Generalised fluxes in matrix compactifications

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In the recent years a lot of attention is focused on unconventional string compactifications. A variety of different but related frameworks was developed in order to address issues such as duality invariance, non-geometry and non-commutativity in string theory. In this contribution we review and clarify the approach that goes through matrix models. Furthermore, we discuss some connections of this framework to other related approaches.

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1. Introduction and discussion

The low-energy limit of perturbative string theories is described by particular ten-dimensional supergravities, which are very useful in the study of string compactifications. However, unlike supergravity, string theory is not a field theory. Therefore it has properties, mainly related to the finite string length, which cannot be directly captured by vanilla low-energy supergravity. Such properties include the existence of winding modes of the string, string dualities, non-geometry and relations to non-commutative geometry to name a few. Their role in string compactifications is under investigation, which led to the development of several new frameworks and techniques. Indeed, the above issues were addressed systematically in the doubled formalism [1] and in particular with the construction of T-folds and twisted doubled tori [2, 3]. Moreover, techniques from generalised complex geometry [4] have also proven to be useful in this context [5, 6, 7]. A powerful framework where both the doubled formalism and generalised geometry come into play is double field theory [8]. However, it should be mentioned that the above frameworks, albeit capturing more properties of the fundamental string theory than vanilla supergravity does, are effective theories too. At a more fundamental level, use of conformal field theory methods and sigma models have also provided interesting insights [9, 10, 11, 12]. Here we would like to argue that a fundamental framework where unconventional string compactifications can be studied is matrix models. Such an approach was initially pursued in Ref. [13] and more recently in Refs. [14, 15].

The motivation to use matrix models in order to study unconventional string compactifications is manifold. Certain matrix models, with prime examples the BFSS and IKKT models [16, 17], grew out from efforts to understand second quantization of strings and there is evidence that they can be considered as non-perturbative definitions of M theory or superstrings (some reviews on the topic are Refs. [18, 19, 20]). It is advantageous that such models are non-perturbative, quantum theories where non-commutative structures naturally appear [21]. Moreover, they are more than abstract constructions since they may be actually utilized for model building in particle physics and cosmology [22, 23, 24] . Regarding compactifications, there are essentially two points of view. One way would be to directly define the matrix model on a compact manifold, while the other one involves imposing conditions to the degrees of freedom corresponding to different compactifications of the matrix model [1]. This second point of view was employed in Ref. [26], where toroidal matrix compactifications were studied and it was shown that there exist many interesting connections to non-commutative geometry. In particular, compactifications of matrix theory on non-commutative tori seem to point at a correspondence between deformation parameters of the tori and moduli of 11D supergravity [26, 27, 28]. Then it is reasonable to ask whether such a correspondence is relevant for compactifications with fluxes. Fluxes in string compactifications can be either purely geometric, related to the internal geometry of the compactification manifold, NSNS or RR fluxes, being expectation values of form fields, or the so-called non-geometric fluxes. The latter are dual cousins of the previous ones and they have been discussed extensively in the literature (see Ref. [29] and its citations). The purpose here is to review and clarify how such quantities can be traced in matrix compactifications.

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[1]It is worth mentioning another approach which is different in spirit than traditional string compactifications [25]. In this approach spacetime structures are viewed as compact brane-like solutions of matrix models embedded in flat ten-dimensional ambient spacetime.

A convenient starting point is to determine and solve the conditions which define a matrix compactification on nilmanifolds. The latter are smooth parallelizable manifolds with non-trivial geometry which incorporate geometric fluxes and they have played a distinguished role in string compactifications [30, 31, 32, 33, 34, 35, 36, 37]. Their study in the matrix model is tractable because they can be described as non-trivial toroidal fibrations. This picture emerges from their description as homogeneous spaces constructed as orbits of a lattice in a nilpotent Lie group [2]. The prototype example of such a construction is the 3D Heisenberg nilmanifold but there exist several more cases with richer geometry. This construction leads to a set of identification conditions, generalizing the ones of a standard torus, which can subsequently be imposed on the degrees of freedom of the matrix model in the same spirit as for the toroidal matrix compactifications of Ref. [26]. Associated to matrix compactifications on nilmanifolds, one obtains particular operator algebras which can be represented in terms of phase space variables. In the T-dual picture one obtains a non-commutative Yang-Mills theory on a dual nilmanifold, as usual in matrix compactifications [26, 28, 39].

