PROCEEDINGS OF SCIENCE

$N = (2,2)$ Non-Linear σ-Models: A Synopsis

Alexander Sevrin*† and Daniel C. Thompson
Theoretische Natuurkunde, Vrije Universiteit Brussel
and
The International Solvay Institutes
Pleinlaan 2, B-1050 Brussels, Belgium
E-mail: Alexandre.Sevrin@vub.ac.be *and* Daniel.Thompson@vub.ac.be

We review $N = (2,2)$ supersymmetric non-linear σ-models in two dimensions and their relation to generalized Kähler and Calabi-Yau geometry. We illustrate this with an explicit non-trivial example.

Proceedings of the Corfu Summer Institute 2012
September 8-27, 2012
Corfu, Greece

*Speaker.
†Also at the Physics Department, Universiteit Antwerpen, Campus Groenenborger, 2020 Antwerpen, Belgium.

PoS(Corfu2012)097

1. Introduction

Moduli stabilization, the AdS-CFT correspondence and string phenomenology all require the study of strings in non-trivial backgrounds. The method of choice to do so uses the supergravity approximation. In order to go beyond supergravity one is currently forced to use $d=2$ conformal field theories, Landau-Ginzburg type descriptions, the pure spinor formalism or the σ-model worldsheet description. The latter is limited to non-trivial NSNS backgrounds only. Particularly interesting are non-linear σ-models in two dimensions with $N=(2,2)$ supersymmetry [1], [2] which play a central role in the worldsheet description of type II string theories in non-trivial NSNS backgrounds. Even at the classical level these models exhibit a rich geometric structure which is the main topic of the current review paper.

This paper closely follows the line of the talk. In the next section we introduce non-linear σ-models and identify the geometries which allow for an $N=(2,2)$ supersymmetry. Subsequently we introduce Hitchin's generalized Kähler geometry, the proper setting for the above mentioned models. After this we present the full off-shell description of these models in $N=(2,2)$ superspace. Having an $N=(2,2)$ superspace description allows one to perform various (T-)duality transformations keeping the supersymmetries manifest. These transformations typically alter the nature of the superfields and we provide an exhaustive list of them. Finally we illustrate all of this using a simple example: the Wess-Zumino-Witten model on $SU(2)\times U(1)$. We end with conclusions and summarize our conventions in the appendix.

2. From $N=(1,1)$ to $N=(2,2)$ supersymmetry

In the absence of boundaries, a non-linear σ-model in two dimensions is classically fully characterized by its D-dimensional target manifold $\mathscr{M}$, a metric on the target manifold g and a closed three-form H, $dH=0$. The latter is known as the torsion or the NS-NS form and can be locally written as $H=d\,b$ with b the Kalb-Ramond two-form. The action is then given by[1],

$$\mathscr{S}=\int d^2\sigma\, \partial_{\ddagger}x^a\big(g_{ab}+b_{ab}\big)\partial_{=}x^b\,, \tag{2.1}$$

where x^a, $a\in\{1,\cdots,D\}$, is a set of local coordinates on $\mathscr{M}$.

Without adding any further geometric structure this action may be supersymmetrized provided that the number of supersymmetries N is bounded by $N\leq(1,1)$. In this paper we will exclusively focus on $N=(1,1)$ and $N=(2,2)$. The $N=(1,1)$ supersymmetric version of eq. (2.1) in $N=(1,1)$ superspace is simply,

$$\mathscr{S}=4\int d^2\sigma\, d^2\theta\, D_+x^a\,(g_{ab}+b_{ab})\,D_-x^b\,. \tag{2.2}$$

Integrating over θ^+ and θ^- we get the action in ordinary space,

$$\begin{aligned}\mathscr{S}=\int d^2\sigma\,\big(&\partial_{\ddagger}x^a(g_{ab}+b_{ab})\partial_{=}x^b+2i\,\psi_+^a\,g_{ab}\,\nabla^{(+)}_{=}\psi_+^b+2i\,\psi_-^a\,g_{ab}\,\nabla^{(-)}_{\ddagger}\psi_-^b\\ &+\psi_-^a\,\psi_-^b\,R^{(-)}_{abcd}\,\psi_+^c\,\psi_+^d+2(F^a-i\Gamma^{\;a}_{(-)cd}\,\psi_-^c\,\psi_+^d)g_{ab}(F^b-i\Gamma^{\;b}_{(-)ef}\,\psi_-^e\,\psi_+^f)\big)\,,\end{aligned} \tag{2.3}$$

[1]Our conventions are given in appendix A.

where we introduced covariant derivatives $\nabla^{(+)}$ and $\nabla^{(-)}$ which use the (Bismut) connections $\Gamma_{(+)}$ and $\Gamma_{(-)}$ resp.,

$$\Gamma_{(\pm)} \equiv \{\} \mp \frac{1}{2} g^{-1} H\,, \tag{2.4}$$

with $\{\}$ the standard Levi-Civita connection. The curvature tensor on $\mathscr{M}$ calculated with the $\Gamma_{(+)}$ and $\Gamma_{(-)}$ connection resp. is denoted by $R^{(+)}$ and $R^{(-)}$, they are related by $R^{(+)}_{abcd} = R^{(-)}_{cdab}$. From eq. (2.3) it is clear that the physical fields x^a and $\psi^a_\pm$ are supplemented with a set of auxiliary fields F^a.

We now investigate under which conditions the action eq. (2.2) exhibits an $N = (2,2)$ supersymmetry [1]-[3]. The most general expression we can write down for the two additional supersymmetries taking into account super-Poincaré invariance and dimensions is given by,

$$\delta x^a = \varepsilon^+ J^a_{+b}(x) D_+ x^b + \varepsilon^- J^a_{-b}(x) D_- x^b\,. \tag{2.5}$$

Calculating the algebra of these transformations yields,

$$\begin{aligned}
[\delta(\varepsilon_1), \delta(\varepsilon_2)] x^a = & -i\varepsilon_1^+\varepsilon_2^+ (J_+^2)^a{}_b \partial_{\mp} x^b - i\varepsilon_1^-\varepsilon_2^- (J_-^2)^a{}_b \partial_{=} x^b \\
& -2\varepsilon_1^+\varepsilon_2^+ \mathscr{N}[J_+,J_+]^a{}_{bc} D_+ x^b D_+ x^c - 2\varepsilon_1^-\varepsilon_2^- \mathscr{N}[J_-,J_-]^a{}_{bc} D_- x^b D_- x^c \\
& +\big(\varepsilon_1^+\varepsilon_2^- + \varepsilon_1^-\varepsilon_2^+\big)[J_+,J_-]^a{}_b \big(D_+D_- x^b + \Gamma^b_{(+)dc} D_+ x^c D_- x^d\big) + \cdots\,,
\end{aligned} \tag{2.6}$$

where we omitted terms linear in $\nabla^{(+)} J_+$ and $\nabla^{(-)} J_-$. We denoted the Nijenhuis tensor[2] by $\mathscr{N}$. The non closure terms in the last line of eq. (2.6) are proportional to the equation of motion for x following from the action in eq. (2.2) and closure of the algebra is generically only guaranteed on-shell. So one concludes that the transformations given in eq. (2.5) satisfy the supersymmetry algebra *on-shell* if J_+ and J_- are both complex structures, *i.e.* $J_+^2 = J_-^2 = -\mathbf{1}$ and $\mathscr{N}[J_\pm, J_\pm] = 0$. In addition the complex structures should be covariantly constant, $\nabla^{(+)} J_+ = \nabla^{(-)} J_- = 0$.

The action eq. (2.2) is invariant under eq. (2.5) provided that the metric is hermitian with respect to *both* complex structures, $g(J_\pm U, J_\pm V) = g(U,V)$ and, as with the algebra, the complex structures should be covariantly constant.

