrees of freedom of black holes [28] and that they are related to non-geometric Q-fluxes [29].

Our methods may be generalized and applied to study the half-supersymmetric branes of supergravities with less supersymmetry. The branes of half-supersymmetric supergravity have already been studied [27]. We hope to report on the half-supersymmetric branes of a quarter-supersymmetric supergravity shortly [30].

Finally, we hope that all this new information on branes will lead to a better understanding of their role in string theory and, most importantly, of the geometry underlying string theory.

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