Matrix compactifications on tori and nilmanifolds show that an important role in this context is played by phase space, as expected in a quantum-mechanical theory. Apart from the original degrees of freedom X, new objects $\tilde{X}$ appear, a situation reminiscent of the appearance of winding modes when strings are compactified. This natural doubling can subsequently be used to examine other duality frames in matrix compactifications. Indeed, we argue that particular transformations lead to matrix compactifications which can be identified with situations where NSNS flux or non-geometric fluxes are present. We present two arguments in favour of this interpretation. The first is that the action of the operator algebras that we obtain on the generalized phase space of $(X, \tilde{X})$ exhibits the structure of a class of twisted doubled tori as constructed in Refs. [2, 3] in order to account for the corresponding geometric and non-geometric fluxes. Let us stress that we do not impose the conditions of these twisted doubled tori but they rather follow from the action of the operator algebras. Secondly, the formulae that provide the flux in each matrix compactification as derivation of the commutator of the matrices are in exact correspondence with the ones found for the same fluxes in the context of double field theory [40, 41, 42, 43]. Therefore, each operator algebra is associated to some matrix compactification and each matrix compactification corresponds to a duality frame of a flux background in supergravity. This provides a picture of duality frames as different superselection sectors of the matrix model, accompanied with flux quantization which is part of the consistency of the compactification.

According to the above, some understanding of unconventional flux compactifications of string theory is possible within the framework of matrix models. Moreover, this formalism bears a lot of resemblance and is in agreement to other approaches. It is then reasonable to ask what more could we learn following this road that we could not learn in another way. This critical viewpoint brings us partially back to our motivation and partially to a discussion on future possibilities. From a motivational point of view, as already mentioned before, matrix models are not supposed to be effective theories but rather constructive definitions of string or M theory. If this is true, then understanding properties of string compactifications in this framework would be more valuable than in an effective theory. However, matrix models still need to prove their merit. A direction

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[2]Relations between nilpotent Lie algebras and matrix models were studied systematically in Ref. [38].

which we find interesting is to make an effort to describe genuinely non-geometric cases in the matrix model. What we mean here is that the non-geometric cases that are usually discussed possess duality frames where they become geometric. Since T-dual backgrounds are physically equivalent this means that they are not really new. What would be totally new is a non-geometry which cannot be geometrized in any duality frame (see the discussion in Ref. [44]). This issue was also discussed in Ref. [45] in the context of general co-dimension 2 objects dubbed exotic branes. It would also be interesting to study the dynamics of such objects and matrix models might play a role in this too.

2. Matrix compactification on the Heisenberg nilmanifold

2.1 Geometry of the unpolarized 3D nilmanifold

Let us examine the 3D nilmanifold based on the nilpotent Lie algebra $\mathfrak{g} = (0,0,12)$ with corresponding group G. This notation means that the algebra has three generators T_i, $i = 1,2,3$ which satisfy the single commutation relation $[T_1, T_2] = T_3$. The only non-vanishing structure constants are therefore $f_{12}^3 = -f_{21}^3 = 1$. The Heisenberg nilmanifold is obtained as an orbit of a lattice $\Gamma \subset G$ in G. It should be noted that there is some freedom on the construction of such an orbit. The most usual choice in the physics literature is the polarised one, where the nilmanifold takes the form of a 2-torus fibration over a circle. Here we consider instead a different parametrization, which is more convenient when abstract index notation is used.

Let us consider the following basis of left-invariant 1-forms,

$$
\begin{aligned}
e^1 &= dx^1, \\
e^2 &= dx^2, \\
e^3 &= dx^3 - \tfrac{1}{2}x^1 dx^2 + \tfrac{1}{2}x^2 dx^1,
\end{aligned}
\tag{2.1}
$$

which satisfy the Maurer-Cartan equations

$$
de^i + \tfrac{1}{2}f_{jk}^i e^j \wedge e^k = 0.
$$

There is a basis of dual vector fields on the tangent bundle of the nilmanifold, which is found to be

$$
\begin{aligned}
e_1 &= \partial_1 - \tfrac{1}{2}x^2 \partial_3, \\
e_2 &= \partial_2 + \tfrac{1}{2}x^1 \partial_3, \\
e_3 &= \partial_3.
\end{aligned}
\tag{2.2}
$$

One can directly confirm that

$$
\langle e^i, e_j \rangle = \delta_j^i,
$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing between the algebra $\mathfrak{g}$ and its dual algebra $\mathfrak{g}^*$. These 1-forms and vectors are defined on the corresponding group manifold G, which is the covering space of the nilmanifold, and they descend on the orbit $\Gamma \backslash G$ under the following identifications:

$$
(x^1, x^2, x^3, \partial_1, \partial_2, \partial_3) \sim (x^1 + c^1, x^2 + c^2, x^3 + \tfrac{c^1}{2}x^2 - \tfrac{c^2}{2}x^1 + c^3, \partial_1 + \tfrac{c^2}{2}\partial_3, \partial_2 - \tfrac{c^1}{2}\partial_3, \partial_3). \tag{2.3}
$$