Given vectors U, V and W we can summarize the conditions under which the non-linear σ-model possesses an $N = (2,2)$ supersymmetry by:

- J_+ and J_- are two complex structures, *i.e.*,

$$J_+^2 = J_-^2 = -\mathbf{1}\,, \qquad [U,V] + J_\pm[J_\pm U, V] + J_\pm[U, J_\pm V] - [J_\pm U, J_\pm V] = 0\,. \tag{2.7}$$

- The metric g is hermitian w.r.t. both complex structures:

$$g(J_\pm U, J_\pm V) = g(U,V)\,. \tag{2.8}$$

[2] Out of two $(1,1)$ tensors $R^a{}_b$ and $S^a{}_b$, one constructs a $(1,2)$ tensor $\mathscr{N}[R,S]^a{}_{bc}$, the Nijenhuis tensor, as $\mathscr{N}[R,S]^a{}_{bc} = R^a{}_d S^d{}_{[b,c]} + R^d{}_{[b} S^a{}_{c],d} + R \leftrightarrow S$. For an almost complex structure J, the vanishing of $\mathscr{N}[J,J]$ is equivalent to $[U,V] + J[JU,V] + J[U,JV] - [JU,JV] = 0$ with U and V two vectors.

PoS(Corfu2012)097

- The exterior derivative of the two-forms $\omega_\pm(U,V) = -g(U,J_\pm V)$ is given by,

$$d\omega_\pm\big(U,V,W\big) = \mp H(J_\pm U,J_\pm V,J_\pm W)\,, \tag{2.9}$$

The last condition is equivalent – using the previous conditions – to the requirement that the complex structures are covariantly constant. A target space possessing these properties is often called a bi-hermitian geometry.

The non-closure terms in the algebra eq. (2.6) are proportional to the commutator of the complex structures, $[J_+,J_-]$, so we expect that no auxiliary fields will be needed along $\ker[J_+,J_-]$ while the requirement of off-shell closure will necessitate the introduction of additional auxiliary fields along $\mathrm{im}\big([J_+,J_-]g^{-1}\big)$. In section 4 we present a full off-shell formulation of the model by reformulating it in $N=(2,2)$ superspace. However we first introduce a geometric formalism in which the above structure finds a natural setting.

3. Generalized Kähler geometry and $N=(2,2)$ non-linear σ-models

Because of the presence of the Kalb-Ramond two-form b, a coordinate transformation often occurs in combination with a b gauge transformation. *E.g.* a coordinate transformation generated by a vector X acts as $g\to g+\mathscr{L}_X g$ and $H\to H+\mathscr{L}_X H$ on g and H. It is an isometry if g and H are invariant. Using that on forms[3] $\mathscr{L}_X = d\iota_x+\iota_X d$ holds, we find $\mathscr{L}_X H = d\mathscr{L}_X b$, so under the coordinate transformation we get that $b\to b+\mathscr{L}_X b+d\xi$ for some one-form ξ. Calculating the commutator of two such transformations generated by (X,ξ) and (Y,η) we find that it closes giving a transformation parameterized by $([X,Y],\mathscr{L}_X\eta-\mathscr{L}_Y\xi+\cdots)$ where the dots stands for closed terms.

This phenomenon was captured by Hitchin and Gualtieri in a natural generalization of Kähler geometry called generalized Kähler Geometry (GKG) [4]. The central object is the bundle $T\oplus T^*$, where T and T^* are the tangent and the cotangent bundle resp., on which one defines a symmetric bilinear pairing,

$$\langle\mathbb{X},\mathbb{Y}\rangle = \frac{1}{2}\left(\iota_X\eta+\iota_Y\xi\right), \tag{3.1}$$

where $\mathbb{X}=X+\xi$, $\mathbb{Y}=Y+\eta\ \in T\oplus T^*$. This pairing has a large isometry group which includes generalized coordinate transformations and b-transformations as subgroups. The latter act as,

$$\mathbb{X}\to e^b\,\mathbb{X} = \mathbb{X}+\iota_X b\,, \tag{3.2}$$

where b is a locally defined two-form. In addition this pairing supports a natural action of a bi-vector.

The Courant bracket replaces the Lie bracket,

$$[\![\mathbb{X},\mathbb{Y}]\!] = [X,Y]+\mathscr{L}_X\eta-\mathscr{L}_Y\xi-\frac{1}{2}d\big(\iota_X\eta-\iota_Y\xi\big)\,. \tag{3.3}$$

[3] We define the interior product so that the vector is always contracted with the first argument of the form. *I.e.* $\iota_X\omega=\omega(X,\cdots)$.

While anti-symmetric, the Courant bracket only satisfies the Jacobi identities on an isotropic subspace[4] of $T \oplus T^*$ provided it acts involutively on that isotropic subspace. The b-transformation eq. (3.2) acts as ,

$$[\![e^b(\mathbb{X}), e^b(\mathbb{Y})]\!] = e^b[\![\mathbb{X}, \mathbb{Y}]\!]_{db}\,, \tag{3.4}$$

where $[\![\mathbb{X}, \mathbb{Y}]\!]_{db} = [\![\mathbb{X}, \mathbb{Y}]\!] + \iota_Y \iota_X db$. Thus the isometries of the of the inner product which hold as symmetries of the Courant bracket consist of $GL(D)$ transformations (evident from the index free formulation) and the action of closed two-forms.

The spinor representation of the isometry group of eq. (3.1) is realized on the exterior algebra on T^*. For $\mathbb{X} = X + \xi \in T \oplus T^*$ we introduce a "gamma matrix" $\Gamma_{\mathbb{X}}$ which act on $\phi \in \wedge^\bullet T^*$ as,

$$\Gamma_{\mathbb{X}} \cdot \phi = \iota_X \phi + \xi \wedge \phi\,. \tag{3.5}$$

A direct check shows that the poly-forms $\wedge^\bullet T^*$ do indeed provide a module for the Clifford algebra since:

$$\big\{\Gamma_{\mathbb{X}}, \Gamma_{\mathbb{Y}}\big\} \cdot \phi = 2\langle \mathbb{X}, \mathbb{Y}\rangle\, \phi\,. \tag{3.6}$$

Having the gamma matrices, one gets the spin representation of the isometry group of the bilinear form in the usual way. *E.g.* for the b transform, see eq. (3.2), one gets,

$$\phi \to e^{-b\wedge}\, \phi\,. \tag{3.7}$$

Similarly a spinor transforms under a coordinate transformation as a density,

$$x \to x'(x) \Rightarrow \phi(x) \to \phi'(x') = \sqrt{\det \frac{\partial x'}{\partial x}}\, \phi(x)\,. \tag{3.8}$$

In fact this transformation law implies the isomorphism between spinors and poly-forms is actually given by

$$S \cong (\det T)^{\frac{1}{2}} \wedge^\bullet T^*\,. \tag{3.9}$$

The Mukai pairing gives an invariant (under the isometry group of $\langle \cdot, \cdot \rangle$ connected to the identity) bilinear for the spinors,

$$\big(\phi_1, \phi_2\big) = \sigma(\phi_1) \wedge \phi_2|_{\mathrm{top}}\,, \tag{3.10}$$

where $\phi_1,\, \phi_2 \in \wedge^\bullet T^*$ and σ acts as,

$$\sigma\big(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^r\big) = dx^r \wedge dx^{r-1} \wedge \cdots \wedge dx^1\,. \tag{3.11}$$

That this bilinear produces a top-form rather than a scalar is reflected by the isomorphism eq. (3.9).

[4]An isotropic subspace $L \subset T \oplus T^*$ is defined by $\forall \mathbb{X},\, \mathbb{Y} \in L : \langle \mathbb{X}, \mathbb{Y}\rangle = 0$. If the dimension of $T \oplus T^*$ is $2m$, then the maximal dimension of L is given by m. Whenever the dimension of L is m, we talk about a maximal isotropic subspace. A simple but important example of a maximal isotropic is $T \subset T \oplus T^*$.

PoS(Corfu2012)097

There is a natural way to associate an isotropic subspace L of $T \oplus T^*$ to a given spinor ϕ:

$$\mathbb{X} \in L \Leftrightarrow \Gamma_{\mathbb{X}} \cdot \phi = 0\,. \tag{3.12}$$

Using eq. (3.6) one immediately shows that L is indeed isotropic. When L is maximally isotropic one calls ϕ a *pure spinor*. As shown by Gualtieri, any non-degenerate pure spinor can be written in the form,

$$\phi = \kappa \wedge e^{i\Omega + \Xi}\,, \tag{3.13}$$

where Ω and Ξ are real two-forms and κ is a complex decomposable k-form where k is the type of the generalized complex structure.