Note that we included the transformations of the derivative operators which follow from eq. (2.2). The above relation may as well be written in the following very compact form:

$$(x^i, \partial_j) \sim (x^i + c^i + \tfrac{1}{2} f^i_{jk} c^j x^k, \partial_i + \tfrac{1}{2} f^k_{ij} c^j \partial_k). \tag{2.4}$$

2.2 Matrix compactification

We consider a set of matrices X^a of a matrix model associated to them, for example the BFSS or the IKKT model. In the formulation employed here the only difference is the number of the matrices; the index a runs from 1 to 9 for the BFSS model, where time is treated separately, and from 1 to 10 for the IKKT model. Moreover, we focus our attention to the bosonic sector. Fermions ψ, being the supersymmetric partners of X^a in each model, are treated similarly as discussed for example in Ref. [26]. The classical action of the BFSS model is

$$S = \frac{1}{2g} \int dt \left[Tr \left(\dot{X}^a \dot{X}_a - \frac{1}{2} [X^a, X^b]^2 \right) + 2\psi^T \dot{\psi} - 2\psi^T \Gamma_a [\psi, X^a] \right], \tag{2.5}$$

where Γ_a furnish a representation of $SO(9)$. This action is written in units of the string length and g is the string coupling [16].

The most standard approach to toroidal compactification of a matrix model involves imposing a set of conditions which correspond to the identifications of the coordinates of the torus under the lattice action. This approach is also appropriate for compactifications on nilmanifolds, as shown in [13, 14, 15]. In the present case it amounts to considering three unitary operators U_i and the following set of conditions, corresponding to the identifications (2.4),

$$\begin{aligned}
U_i X^i U_{-i} &= X^i + R^i, \quad i = 1, 2, 3, \\
U_1 X^3 U_{-1} &= X^3 + \frac{R^1}{2} X^2, \\
U_2 X^3 U_{-2} &= X^3 - \frac{R^2}{2} X^1, \\
U_i X^a U_{-i} &= X^a, \quad a \neq i, \quad a = 1, \ldots, 9, \quad (a,i) \neq \{(3,1),(3,2)\}.
\end{aligned} \tag{2.6}$$

Note that we adopt the notation $U_{-i} \equiv U_i^{-1}$. R^i are constant matrices which may be thought of as radii (more accurately, as radii multiplied by $2\pi)^3$. Let us define the Q-operator as

$$Q_{ij} = U_i U_j U_{-i} U_{-j}. \tag{2.7}$$

In the present case we find the following commutation relations:

$$[Q_{ij}, X^k] = -R^i R^j f^{\ k}_{ij} Q_{ij}, \tag{2.8}$$

where no summation is implied on the RHS. Now, these relations imply that Q_{13} and Q_{23} are scalar operators and therefore we may write

$$Q_{13} = e^{-i\theta_{13}} \quad \text{and} \quad Q_{23} = e^{-i\theta_{23}}, \tag{2.9}$$

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[3] The reader should avoid confusion of indices and powers. The superscript in R^2 is an index and not a power. In this paper there will never appear a power of such a quantity, therefore one should always think of superscripts as indices.

for some real parameters θ_{13}, θ_{23}. The third Q-operator, Q_{12}, does not commute with X^3 and therefore we cannot directly write its form as for the previous two. Notice, however, that the relation (2.8) can be rewritten as

$$Q_{12}X^3Q_{-12} = X^3 - R^1R^2f_{12}^3.$$

Comparing this expression with conditions (2.6) we conclude that

$$Q_{12} \propto U_{-3}. \tag{2.10}$$

Let us set all constants $R^i = 1$ for the moment. Now, in order to solve the conditions (2.6), we write the unitary operators in a rather generic form $U_i = e^{iY_i}$, where Y_i are arbitrary hermitean operators. Note that the unitary operators act on functions of a Hilbert space and their form is specified by this action; this does not directly mean that the operators Y_i can be written down explicitly. However, in the present case it turns out that this is indeed possible. It is directly observed that the algebra of Us takes the following form,

$$\begin{aligned} U_1U_3 &= e^{-i\theta_{13}}U_3U_1, \\ U_2U_3 &= e^{-i\theta_{23}}U_3U_2, \\ U_2U_1 &= U_3U_1U_2. \end{aligned} \tag{2.11}$$

Let us mention that this is a C^* algebra known in the mathematical literature from the work of Packer [46]. It is now simple to determine the algebra of (X, Y). The "diagonal" compactification conditions imply that

$$[Y_i, X^i] = -i.$$

Moreover, the "off-diagonal" ones imply that

$$[Y_1, X^3] = -\tfrac{i}{2}X^2 \quad \text{and} \quad [Y_2, X^3] = \tfrac{i}{2}X^1.$$

Furthermore, Eq. (2.10) gives

$$[Y_1, Y_2] = iY_3.$$

Similarly, we obtain

$$[Y_1, Y_3] = i\theta_{13} \quad \text{and} \quad [Y_2, Y_3] = i\theta_{23}.$$

This procedure does not fix the commutation relations among X^i. Here we restrict to trivial gauge bundles and thus the full algebra of X and Y reads as

$$\begin{aligned} [X^i, X^j] &= 0, \\ [X^i, Y_j] &= i\delta_i^j + \tfrac{i}{2}f_{jk}^{\;i}X^k, \\ [Y_i, Y_j] &= if_{ij}^{\;k}Y_k + i\theta_{ij}|_{\gamma 2}, \end{aligned} \tag{2.12}$$

where $f_{12}^3 = 1$ and the notation $|_{\gamma 2}$ means that θ_{12} vanishes[4].

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[4]The reason that θ_{13} and θ_{23} are non-vanishing but θ_{12} vanishes may be formulated in terms of Poisson geometry. Indeed, Rieffel in Ref. [47] shows that the most general invariant Poisson structure on the Heisenberg nilmanifold is $\Lambda = \mu\partial_1 \wedge \partial_3 + \nu\partial_2 \wedge \partial_3$, where μ, ν are constants. Quantization of this Poisson structure would then lead to non-vanishing commutators only for the corresponding pairs of indices.

One question that arises is whether an explicit expression for the solutions may be found in terms of coordinate and momentum operators satisfying the standard Heisenberg relation

$$[\hat{x}^i, \hat{p}_j] = i\delta^i_j$$

but otherwise commuting, i.e.

$$[\hat{x}^i, \hat{x}^j] = 0 \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = 0.$$

This is indeed possible. For simplicity, let us consider the case of $\theta_{ij} = 0$. Then a particular solution may be written as

$$\begin{aligned} X^i &= \hat{x}^i, \\ Y_i &= \hat{p}_i + \tfrac{1}{2} f_{ij}{}^k \hat{x}^j \hat{p}_k. \end{aligned} \tag{2.13}$$

Note that the Ys are related to the right-invariant vector fields on the nilmanifold.

As shown in Ref. [26], having found a particular solution, the general solution is given as

$$X^i = \hat{x}^i + \hat{A}^i, \tag{2.14}$$

where $\hat{A}^i = \hat{A}^i(\hat{U}_j)$ depend [5] on a set of unitary operators $\hat{U}_i$ which commute with the U_i, i.e.

$$U_i \hat{U}_j = \hat{U}_j U_i.$$

The expression (2.14) for the general solution is reminiscent of the covariant coordinates which are used in the study of non-commutative gauge theories [48]. They represent a non-commutative connection which gives rise to the gauge fields of the resulting theory. Finding these hatted operators in the present case is a straightforward task. One anticipates that they are related to the left-invariant vector fields of the nilmanifold; indeed it is found that $\hat{U}_i = e^{i\hat{Y}_i}$, where

$$\hat{Y}_i = \hat{p}_i - \tfrac{1}{2} f_{ij}{}^k \hat{x}^j \hat{p}_k. \tag{2.15}$$

Then the algebra of the hatted operators, which is essentially the gauge algebra of the theory, is found to be

$$\begin{aligned} \hat{U}_1 \hat{U}_3 &= e^{-i\theta'_{13}} \hat{U}_3 \hat{U}_1, \\ \hat{U}_2 \hat{U}_3 &= e^{-i\theta'_{23}} \hat{U}_3 \hat{U}_2, \\ \hat{U}_1 \hat{U}_2 &= \hat{U}_3 \hat{U}_2 \hat{U}_1. \end{aligned} \tag{2.16}$$

In the case of trivial gauge bundles, the primed parameters are related to the unprimed ones as $\theta'_{ij} = -\theta_{ij}$. This shows that the C^* algebra given by $\hat{U}$s, where the gauge fields $\hat{A}^i$ live, is dual to the one of the operators U^i. In particular, it was shown in Ref. [46] that these C^* algebras are Morita equivalent.

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[5]Imposing compactification conditions (2.6) on the general solution (2.14) implies actually that $\hat{A}^i = A^i(\hat{U}_j) + \tfrac{1}{2} f_{jk}{}^i \hat{x}^j A^k(\hat{U}_j)$, see Ref. [15] for details.