A *generalized complex structure* (GCS) is defined as a linear map $\mathcal{J} : T \oplus T^* \to T \oplus T^*$, satisfying $\mathcal{J}^2 = -1$, for which the natural pairing is "hermitian", $\langle \mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}\rangle = \langle \mathbb{X}, \mathbb{Y}\rangle$ for all $\mathbb{X}, \mathbb{Y} \in T \oplus T^*$ and for which the $+i$ eigenbundle is involutive under the Courant bracket,

$$[\![\mathbb{X},\mathbb{Y}]\!] + \mathcal{J}[\![\mathcal{J}\mathbb{X},\mathbb{Y}]\!] + \mathcal{J}[\![\mathbb{X},\mathcal{J}\mathbb{Y}]\!] - [\![\mathcal{J}\mathbb{X},\mathcal{J}\mathbb{Y}]\!] = 0. \tag{3.14}$$

In order for a GCS to exist the manifold $\mathcal{M}$ should be even dimensional and we take $\dim \mathcal{M} = 2m$.

Note that if $\mathcal{J}$ is integrable w.r.t. the bracket $[\![\cdot,\cdot]\!]$ then $e^{-b}\mathcal{J}e^{b}$ is integrable w.r.t. the bracket $[\![\cdot,\cdot]\!]_{db}$. Writing $(T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L}$ where L ($\bar{L}$) is the $+i$ ($-i$) eigenbundle of $\mathcal{J}$, we see that L is a maximal isotropic subspace of $(T \oplus T^*) \otimes \mathbb{C}$. As a consequence, any GCS comes with a pure spinor ϕ defined by $\Gamma_{\mathbb{X}} \cdot \phi = 0$, $\forall \mathbb{X} \in L$. For a physicist this is just the highest weight vector of the spinor representation.

Both a regular complex structure J and a symplectic structure (a closed non-degenerate two-form) ω give rise to a GCS,

$$\mathcal{J}_c = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \qquad \mathcal{J}_s = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{3.15}$$

A generic GCS interpolates between these two extreme cases. Gualtieri extended the Newlander-Nirenberg (for complex structures) and the Darboux theorem (for symplectic structures) to an arbitrary GCS: by an appropriate diffeomorphism and b-transformation one can always turn a GCS to the standard product GCS $\mathbb{C}^k \times (\mathbb{R}^{2m-2k}, \omega)$. The integer k is called the *type* of the GCS. So a $2m$ dimensional manifold with a generalized complex structure is foliated by $2m-2k$ dimensional leaves of the form $\mathbb{R}^{2m-2k} \times \{\text{point}\}$ on which a symplectic form ω is properly defined. Transverse to the leaves, complex coordinates z_i with $i \in \{1,\dots,k\}$ are introduced such that the leaves are located at $z_i = \text{constant}\,(\forall i)$. A generic feature of generalized complex geometry is that loci might exist where the type jumps, one calls this phenomenon *type changing*. Gualtieri's theorem holds for neighborhoods of regular points (*i.e.* where no type changing occurs).

A *generalized Kähler structure* (GKS) requires two mutually commuting GCS's, $\mathcal{J}_+$ and $\mathcal{J}_-$ such that,

$$\mathcal{G}\big(\mathbb{X},\mathbb{Y}\big) = \langle \mathcal{J}_+\mathbb{X}, \mathcal{J}_-\mathbb{Y}\rangle, \tag{3.16}$$

defines a positive definite metric on $T\oplus T^*$. Gualtieri showed that a GKS is exactly equivalent to the previously introduced $N=(2,2)$ non-linear σ-model gemometry (see section 2),

$$\mathscr{J}_\pm=\frac{1}{2}\begin{pmatrix}1&0\\-b&1\end{pmatrix}\begin{pmatrix}J_+\pm J_-&\omega_+^{-1}\mp\omega_-^{-1}\\-(\omega_+\mp\omega_-)&-(J_+^t\pm J_-^t)\end{pmatrix}\begin{pmatrix}1&0\\+b&1\end{pmatrix}. \tag{3.17}$$

We call the types of the generalized complex structures $\mathscr{J}_+$ and $\mathscr{J}_-$, k_+ and k_- resp. From eq. (3.17) we get that

$$\big(k_+,k_-\big)=\frac{1}{2}\big(\dim\ker(J_+-J_-),\dim\ker(J_++J_-)\big). \tag{3.18}$$

As $\mathscr{J}_+$ and $\mathscr{J}_-$ commute, we can write,

$$\big(T\oplus T^*\big)\otimes\mathbb{C}=L_{++}\oplus L_{+-}\oplus L_{--}\oplus L_{-+}, \tag{3.19}$$

where L_{++} is the $+i$ eigenbundle for both $\mathscr{J}_+$ and $\mathscr{J}_-$; L_{+-} is the $+i$ eigenbundle for $\mathscr{J}_+$ and the $-i$ eigenbundle for $\mathscr{J}_-$; etc. Denoting $\mathbb{X}_+\in L_{++}$, $\mathbb{X}_-\in L_{+-}$, $\bar{\mathbb{X}}_+\in L_{--}$ and $\bar{\mathbb{X}}_-\in L_{-+}$, we can introduce pure spinors ϕ_+ and ϕ_- for $\mathscr{J}_+$ and $\mathscr{J}_-$ resp. defined by,

$$\Gamma_{\mathbb{X}_+}\cdot\phi_+=\Gamma_{\mathbb{X}_-}\cdot\phi_+=0,\qquad \Gamma_{\mathbb{X}_+}\cdot\phi_-=\Gamma_{\bar{\mathbb{X}}_-}\cdot\phi_-=0. \tag{3.20}$$

Eq. (3.20) does not fix the normalization of the pure spinors ϕ_+ and ϕ_-. It can be shown [4] that the integrability of the generalized complex structures guarantees the existence of $\mathbb{Y}_+$ and $\mathbb{Y}_-$ such that,

$$d\phi_\pm=\Gamma_{\mathbb{Y}_\pm}\cdot\phi_\pm\,. \tag{3.21}$$

Finally, a *generalized Calabi-Yau geometry* is a generalized Kähler geometry for which the pure spinors ϕ_+ and ϕ_- are *globally defined*, *closed* and they satisfy,

$$\big(\phi_+,\bar\phi_+\big)=c\,\big(\phi_-,\bar\phi_-\big)\neq 0, \tag{3.22}$$

with c constant.

4. The off-shell realization of $N=(2,2)$ supersymmetry

We now pass to $N=(2,2)$ superspace. Because of dimensional reasons the Lagrange density for the $N=(2,2)$ non-linear σ-model will be a function of scalar superfields so all dynamics will arise from the constraints satisfied by the superfields. We will only consider constraints linear in the derivatives. The maximal set of constraints we can impose on a set of scalar superfields x^a consistent with dimensions and super-Lorentz invariance is [5],

$$\hat D_+x^a=J^a_{+b}(x)D_+x^b\,,\qquad \hat D_-x^a=J^a_{-b}(x)D_-x^b\,. \tag{4.1}$$

Eqs. (A.4) then imply integrability conditions which state that J_+ and J_- are complex structures, *i.e.* $J_+^2=J_-^2=-\mathbf{1}$ and $\mathscr{N}[J_\pm,J_\pm]=0$. From $\{\hat D_+,\hat D_-\}=0$ follows that J_+ and J_- commute, $[J_+,J_-]=0$ and that $\mathscr{M}[J_+,J_-]$ vanishes[5].

[5] Out of two *commuting* $(1,1)$ tensors R and S one constructs the tensor $\mathscr{M}[R,S]$, $\mathscr{M}[R,S](U,V)=[RU,SV]-R[U,SV]-S[RU,V]+RS[U,V]$. It is related to the Nijenhuis tensor by $\mathscr{N}[R,S](U,V)=\frac{1}{2}\big(\mathscr{M}[R,S](U,V)+\mathscr{M}[S,R](U,V)\big)$.