3. The case of general fluxes

One of our main motivations is the situation described by Jackiw in Ref. [49], exhibiting a very interesting interplay among a physical quantum-mechanical system and the mathematics of cocycles. Studying a quantum-mechanical particle moving under the influence of a magnetic field one may postulate non-canonical commutation relations of the form

$$
\begin{aligned}
[\hat{x}^i, \hat{x}^j] &= 0, \\
[\hat{x}^i, \hat{p}_j] &= i\delta^i_j, \\
[\hat{p}_i, \hat{p}_j] &= i\varepsilon_{ijk}B^k,
\end{aligned}
\tag{3.1}
$$

where $\hat{p}_i$ is the mechanical momentum and B is the magnetic field. Then the Jacobiator of the momenta is computed as

$$
[\hat{p}_i, \hat{p}_j, \hat{p}_k] \equiv [\hat{p}_i, [\hat{p}_j, \hat{p}_k]] + \text{c.p.} = \text{div}B.
\tag{3.2}
$$

In the absence of magnetic sources the Jacobi identity is satisfied. However, this fails to happen when the magnetic field has non-vanishing divergence, in which case a 3-cocycle is present and therefore some non-associativity too. This non-associativity is lifted and the cocycle is removed at the level of the unitary (translation) operators due to a Dirac quantization condition. In the following, this structure serves as motivation to consider in the matrix model more general commutation relations than before and investigate their physical meaning. Moreover, the special role of the phase space in the previous section is another motivation to go further and explore its full potential. Finally, according to Ref. [26], the parameters θ_{ij} of non-commutativity correspond to background values of the three-form of 11-dimensional supergravity or, equivalently, to the B field of 10-dimensional type IIA supergravity. It is then interesting to examine the way that fluxes can be described in this formalism.

3.1 Geometric flux revisited

Let us begin by revisiting the geometric flux case which was described in the previous section. Here we would like to pay more attention to the full phase space structure. Previously we considered the action of the unitary operators on the coordinate space but their action on the momentum space was not made explicit. Now we reconsider these operators, acting on functions on the phase space as

$$
\begin{aligned}
(U_1 f)(\hat{x}^i; \hat{p}_i) &= f(\hat{x}^1 + 1, \hat{x}^3 + \tfrac{1}{2}\hat{x}^2; \hat{p}_2 - \tfrac{1}{2}\hat{p}_3) = e^{i\hat{p}_1} f(\hat{x}^3 + \tfrac{1}{2}\hat{x}^2; \hat{p}_2 - \tfrac{1}{2}\hat{p}_3), \\
(U_2 f)(\hat{x}^i; \hat{p}_i) &= f(\hat{x}^2 + 1, \hat{x}^3 - \tfrac{1}{2}\hat{x}^1; \hat{p}_1 + \tfrac{1}{2}\hat{p}_3) = e^{i\hat{p}_2} f(\hat{x}^3 - \tfrac{1}{2}\hat{x}^1; \hat{p}_1 + \tfrac{1}{2}\hat{p}_3), \\
(U_3 f)(\hat{x}^i; \hat{p}_i) &= f(\hat{x}^3 + 1) = e^{i\hat{p}_3} f(\hat{x}^i; \hat{p}_i),
\end{aligned}
\tag{3.3}
$$

where we write explicitly only the directions which are subject to some transformation. These operators when acting on X^i reproduce the conditions (2.6) as they should. Moreover, from (3.3) we can deduce their action on the momentum space, $\tilde{X}_i \equiv \hat{p}_i$. The non-trivial relations are

$$
\begin{aligned}
U_1 \tilde{X}_2 U_{-1} &= \tilde{X}_2 - \tfrac{1}{2}\tilde{X}_3, \\
U_2 \tilde{X}_1 U_{-2} &= \tilde{X}_1 + \tfrac{1}{2}\tilde{X}_3.
\end{aligned}
\tag{3.4}
$$

Since we are working in phase space, let us also consider the operators $\tilde{U}^i$ which are dual to the U_i ones. The former are associated to translations in momentum space much like the latter are translation operators in position space[6]. Therefore we consider

$$
\begin{aligned}
(\tilde{U}^1 f)(\hat{x}^i; \hat{p}_i) &= f(\hat{p}_1 + 1) = e^{-i\hat{x}^1} f(\hat{x}^i; \hat{p}_i), \\
(\tilde{U}^2 f)(\hat{x}^i; \hat{p}_i) &= f(\hat{p}_2 + 1) = e^{-i\hat{x}^2} f(\hat{x}^i; \hat{p}_i), \\
(\tilde{U}^3 f)(\hat{x}^i; \hat{p}_i) &= f(\hat{p}_1 - \tfrac{1}{2}\hat{x}^2, \hat{p}_2 + \tfrac{1}{2}\hat{x}^1, \hat{p}_3 + 1) = e^{-i\hat{x}^3} f(\hat{p}_1 - \tfrac{1}{2}\hat{x}^2, \hat{p}_2 + \tfrac{1}{2}\hat{x}^1).
\end{aligned}
\tag{3.5}
$$