So J_+ and J_- are mutual commuting complex structures which can be simultaneously diagonalized. This gives rise to two cases: either both J_+ and J_- have the same eigenvalue or they have opposite eigenvalues. In the first case we call them **chiral superfields**, in a complex basis they satisfy,

$$\bar{\mathbb{D}}_+ z = \bar{\mathbb{D}}_- z = 0\,, \qquad \mathbb{D}_+ \bar{z} = \mathbb{D}_- \bar{z} = 0\,. \tag{4.2}$$

The latter case gives rise to **twisted chiral superfields** [1] satisfying,

$$\bar{\mathbb{D}}_+ w = \mathbb{D}_- w = 0\,, \qquad \mathbb{D}_+ \bar{w} = \bar{\mathbb{D}}_- \bar{w} = 0\,. \tag{4.3}$$

The only possibility left is constraining a single chirality. In order to get a σ-model action they need to come in pairs satisfying constraints of opposite chirality. They are called **semi-chiral superfields** [6] and they are defined by,

$$\bar{\mathbb{D}}_+ l = \bar{\mathbb{D}}_- r = 0, \qquad \mathbb{D}_+ \bar{l} = \mathbb{D}_- \bar{r} = 0, \tag{4.4}$$

An unconstrained $N=(2,2)$ superfield consists of four independent $N=(1,1)$ superfields. When dealing with chiral or twisted chiral superfields, the constraints reduce the number of components of an $N=(2,2)$ superfield to those of an $N=(1,1)$ superfield. A semi-chiral superfield describes twice as many degrees of freedom compared to an $N=(1,1)$ superfield, half of which will turn out to be auxiliary.

Decomposing the tangent space as $T=\ker(J_+-J_-)\oplus\ker(J_++J_-)\oplus\mathrm{im}\big([J_+,J_-]g^{-1}\big)$, one can show that $\ker(J_--J_-)$, $\ker(J_++J_-)$ and $\mathrm{im}\big([J_+,J_-]g^{-1}\big)$ can be integrated to chiral, twisted chiral and semi-chiral fields resp. (this was conjectured in [7] and proven in [8]).

The most general action involving these superfields in $N=(2,2)$ superspace is given by,

$$\mathscr{S} = 4\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\, V(l,\bar{l},r,\bar{r},w,\bar{w},z,\bar{z}), \tag{4.5}$$

where the Lagrange density $V(l,\bar{l},r,\bar{r},w,\bar{w},z,\bar{z})$ is an arbitrary real function of the semi-chiral, $l^{\tilde{\alpha}}$, $l^{\bar{\tilde{\alpha}}}$, $r^{\tilde{\mu}}$, $r^{\bar{\tilde{\mu}}}$, $\tilde{\alpha}$, $\bar{\tilde{\alpha}}$, $\tilde{\mu}$, $\bar{\tilde{\mu}}\in\{1,\cdots n_s\}$, the twisted chiral, w^{μ}, $w^{\bar{\mu}}$, μ, $\bar{\mu}\in\{1,\cdots n_t\}$, and the chiral superfields, z^{α}, $z^{\bar{\alpha}}$, α, $\bar{\alpha}\in\{1,\cdots n_c\}$. It is defined modulo a generalized Kähler transformation,

$$V \to V + F(l,w,z) + \bar{F}(\bar{l},\bar{w},\bar{z}) + G(\bar{r},w,\bar{z}) + \bar{G}(r,\bar{w},z). \tag{4.6}$$

As for the usual Kähler case these generalized Kähler transformations are essential for the global consistency of the model, see *e.g.* [9]. Finally, there is the local mirror transformation which sends J_- to $-J_-$ which at the level of the generalized Kähler potential amounts to,

$$V(l,\bar{l},r,\bar{r},w,\bar{w},z,\bar{z}) \to V(l,\bar{l},\bar{r},r,z,\bar{z},w,\bar{w})\,. \tag{4.7}$$

Before proceeding, we introduce some notation. We write,

$$M_{AB} = \begin{pmatrix} V_{ab} & V_{a\bar{b}} \\ V_{\bar{a}b} & V_{\bar{a}\bar{b}} \end{pmatrix}, \tag{4.8}$$

where, $(A,a) \in \{(l,\tilde{\alpha}), (r,\tilde{\mu}), (w,\mu), (z,\alpha)\}$ and $(B,b) \in \{(l,\tilde{\beta}), (r,\tilde{\nu}), (w,\nu), (z,\beta)\}$. The subindices on V denote derivatives with respect to those coordinates. In this way *e.g.* we get that M_{wl} is the $2n_t \times 2n_s$ matrix given by,

$$M_{wl} = \begin{pmatrix} V_{\mu\tilde{\beta}} & V_{\mu\bar{\tilde{\beta}}} \\ V_{\bar{\mu}\tilde{\beta}} & V_{\bar{\mu}\bar{\tilde{\beta}}} \end{pmatrix}. \tag{4.9}$$

We also introduce the matrix $\mathbb{J}$,

$$\mathbb{J} \equiv i \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{4.10}$$

with $\mathbf{1}$ the unit matrix and using this we write,

$$C_{AB} \equiv \mathbb{J} M_{AB} - M_{AB}\mathbb{J}, \qquad A_{AB} \equiv \mathbb{J} M_{AB} + M_{AB}\mathbb{J}. \tag{4.11}$$

Starting from the action eq. (4.5) one passes to $N=(1,1)$ superspace and eliminates the auxiliary fields. Comparing this to eqs. (4.1) and (2.2) one gets explicit expressions for the complex structures, the metric and the Kalb-Ramond two-form. The complex structures are given by,

$$\begin{aligned} J_+ &= \begin{pmatrix} \mathbb{J} & 0 & 0 & 0 \\ M_{lr}^{-1}C_{ll} & M_{lr}^{-1}\mathbb{J}M_{lr} & M_{lr}^{-1}C_{lw} & M_{lr}^{-1}C_{lz} \\ 0 & 0 & \mathbb{J} & 0 \\ 0 & 0 & 0 & \mathbb{J} \end{pmatrix}, \\ J_- &= \begin{pmatrix} M_{rl}^{-1}\mathbb{J}M_{rl} & M_{rl}^{-1}C_{rr} & M_{rl}^{-1}A_{rw} & M_{rl}^{-1}C_{rz} \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & -\mathbb{J} & 0 \\ 0 & 0 & 0 & \mathbb{J} \end{pmatrix}, \end{aligned} \tag{4.12}$$

where we labeled rows and columns in the order l, $\bar{l}$, r, $\bar{r}$, w, $\bar{w}$, z, $\bar{z}$. Note that once semi-chiral fields are present, neither of the complex structures is diagonal. One easily shows [7] that making a coordinate transformation,

$$r^{\tilde{\mu}} \to V_{\tilde{\alpha}}\,, \qquad r^{\bar{\tilde{\mu}}} \to V_{\bar{\tilde{\alpha}}}\,, \tag{4.13}$$

while keeping the other coordinates unchanged diagonalizes J_+. Similarly, the coordinate transformation,

$$l^{\tilde{\alpha}} \to V_{\tilde{\mu}}\,, \qquad l^{\bar{\tilde{\alpha}}} \to V_{\bar{\tilde{\mu}}}\,, \tag{4.14}$$

with the other coordinates fixed diagonalizes J_-. This observation led to a reinterpretation in [8] of the generalized Kähler potential as the generating function for a canonical transformation which interpolates between the coordinate system in which J_+ is diagonal and the one in which J_- is diagonal.

From the second order action one reads off the metric g and the torsion two-form potential b. One finds two natural expressions,

$$\begin{aligned}\big(g-b_+\big)(X,Y)&=\Omega^+(X,J_+Y)=dB^+(X,J_+Y)\,,\\ \big(g+b_-\big)(X,Y)&=\Omega^-(X,J_-Y)=dB^-(X,J_-Y)\,.\end{aligned}\tag{4.15}$$

where Ω^+ and Ω^- are two (locally defined) closed two-forms linear in the generalized Kähler potential and $H=db_+=db_-$. From eq. (4.15) one gets that $b_\pm$ are (2,0)+(0,2) forms w.r.t. $J_\pm$. Explicitly we get,

$$\Omega^+=-\frac{1}{2}\begin{pmatrix}C_{ll}&A_{lr}&C_{lw}&A_{lz}\\-A_{rl}&-C_{rr}&-A_{rw}&-C_{rz}\\C_{wl}&A_{wr}&C_{ww}&A_{wz}\\-A_{zl}&-C_{zr}&-A_{zw}&-C_{zz}\end{pmatrix},\qquad \Omega^-=\frac{1}{2}\begin{pmatrix}C_{ll}&C_{lr}&C_{lw}&C_{lz}\\C_{rl}&C_{rr}&C_{rw}&C_{rz}\\C_{wl}&C_{wr}&C_{ww}&C_{wz}\\C_{zl}&C_{zr}&C_{zw}&C_{zz}\end{pmatrix}.\tag{4.16}$$

where we labeled rows and columns in the order $(l,\bar l,r,\bar r,w,\bar w,z,\bar z)$ and locally we get $\Omega^\pm=dB^\pm$ where[6],

$$\begin{aligned}2B^+&=iV_l\,dl-iV_{\bar l}\,d\bar l-iV_r\,dr+iV_{\bar r}\,d\bar r+iV_w\,dw-iV_{\bar w}\,d\bar w-iV_z\,dz+iV_{\bar z}\,d\bar z\,,\\ 2B^-&=-iV_l\,dl+iV_{\bar l}\,d\bar l-iV_r\,dr+iV_{\bar r}\,d\bar r-iV_w\,dw+iV_{\bar w}\,d\bar w-iV_z\,dz+iV_{\bar z}\,d\bar z\,.\end{aligned}\tag{4.17}$$

There is a certain freedom in the definition of $\Omega^\pm$,

$$\Omega^\pm(X,Y)\simeq\Omega^\pm(X,Y)+d\xi^\pm(X,J_\pm Y)\,,\tag{4.18}$$

where $d\xi^+$ and $d\xi^-$ are $(2,0)+(0,2)$ forms w.r.t. J_+ and J_- resp. Examples of this are $\xi^+=V_l\,dl+V_{\bar l}\,d\bar l$, $\xi^+=iV_l\,dl-iV_{\bar l}\,d\bar l$, $\xi^-=V_r\,dr+V_{\bar r}\,d\bar r$ and $\xi^-=iV_r\,dr-iV_{\bar r}\,d\bar r$. For the choice in eq. (4.16) one gets that when no chiral fields are present we can write Ω^+ as [10],

$$\Omega^+(X,Y)=2\,g\big(X,(J_+-J_-)^{-1}Y\big)\,.\tag{4.19}$$

Similarly, when there are no twisted chiral fields we get,

$$\Omega^-(X,Y)=2\,g\big(X,(J_++J_-)^{-1}Y\big)\,.\tag{4.20}$$

Note that these expressions only exist in regular neighborhoods, at loci where type changing occurs they might not exist. Using the previous expressions one also finds,

$$b_--b_+=\frac{1}{2}d\big(-V_l\,dl-V_{\bar l}\,d\bar l+V_r\,dr+V_{\bar r}\,d\bar r-V_w\,dw-V_{\bar w}\,d\bar w+V_z\,dz+V_{\bar z}\,d\bar z\big)\,.\tag{4.21}$$

In [11] the 1-loop β-function for a general $N=(2,2)$ σ-model in superspace was calculated. The counter term which was found reads,

$$\mathscr{S}_{1-\text{loop}}\sim\frac{1}{\varepsilon}\int d^2\sigma\,d^2\theta\,d^2\hat\theta\,\ln\frac{\det\big(N_+\big)}{\det\big(N_-\big)}\,,\tag{4.22}$$

[6] The notation we use is such that $V_l\,dl$ stands for $\partial_{l^{\tilde\alpha}}V\,dl^{\tilde\alpha}$, $V_{\bar l}\,d\bar l$ for $\partial_{l^{\bar{\tilde\alpha}}}V\,dl^{\bar{\tilde\alpha}}$, etc.

where,

$$N_+ = \begin{pmatrix} V_{l\bar{l}} & V_{lr} & V_{l\bar{w}} \\ V_{\bar{r}\bar{l}} & V_{\bar{r}r} & V_{\bar{r}\bar{w}} \\ V_{w\bar{l}} & V_{wr} & V_{w\bar{w}} \end{pmatrix}, \tag{4.23}$$

and,

$$N_- = \begin{pmatrix} V_{l\bar{l}} & V_{l\bar{r}} & V_{l\bar{z}} \\ V_{r\bar{l}} & V_{r\bar{r}} & V_{r\bar{z}} \\ V_{z\bar{l}} & V_{z\bar{r}} & V_{z\bar{z}} \end{pmatrix}. \tag{4.24}$$

It vanishes if,

$$\frac{\det\left(N_+\right)}{\det\left(N_-\right)} = \pm|f_+(l,w,z)|^2|f_-(r,\bar{w},z)|^2, \tag{4.25}$$

for some functions f_+ and f_-.

Let us now return to the generalized Kähler geometry. The closed pure spinors, eqs. (3.20), are given by [12], [13],

$$\begin{aligned} \phi_+ &= d\bar{z}^1\wedge d\bar{z}^2\wedge\cdots\wedge d\bar{z}^{n_c}\wedge e^{i\Omega^+ + \Xi^+}, \\ \phi_- &= d\bar{w}^1\wedge d\bar{w}^2\wedge\cdots\wedge d\bar{w}^{n_t}\wedge e^{i\Omega^- + \Xi^-}, \end{aligned} \tag{4.26}$$

where $\Omega^\pm$ are given in eq. (4.16) and $\Xi^\pm$ are given by,

$$\begin{aligned} \Xi^+ &= \frac{1}{2}d\big(V_l\,dl + V_{\bar{l}}\,d\bar{l} - V_z\,dz - V_{\bar{z}}\,d\bar{z}\big), \\ \Xi^- &= \frac{1}{2}d\big(V_r\,dr + V_{\bar{r}}\,d\bar{r} - V_w\,dw - V_{\bar{w}}\,d\bar{w}\big). \end{aligned} \tag{4.27}$$

Using this one gets the Mukai pairings,

$$\begin{aligned} \big(\phi_+,\bar{\phi}_+\big) &= (-1)^{n_c(n_c+1)/2+n_t+n_s}\,2^{n_t+2n_s}\,\det N_+ \\ \big(\phi_-,\bar{\phi}_-\big) &= (-1)^{n_t(n_t+1)/2}\,2^{n_c+2n_s}\,\det N_-, \end{aligned} \tag{4.28}$$

from which one gets the generalized Calabi-Yau condition eq. (3.22),

$$\frac{\det\left(N_+\right)}{\det\left(N_-\right)} = \text{constant } \neq 0. \tag{4.29}$$

It is clear that eq. (4.29) is stronger than eq. (4.25) which reflects the fact that the vanishing of the β-functions is necessary but not sufficient for the background to solve the supergravity equations of motion. For a more detailed discussion see *e.g.* [12] and references therein.

5. Duality transformations

In $N = (2,2)$ supersymmetric models there exists a variety of duality transformations which allows one to change the nature of the superfields. A complete catalogue of duality transformations in $N=(2,2)$ superspace was obtained in [14], the extension of this to the case where boundaries are present was developed in [15]. A first class of duality transformations is always possible and is a consequence of the constraints satisfied by $N=(2,2)$ superfields. One imposes the constraints on the superfields through Lagrange multipliers (unconstrained superfields). In such a first order formulation the original fields are then unconstrained superfields. Integrating over the Lagrange multipliers results in the original model; integrating over the original unconstrained fields yields the dual formulation. In this way a (twisted) chiral field is dual to a (twisted) complex linear superfield, so the dual models involve superfields satisfying constraints which are quadratic in the derivatives, a case we do not consider any further here. More interesting are semi-chiral fields where one has a total of 4 dual formulations.