These operators act trivially on the X^i, however they have the following non-trivial action on $\tilde{X}_i$:

$$
\begin{aligned}
\tilde{U}^i \tilde{X}_i \tilde{U}^{-i} &= \tilde{X}_i + 1, \\
\tilde{U}^3 \tilde{X}_1 \tilde{U}^{-3} &= \tilde{X}_1 - \tfrac{1}{2} X^2, \\
\tilde{U}^3 \tilde{X}_2 \tilde{U}^{-3} &= \tilde{X}_2 + \tfrac{1}{2} X^1.
\end{aligned}
\tag{3.6}
$$

Setting for simplicity all the constant parameters θ_{ij} to zero, the extended algebra of the operators U_i and $\tilde{U}^i$ has the following non-trivial relations:

$$
U_2 U_1 = U_3 U_1 U_2, \quad U_1 \tilde{U}^3 = \tilde{U}^2 \tilde{U}^3 U_1, \quad \tilde{U}^3 U_2 = \tilde{U}^1 U_2 \tilde{U}^3.
\tag{3.7}
$$

This is an extension of the C^* algebra of the previous section. Moreover, as previously, one has to determine the operators $\hat{U}_i$ and $\hat{\tilde{U}}^i$ commuting with U_i and $\tilde{U}^i$ and thus providing the dependence of the gauge fields. We are not going to write their explicit form here but it is instructive to write their commutation relations which read as

$$
\hat{U}_1 \hat{U}_2 \hat{U}_{-1} \hat{U}_{-2} = \hat{U}_3, \quad \hat{\tilde{U}}^3 \hat{U}_1 \hat{\tilde{U}}^{-3} \hat{U}_{-1} = \hat{\tilde{U}}^2, \quad \hat{\tilde{U}}^3 \hat{U}_2 \hat{\tilde{U}}^{-3} \hat{U}_{-2} = \hat{\tilde{U}}^{-1}.
\tag{3.8}
$$

These relations represent the gauge algebra of the resulting theory and they are reminiscent of the gauge algebra of a 4-dimensional supergravity obtained from a dimensional reduction on a twisted torus [31].

3.2 H flux

Let us now perform the following transformation,

$$
(X^3, \tilde{X}_3) \to (\tilde{X}_3, -X^3),
\tag{3.9}
$$

which preserves the Heisenberg relation $[X^i, \tilde{X}_i] = i$, accompanied with the corresponding transformation on the unitary operators,

$$
(U_3, \tilde{U}^3) \to (\tilde{U}^3, U_{-3}).
\tag{3.10}
$$

Computing the action of the unitary operators U_i on X^i we find

$$
U_i X^j U_{-i} = X^j + \delta_i^j.
\tag{3.11}
$$

---

[6]Loosely speaking, one can think of these operators as coming from (2.13) after exchanging $\partial/\partial x^i$ with $\partial/\partial p_i$.

This set of conditions look exactly like the ones for a simple toroidal compactification of the model. However, there is more structure than just a torus. Indeed, writing $U_i = e^{iY_i}$ and $\tilde{U}^i = e^{i\tilde{Y}^i}$, we find the relations

$$U_i U_j = e^{-iF_{ijk}\tilde{Y}^k} U_j U_i, \tag{3.12}$$

where F_{ijk} is fully antisymmetric. These relations exhibit a 3-cocycle. Therefore we interpret the situation as presence of some flux, which is obtained as a derivation of the cocycle. Such derivations are actually commutators and therefore in the present case the flux is obtained as

$$F_{ijk} = [Y_{[i}, [Y_j, Y_{k]}]]. \tag{3.13}$$

This flux is interpreted as a NSNS H flux in the corresponding supergravity compactification. There are at least two arguments which support this interpretation. First, as advocated in Ref. [26], the cocycle appearing in Eq. (3.12) is related to the B field of type IIA supergravity. In supergravity compactifications an H flux may be obtained from a B field with the general structure

$$B = \tfrac{1}{3}(x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2).$$

The H flux is then obtained by

$$H_{ijk} = \partial_{[i} B_{jk]},$$

and the analogy to Eq. (3.13) is evident.

Another argument for the interpretation of the above situation as an H flux is the following. With the above data we may determine conditions from the action of Us on $\tilde{X}$s. In particular we obtain that

$$U_i \tilde{X}_j U_{-i} = \tilde{X}_j - F_{ijk} X^k. \tag{3.14}$$

Moreover, the action of $\tilde{U}^i$ on Xs and $\tilde{X}$s can be determined in a similar way and it is found to be

$$\tilde{U}^i X^j \tilde{U}^{-i} = X^j, \tag{3.15}$$
$$\tilde{U}^i \tilde{X}_j \tilde{U}^{-i} = \tilde{X}_j + \delta^i_j. \tag{3.16}$$

The tilded operators act trivially on Xs and as translations on $\tilde{X}$s. Collecting together the conditions (3.11), (3.14), (3.15), (3.16), we see directly that these are the conditions that should be imposed for the compactification on a particular twisted doubled torus [2, 3]. This twisted doubled torus is exactly the one associated with an H flux situation.

3.3 Q flux

A further transformation,

$$(X^2, \tilde{X}_2) \to (\tilde{X}_2, -X^2), \quad (U_2, \tilde{U}^2) \to (\tilde{U}^2, U_{-2}). \tag{3.17}$$

leads to the following action of Us and $\tilde{U}$s on Xs and $\tilde{X}$s:

$$U_i X^i U_{-i} = X^i + 1, \quad U_1 X^2 U_{-1} = X^2 + \tfrac{1}{2}\tilde{X}_3, \quad U_1 X^3 U_{-1} = X^3 - \tfrac{1}{2}\tilde{X}_2, \tag{3.18}$$

and

$$\tilde{U}^i \tilde{X}_i \tilde{U}^{-i} = \tilde{X}_i + 1,$$
$$\tilde{U}^2 X^3 \tilde{U}^{-2} = X^3 + \tfrac{1}{2}X^1, \quad \tilde{U}^3 X^2 \tilde{U}^{-3} = X^2 - \tfrac{1}{2}X^1,$$
$$\tilde{U}^2 \tilde{X}_1 \tilde{U}^{-2} = \tilde{X}_1 + \tfrac{1}{2}\tilde{X}_3, \quad \tilde{U}^3 \tilde{X}_1 \tilde{U}^{-3} = \tilde{X}_1 - \tfrac{1}{2}\tilde{X}_2. \tag{3.19}$$

The corresponding operator algebra is specified by the non-trivial relations

$$U_i \tilde{U}^j = e^{F_i^{jk} Y_k} \tilde{U}^j U_i, \quad \tilde{U}^i \tilde{U}^j = e^{-F_k^{ij} \tilde{Y}^k} \tilde{U}^j \tilde{U}^i. \tag{3.20}$$

In the same spirit as before, the above compactification conditions correspond to the global identifications for the coordinates of a twisted doubled torus associated to non-geometric Q flux in Refs. [2, 3]. This, along with a further argument that we discuss later, suggests that the generalised flux F_k^{ij} should be identified with a Q_k^{ij} non-geometric flux.

It should be noted that in the present case the sector of Xs does not close in itself and therefore a truncation of the compactification to the sector of Xs and Us is not plausible. We discuss how this situation should be interpreted in the following section.

3.4 R flux

As before, let us consider the last remaining possible transformation,

$$(X^1, \tilde{X}_1) \to (\tilde{X}_1, -X^1), \quad (U_1, \tilde{U}^1) \to (\tilde{U}^1, U_{-1}). \tag{3.21}$$

which preserves the Heisenberg relation. The action of Us and $\tilde{U}$s on Xs and $\tilde{X}$s is now given as

$$U_i X^j U_{-i} = X^j + \delta_i^j,$$
$$U_i \tilde{X}_j U_{-i} = \tilde{X}_j,$$
$$\tilde{U}^i X^j \tilde{U}^{-i} = X^j + F^{ijk} \tilde{X}_k,$$
$$\tilde{U}^i \tilde{X}_j \tilde{U}^{-i} = \tilde{X}_j + \delta_j^i, \tag{3.22}$$

where F^{ijk} is fully antisymmetric. These relations correspond to the identification conditions of a twisted doubled torus with R flux as in Refs. [2, 3]. The non-trivial relations of the operator algebra are

$$\tilde{U}^i \tilde{U}^j = e^{-F^{ijk} Y_k} \tilde{U}^j \tilde{U}^i. \tag{3.23}$$

Further arguments related to this case are discussed below.

4. Interpretation and relations to other frameworks

Relation to generalised fluxes in Double Field Theory. As we mentioned in the introduction, there is a variety of frameworks addressing the issues of non-geometry and non-geometric fluxes. A mathematical object which plays a central role in most of them is the generalised metric. Here we are not going to provide any details on generalised geometry due to lack of space. Instead we collect just the necessary ideas and formulas for our interpretation and refer the reader to other self-contained texts.