The starting point is the first order action,

$$\mathscr{S} = -\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\, \Big(V(l,\bar{l},r,\bar{r},\cdots) - \Lambda^+\,\bar{\mathbb{D}}_+ l - \bar{\Lambda}^+\,\mathbb{D}_+\bar{l} - \Lambda^-\,\bar{\mathbb{D}}_- r - \bar{\Lambda}^-\,\mathbb{D}_-\bar{r}\Big), \quad (5.1)$$

where l, $\bar{l}$, r and $\bar{r}$ are unconstrained bosonic complex superfields and $\Lambda^\pm$ and $\bar{\Lambda}^\pm$ are unconstrained complex fermionic superfields. Integrating over the Lagrange multipliers constrains l and r to form a semi-chiral multiplet. Upon partial integration we can rewrite the action in three ways,

$$\begin{aligned}\mathscr{S} &= -\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\, \Big(V(l,\bar{l},r,\bar{r},\cdots) - l\,l' - \bar{l}\,\bar{l}' - \Lambda^-\,\bar{\mathbb{D}}_- r - \bar{\Lambda}^-\,\mathbb{D}_-\bar{r}\Big)\\ &= -\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\, \Big(V(l,\bar{l},r,\bar{r},\cdots) - \Lambda^+\,\bar{\mathbb{D}}_+ l - \bar{\Lambda}^+\,\mathbb{D}_+\bar{l} - r\,r' - \bar{r}\,\bar{r}'\Big)\\ &= -\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\, \Big(V(l,\bar{l},r,\bar{r},\cdots) - l\,l' - \bar{l}\,\bar{l}' - r\,r' - \bar{r}\,\bar{r}'\Big), \end{aligned} \quad (5.2)$$

where we introduced the notation $l' = \bar{\mathbb{D}}_+\Lambda^+$, $\bar{l}' = \mathbb{D}_+\bar{\Lambda}^+$, $r' = \bar{\mathbb{D}}_-\Lambda^-$, $\bar{r}' = \mathbb{D}_-\bar{\Lambda}^-$. Integrating over the unconstrained fields $(l,\bar{l},\Lambda^-,\bar{\Lambda}^-)$, $(\Lambda^+,\bar{\Lambda}^+,r,\bar{r})$ or $(l,\bar{l},r,\bar{r})$ resp. yields three dual formulations of the model. This is perfectly consistent with the interpretation of the generalized Kähler potential as the generating function for a canonical transformation interpolating between coordinates where J_+ or J_- is diagonal as the generating function of a canonical transformation also exists in four versions connected by Legendre transformations.

In fact when considering the global picture of an $N=(2,2)$ model one should allow on the overlap of two coordinate patches not only for a generalized Kähler transformation eq. (4.6) but for Legendre transformations of the potential as well. We will see an example of this in section 6.

A second class of dualities consists of T-dualities where the $N=(2,2)$ supersymmetry is kept manifest. These dualities require the existence of an isometry and the idea is to gauge the isometry while – using Lagrange multipliers – enforcing the gauge fields to be pure gauge. Integrating over the Lagrange multipliers returns us to the original model while integrating over the gauge fields yields the dual model. There are three main cases.

5.1 The duality between a pair of chiral and twisted chiral fields and a semi-chiral multiplet

A first class of T-duality transformations exchanges a pair of chiral and twisted chiral superfields for a semi-chiral multiplet or vice-versa[7]. The starting point is a potential of the form

[7] While this duality transformation was already found in [14], the elucidation of the underlying gauge structure was done in [16].

$V\big(z+\bar z, w+\bar w, i(z-\bar z-w+\bar w),\cdots\big)$. This clearly exhibits the isometry $z \to z+ia$, $w \to w+ia$, with a an arbitrary real constant. The first order action is,

$$\begin{aligned}\mathscr{S} = &-\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(Y,\tilde Y,\hat Y,\cdots\big)+\Lambda^+\bar{\mathbb{D}}_+\big(Y-\tilde Y-i\hat Y\big)+\bar\Lambda^+\mathbb{D}_+\big(Y-\tilde Y+i\hat Y\big)\\ &-\Lambda^-\bar{\mathbb{D}}_-\big(Y+\tilde Y-i\hat Y\big)-\bar\Lambda^-\mathbb{D}_-\big(Y+\tilde Y+i\hat Y\big)\Big)\end{aligned} \tag{5.3}$$

where $\Lambda^\pm$ and $\bar\Lambda^\pm$ are unconstrained complex fermionic superfields and Y, $\tilde Y$ and $\hat Y$ are unconstrained real bosonic superfields. Integrating over the Lagrange multipliers $\Lambda^\pm$ and $\bar\Lambda^\pm$ returns us to the original model. Integrating by parts results in,

$$\begin{aligned}\mathscr{S} = &-\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(Y,\tilde Y,\hat Y,\cdots\big)+Y\big(l+\bar l-r-\bar r\big)-\tilde Y\big(l+\bar l+r+\bar r\big)\\ &-i\hat Y\big(l-\bar l-r+\bar r\big)\Big)\end{aligned} \tag{5.4}$$

where we introduced the semi-chiral multiplet $l=\bar{\mathbb{D}}_+\Lambda^+$, $\bar l=\mathbb{D}_+\bar\Lambda^+$, $r=\bar{\mathbb{D}}_-\Lambda^-$ and $\bar r=\mathbb{D}_-\bar\Lambda^-$. Integrating over Y, $\tilde Y$ and $\hat Y$ yields the dual model.

The inverse transformation starts from a potential of the form $V\big(l+\bar l, r+\bar r, i(l-\bar l-r+\bar r),\cdots\big)$. The basic relations are given by,

$$\begin{aligned}\mathscr{S} = &-\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(Y,\tilde Y,\hat Y,\cdots\big)+iu\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\big(Y-\tilde Y-i\hat Y\big)\\ &+i\bar u\mathbb{D}_+\mathbb{D}_-\big(Y-\tilde Y+i\hat Y\big)-iv\bar{\mathbb{D}}_+\mathbb{D}_-\big(Y+\tilde Y-i\hat Y\big)-i\bar v\mathbb{D}_+\bar{\mathbb{D}}_-\big(Y+\tilde Y+i\hat Y\big)\Big)\\ = &-\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(Y,\tilde Y,\hat Y,\cdots\big)+Y\big(z+\bar z-w-\bar w\big)-\tilde Y\big(z+\bar z+w+\bar w\big)\\ &-i\hat Y\big(z-\bar z-w+\bar w\big)\Big)\end{aligned} \tag{5.5}$$

where $u, v\in\mathbb{C}$ and $Y, \tilde Y, \hat Y\in\mathbb{R}$ are unconstrained superfields and where we defined $z=i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-u$, $\bar z=i\mathbb{D}_+\mathbb{D}_-\bar u$, $w=i\bar{\mathbb{D}}_+\mathbb{D}_-v$ and $\bar w=i\mathbb{D}_+\bar{\mathbb{D}}_-\bar v$.

5.2 The duality between a chiral and a twisted chiral field

The duality transformation between a chiral and a twisted chiral superfield was proposed in [1]. Starting from a potential of the form $V(z+\bar z,\cdots)$ we write the first order action,

$$\mathscr{S} = -\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(Y,\cdots\big)-iu\bar{\mathbb{D}}_+\mathbb{D}_-Y-i\bar u\mathbb{D}_+\bar{\mathbb{D}}_-Y\Big). \tag{5.6}$$

Integrating over the complex unconstrained Lagrange multipliers u and $\bar u$ brings us back to the original model. Upon integrating by parts one gets,

$$\mathscr{S} = -\int d^2\sigma\, d^2\theta\, d^2\theta'\,\Big(V\big(Y,\cdots\big)-Y\big(w+\bar w\big)\Big), \tag{5.7}$$

where we introduced the twisted chiral fields $w=i\bar{\mathbb{D}}_+\mathbb{D}_-u$ and $\bar w=i\mathbb{D}_+\bar{\mathbb{D}}_-\bar u$. Integrating over the unconstrained gauge field Y gives the dual model in terms of a twisted chiral field w.

The inverse transformation starts from potentials of the form $V(w+\bar w,\cdots)$. One has

$$\begin{aligned}&-\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(\tilde Y,\cdots\big)-iu\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde Y-i\bar u\mathbb{D}_+\mathbb{D}_-\tilde Y\Big)\\ &=-\int d^2\sigma\, d^2\theta\, d^2\hat\theta\,\Big(V\big(\tilde Y,\cdots\big)-\tilde Y\big(z+\bar z\big)\Big),\end{aligned} \tag{5.8}$$

where we have put $z=i\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-u$ and $\bar z=i\mathbb{D}_+\mathbb{D}_-\bar u$.

6. The Hopf surface: a canonical example

Wess-Zumino-Witten models on even dimensional reductive Lie group manifolds all allow for at least $N = (2,2)$ supersymmetry [3]. A complex structure on the group is fully determined by its action on the Lie algebra and it is almost equivalent to a choice for a Cartan decomposition of the Lie algebra. Indeed on generators corresponding to positive (negative) roots it has eigenvalue $+i$ $(-i)$ and the only freedom left is on its action on the Cartan subalgebra (CSA) which is mapped to itself and restricted by the condition that the invariant metric on the CSA is hermitian.