The generalised metric collects a d-dimensional metric g and the 2-form field B in a 2d-dimensional matrix which has the form

$$H = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}. \tag{4.1}$$

This generalised metric is obtained from a generalised vielbein E as $H = E^T E$, where

$$E = \begin{pmatrix} e & 0 \\ -e^{-T}B & e^{-T} \end{pmatrix}, \tag{4.2}$$

and e is the standard vielbein out of which the metric g is constructed. The above parametrization of the generalised metric is not unique; assuming that $B\beta = 0$ and writing the generalised vielbein as

$$E = \begin{pmatrix} e & e\beta \\ -e^{-T}B & e^{-T} \end{pmatrix}, \tag{4.3}$$

leads to the following parametrization of the generalised metric

$$H = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} + g\beta \\ -g^{-1}B - \beta g & g^{-1} - \beta g\beta \end{pmatrix}. \tag{4.4}$$

In the latter expressions $\beta = \beta^{ij}\partial_i \wedge \partial_j$ is a bivector, which appears naturally in generalised geometry. Moreover, it is very useful in the study of non-geometric cases in this framework [5]. In order to relate this discussion with the formalism that was presented in the previous section we would like to mention the case of generalised Scherk-Schwarz compactifications of double field theory. These were studied in Refs. [40, 41, 43, 50, 51] and led to expressions of generalised geometric and non-geometric fluxes in terms of the vielbein e, the 2-form B and the bivector β. Although in the general case these expressions are complicated, for the simple case of the Heisenberg nilmanifold they simplify significantly. Then the non-geometric fluxes Q and R are obtained as the derivative and the dual derivative of the bivector respectively, where the dual derivative is with respect to the dual coordinates of double field theory. We summarise these expressions in the following table, where we also show how the corresponding generalised fluxes are obtained in our formalism.

generalised Flux	Matrix Model	DFT
H_{ijk}	$[Y_{[i}, [Y_j, Y_{k]}]]$	$\partial_{[i}B_{jk]}$
f^k_{ij}	$[\tilde{Y}^k, [Y_i, Y_j]]$	$\tilde{\partial}^k B_{ij}$
Q^{ij}_k	$[Y_k, [\tilde{Y}^i, \tilde{Y}^j]]$	$\partial_k \beta^{ij}$
R^{ijk}	$[\tilde{Y}^{[i}, [\tilde{Y}^j, \tilde{Y}^{k]}]]$	$\tilde{\partial}^{[i}\beta^{jk]}$

We observe that the structure of these relations in the matrix model and in double field theory agree and they are in direct correspondence. Let us remember that according to Ref. [26] the quantity $[Y,Y]$ corresponds to the B field in supergravity. The first two lines of the above table are reminiscent of this relation. What is more, the last two lines of the table indicate a further relation,

which is that the quantity $[\tilde{Y}, \tilde{Y}]$ corresponds to the bivector β in the generalised geometry approach to supergravity.

We would like to point out that in richer situations, more complex that the Heisenberg nil-manifold, the above simplification of the form of the generalised fluxes is not always possible. In other words, there exist higher-dimensional nilmanifolds (six-dimensional in particular) where in certain duality frames the generalised metric cannot be written in terms of g and B or g and β only but the general structure of Eq. (4.4) is necessary. We shall examine such cases in a forthcoming publication.

Phase space and non-geometric matrix compactifications.   It should be obvious from the previous sections that phase space plays an important role in the present formalism. The pair of $(X, \tilde{X})$ is reminiscent of coordinates and canonical momenta in phase space. Moreover, as it was noticed in section 3, while for the cases with H and f flux it is possible to project the compactification to the X-sector, this is no longer true for the cases with Q and R flux. The interpretation of this situation is that in the latter cases the meaningful projection is on the $\tilde{X}$-sector. In quantum-mechanical terms this is like shifting to the momentum representation. The picture then is that in the matrix model formalism geometric compactifications with H or f flux are well-defined in position space (X-sector), while the non-geometric ones with Q or R flux are well-defined in momentum space ($\tilde{X}$-sector). A similar correspondence follows for the non-commutativity parameters $i\tilde{\theta}^{ij} = [\tilde{Y}^i, \tilde{Y}^j]$ and $i\theta_{ij} = [Y_i, Y_j]$. In position space there are compactifications with non-constant θ_{ij} corresponding to the geometric cases and in momentum space there are compactifications with non-constant $\tilde{\theta}^{ij}$ which correspond to the non-geometric cases. A similar result in the context of generalised complex geometry was obtained in Ref. [7].

Flux Quantization.   Another property of the models that we discuss here, stemming from their quantum-mechanical nature, is related to the quantization of flux. It is known that in quantum-mechanical problems with non-constant fields and non-canonical commutation relations there are Dirac quantization conditions which follow from consistency requirements [49]. This is essentially due to the fact that at the level of finite translations any 3-cocycle, which is also a signature of failure of the Jacobi identity, should be removed. In the present formalism and taking as an example the case of H flux, the translation operators are the U_is and they satisfy the relation

$$U^i(U^j U^k) = e^{\frac{i}{2} H^{ijk}} (U^i U^j) U^k. \tag{4.5}$$

This cocycle condition is rendered trivial by the quantization condition

$$H = 4\pi n, \quad n \in \mathbb{Z}. \tag{4.6}$$

Needless to say that this is in accord with string theory, where all charges have to be quantized.

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