The group $SU(2)\times U(1)$ is the canonical example. As this model actually has an $N = (4,4)$ supersymmetry it has two 2-spheres of complex structures, one for each chirality. A detailed analysis [13] shows that this gives rise to two superspace formulations of the model: one in terms of a chiral and a twisted chiral superfield [17] and one in terms of a semi-chiral multiplet [7].

We parameterize a group element $g \in SU(2)\times U(1)$,

$$g = e^{i\rho}\begin{pmatrix} \cos\psi\, e^{i\varphi_1} & \sin\psi\, e^{i\varphi_2} \\ -\sin\psi\, e^{-i\varphi_2} & \cos\psi\, e^{-i\varphi_1} \end{pmatrix}, \tag{6.1}$$

where $\varphi_1,\,\varphi_2,\,\rho \in \mathbb{R}\,\mathrm{mod}\,2\pi$ and $\psi \in [0,\pi/2]$. The group manifold $S^3\times S^1$ can also be viewed as a rational Hopf surface defined by $\big(\mathbb{C}^2\backslash(0,0)\big)/\Gamma$ where elements of Γ act on $(w,z)\in\mathbb{C}^2\backslash(0,0)$ as $(w,z)\to e^{2\pi n}(w,z)$, $n\in\mathbb{Z}$, which is clear from the following identification,

$$\begin{aligned} w &= \cos\psi\, e^{-\rho - i\varphi_1}, \\ z &= \sin\psi\, e^{-\rho + i\varphi_2}. \end{aligned} \tag{6.2}$$

It turns out that these coordinates can be identified with a twisted chiral superfield w and a chiral superfield z. The resulting potential is given by,

$$V_{w\neq 0}(z,w,\bar z,\bar w) = \int^{\frac{z\bar z}{w\bar w}} \frac{dq}{q}\ln(1+q) - \frac{1}{2}(\ln w\bar w)^2 , \tag{6.3}$$

which is well defined as long as $w \neq 0$ (or $\psi \neq \pi/2$). The "mirror" potential (see eq. (4.7)),

$$V_{z\neq 0}(w,\bar w,z,\bar z) = -\int^{\frac{w\bar w}{z\bar z}} \frac{dq}{q}\ln(1+q) + \frac{1}{2}(\ln z\bar z)^2 , \tag{6.4}$$

is well defined as long as $z \neq 0$ (or $\psi \neq 0$). On the overlap of the two patches the potentials are related by a generalized Kähler transformation,

$$V_{w\neq 0} - V_{z\neq 0} = -\ln\big(z\bar z\big)\ln\big(w\bar w\big). \tag{6.5}$$

The type is $(k_+,k_-) = (1,1)$ and no type changing occurs. The closed pure spinors can be calculated [13] and their Mukai pairings are given by,

$$\big(\phi_+,\bar\phi_+\big) = \big(\phi_-,\bar\phi_-\big) = -\frac{2}{z\bar z + w\bar w}, \tag{6.6}$$

which vanish nowhere and which satisfy the generalized Calabi-Yau condition eq. (4.29). However a more careful investigation shows that they are not globally well defined defined implying that the current model does not provide an appropriate supergravity solution.

The second formulation of the model in terms of a semi-chiral multiplet is given by,

$$V_{\psi\neq 0}=\ln\frac{l}{\bar r}\,\ln\frac{\bar l}{r}-\int^{r\bar r}\frac{dq}{q}\,\ln\big(1+q\big)\,, \tag{6.7}$$

where the semi-chiral coordinates are related to the previous one by,

$$l=w\,,\qquad \bar l=\bar w\,,\qquad r=\frac{\bar w}{z}\,,\qquad \bar r=\frac{w}{\bar z}\,. \tag{6.8}$$

In order to obtain a potential which is well defined in $\psi=0$ we perform a Legendre transformation,

$$V_{\psi\neq 0}(l,\bar l,r,\bar r)-\ln l\,\ln l'-\ln\bar l\,\ln\bar l'\,, \tag{6.9}$$

where l is now unconstrained and l' satisfies $\mathbb{D}_+l'=0$. Varying w.r.t. l and $\bar l$ one finds that $l'=\bar l/r$ and $\bar l'=l/\bar r$. Making a final coordinate transformation $r'=1/r$ and $\bar r'=1/\bar r$ we get,

$$V_{\psi\neq 0}(l,\bar l,r,\bar r)-\ln l\,\ln l'-\ln\bar l\,\ln\bar l'=V_{\psi\neq\frac{\pi}{2}}(l',\bar l',r',\bar r')-\frac{1}{2}\big(\ln r'\big)^2-\frac{1}{2}\big(\ln\bar r'\big)^2\,, \tag{6.10}$$

where the last two terms can be removed by a generalized Kähler transformation. The resulting potential which is now well defined in $\psi=0$ is explicitly given by,

$$V_{\psi\neq\frac{\pi}{2}}=-\ln\frac{l'}{r'}\,\ln\frac{\bar l'}{\bar r'}+\int^{r'\bar r'}\frac{dq}{q}\,\ln\big(1+q\big)\,, \tag{6.11}$$

and it is once more the mirror transform of the original potential. This is an explicit example of a situation where the generalized potentials on the overlap of two coordinate patches are not only related by a generalized Kähler transformation but by a Legendre transformation as well.

Using the potentials to calculate the complex structures one verifies that type-changing occurs at $\psi=\pi/2$ where the type jumps as $(0,0)\to(0,2)$ and at $\psi=0$ where we have that the type goes as $(0,0)\to(2,0)$. The generalized Calabi-Yau condition eq. (4.29) does not hold anymore however one verifies that the weaker UV-finiteness condition eq. (4.25) is still satisfied.

The formulation of the model in terms of a pair of twisted and chiral superfields and the one in terms of a semi-chiral multiplet are related by a T-duality transformation (as originally suggested in [18] and worked out explicitly in [13]). Indeed, performing a T-duality transformation along the S^1 of $S^1\times S^3$ brings one back to $S^1\times S^3$. The isometry which will be gauged acts as $\rho\to\rho+$constant which is easily verified to be of the type discussed in section 5.1. This T-duality transformation provides an explanation for the simple relation between the chiral/twisted chiral and the semi-chiral parameterization eq. (6.8).

7. Discussion and outlook

In the current review we limited ourselves to non-linear σ-models without boundaries. However boundaries can be added to $N=(2,2)$ superspace which gives rise to an intricate and powerful worldsheet description of supersymmetric D-branes including configurations interpolating between A-branes and B-branes [19], [15].

The previous shows that supersymmetric non-linear σ-models exhibit a rich geometric structure, both at the classical and at the quantum level. Mathematically the resulting bi-hermitian geometry is perfectly captured by Hitchin's generalized Kähler geometry. The analysis of the one loop quantum corrections leads to expressions closely related to Hitchin's generalized Calabi-Yau geometry. Knowing that higher loop corrections result in a deformation of the one-loop β-functions, a natural and interesting question arises which is whether those can be put in a generalized geometrical setting as well, leading to a better and more systematic control of the perturbation series.

Another interesting question deals with the doubled formalism [20]. The doubled formalism provides a concrete proposal to tackle non-geometric or T-fold compactifications. Although the underlying geometry is still being developed, hints at a very rich framework beyond conventional Riemannian geometry are emerging (see for instance [21]). Many expressions are reminiscent of, and indeed motivated by, Hitchin's generalized complex geometry. As supersymmetric non-linear σ-models in $N=(2,2)$ superspace provide a very explicit local realization of generalized Kähler geometry, one could expect that the development of a doubled formalism in $N=(2,2)$ superspace will shed light on these novel geometric structures. Such a doubled formalism certainly exists but has a slightly different structure from what is customary.

A simple example of a σ-model on T^2 illustrates this point. The potentials $V=(z+\bar{z})^2/2$ and $W=-(w+\bar{w})^2/2$ are T-dual. In a manifest $N=(2,2)$ supersymmetric doubled formalism one has to "overdouble" the coordinates, *i.e.* all superfields involved in the T-duality transformation are doubled. In this case the doubled formalism is parameterized by z, $\bar{z}$, w and $\bar{w}$. The overdoubled coordinate is eliminated by the condition,

$$w+\bar{w}=z+\bar{z}, \tag{7.1}$$

which singles out a kind of co-isotropic brane in the overdoubled space. Hull's constraints follow immediately from eq. (7.1), indeed,

$$\hat{D}_\pm(w+\bar{w})=\hat{D}_\pm(z+\bar{z}) \Rightarrow D_\pm(w-\bar{w})=\pm D_\pm(z-\bar{z})\,. \tag{7.2}$$

The expression above implies that we are dealing with superfields which are truly chiral: from eq. (7.1) we get that $\mathbb{D}_+(z-w)=\mathbb{D}_-(z-\bar{w})=0$. From the normal superfield constraints we get $\bar{\mathbb{D}}_+(z-w)=\bar{\mathbb{D}}_-(z-\bar{w})=0$ implying that $\partial_{\neq}(z-w)=\partial_{=}(z-\bar{w})=0$ as well. Chiral bosons in the sense just mentioned are currently being developed together with a full treatment of the doubled formalism in $N=(2,2)$ superspace [22].

Finally, it would be most useful to have some more non-trivial examples of generalized Kähler geometries where the local structure – read the generalized Kähler potential – is explicitly known. WZW-models on even dimensional reductive Lie group manifolds might provide them however at this point explicit results have only been obtained for the $SU(2)\times U(1)$ and the $SU(2)\times SU(2)$ models.

Acknowledgments

AS would like to thank the organizers of the Corfu Summer Institute and the XVIIIth European Workshop on String Theory for the nice scientific environment. This work is supported by the

FWO-Vlaanderen, Project G.0114.10N, by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P7/37 and by the Vrije Universiteit Brussel through the Strategic Research Program "High-Energy Physics". DT is a Postdoctoral Researcher FWO-Vlaanderen.

A. Notation and conventions

We denote the worldsheet coordinates by $\tau,\sigma\in\mathbb{R}$, and the worldsheet light-cone coordinates are defined by,

$$\sigma^{\ddagger}=\tau+\sigma,\qquad \sigma^{=}=\tau-\sigma. \tag{A.1}$$

The $N=(1,1)$ (real) fermionic coordinates are denoted by θ^{+} and θ^{-} and the corresponding derivatives satisfy,

$$D_{+}^{2}=-\frac{i}{2}\partial_{\ddagger},\qquad D_{-}^{2}=-\frac{i}{2}\partial_{=},\qquad \{D_{+},D_{-}\}=0. \tag{A.2}$$

The $N=(1,1)$ integration measure is explicitely given by,

$$\int d^{2}\sigma\, d^{2}\theta=\int d\tau\, d\sigma\, D_{+}D_{-}. \tag{A.3}$$

Passing from $N=(1,1)$ to $N=(2,2)$ superspace requires the introduction of two more real fermionic coordinates $\hat{\theta}^{+}$ and $\hat{\theta}^{-}$ where the corresponding fermionic derivatives satisfy,

$$\hat{D}_{+}^{2}=-\frac{i}{2}\partial_{\ddagger},\qquad \hat{D}_{-}^{2}=-\frac{i}{2}\partial_{=}, \tag{A.4}$$

and again all other – except for (A.2) – (anti-)commutators do vanish. The $N=(2,2)$ integration measure is,

$$\int d^{2}\sigma\, d^{2}\theta\, d^{2}\hat{\theta}=\int d\tau\, d\sigma\, D_{+}D_{-}\hat{D}_{+}\hat{D}_{-}. \tag{A.5}$$

Regularly a complex basis is used,

$$\mathbb{D}_{\pm}\equiv\hat{D}_{\pm}+iD_{\pm},\qquad \bar{\mathbb{D}}_{\pm}\equiv\hat{D}_{\pm}-iD_{\pm}, \tag{A.6}$$

which satisfy,

$$\{\mathbb{D}_{+},\bar{\mathbb{D}}_{+}\}=-2i\partial_{\ddagger},\qquad \{\mathbb{D}_{-},\bar{\mathbb{D}}_{-}\}=-2i\partial_{=}, \tag{A.7}$$

and all other anti-commutators do vanish.

References

[1] S. J. Gates, C. M. Hull and M. Roček, *Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models*, *Nucl. Phys.* **B 248** (1984) 157.

[2] P. S. Howe and G. Sierra, *Two-Dimensional Supersymmetric Nonlinear Sigma Models With Torsion*, *Phys. Lett.* **B 148** (1984) 451.

[3] Ph. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Complex structures on parallelized group manifolds and supersymmetric sigma models*, *Phys. Lett.* **B 206** (1988) 71 and *Extended Supersymmetric Sigma Models on Group Manifolds. 1. The Complex Structures*, *Nucl. Phys.* **B 308** (1988) 662. .

[4] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math. Oxford Ser.* **54**, 281 (2003) [arXiv:math/0209099]; M. Gualtieri, *Generalized complex geometry*, Oxford University DPhil thesis, 2003, arXiv:math/0401221; M. Gualtieri, *Generalized complex geometry*, arXiv:math/0703298, M. Gualtieri, *Generalized Kahler geometry*, [arXiv:1007.3485 [math.DG]]. A particularly lucid presentation is given in N. Hitchin, *Lectures on generalized geometry*, [arXiv:1008.0973 [math.DG]].

[5] J. Maes and A. Sevrin, *A Note on N=(2,2) superfields in two dimensions*, Phys. Lett. B **642** (2006) 535 [hep-th/0607119].

[6] T. Buscher, U. Lindström and M. Roček, *New Supersymmetric Sigma Models With Wess-Zumino Terms*, *Phys. Lett.* **B 202** (1988) 94.

[7] A. Sevrin and J. Troost, *Off-shell formulation of N = 2 non-linear sigma-models*, *Nucl. Phys.* **B 492** (1997) 623, hep-th/9610102.

[8] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry*, *Commun. Math. Phys.* **269** (2007) 833, hep-th/0512164.

[9] C. M. Hull, U. Lindstrom, M. Roček, R. von Unge and M. Zabzine, *Generalized Kahler geometry and gerbes*, *J. High Energy Phys.* **0910** (2009) 062, arXiv:0811.3615 [hep-th].

[10] J. Bogaerts, A. Sevrin, S. van der Loo and S. Van Gils, *Properties of semi-chiral superfields*, *Nucl. Phys.* **B 562** (1999) 277, hep-th/9905141.

[11] M. T. Grisaru, M. Massar, A. Sevrin and J. Troost, *The quantum geometry of N = (2,2) non-linear sigma-models*, *Phys. Lett.* **B 412** (1997) 53, [arXiv:hep-th/9706218].

[12] C. M. Hull, U. Lindstrom, M. Roček, R. von Unge and M. Zabzine, *Generalized Calabi-Yau metric and Generalized Monge-Ampere equation*, *J. High Energy Phys.* **1008** (2010) 060, arXiv:1005.5658 [hep-th].

[13] A. Sevrin, W. Staessens and D. Terryn, *The Generalized Kahler geometry of N=(2,2) WZW-models*, JHEP **1112** (2011) 079 [arXiv:1111.0551 [hep-th]].

[14] M. T. Grisaru, M. Massar, A. Sevrin and J. Troost, *Some aspects of N = (2,2), D = 2 supersymmetry*, *Fortschr. Phys.* **47** (1999) 301, hep-th/9801080.

[15] A. Sevrin, W. Staessens and A. Wijns, *An N=2 worldsheet approach to D-branes in bihermitian geometries: II. The general case*, *J. High Energy Phys.* **0909** (2009) 105, [arXiv:0908.2756 [hep-th]].

[16] U. Lindström, M. Roček, I. Ryb, R. von Unge and M. Zabzine, *New N = (2, 2) vector multiplets*, *J. High Energy Phys.* **0708** (2007) 008, [arXiv:0705.3201 [hep-th]]; S. J. J. Gates and W. Merrell, *D=2 N=(2,2) Semi Chiral Vector Multiplet*, *J. High Energy Phys.* **0710** (2007) 035, [arXiv:0705.3207 [hep-th]]; U. Lindström, M. Roček, I. Ryb, R. von Unge and M. Zabzine, *T-duality and Generalized Kahler Geometry*, *J. High Energy Phys.* **0802** (056) 2008, hep-th/0707.1696; W. Merrell and D. Vaman, *T-duality, quotients and generalized Kähler geometry*, *Phys. Lett.* **B 665** (401) 2008, hep-th/0707.1697.

[17] M. Roček, K. Schoutens and A. Sevrin, *Off-shell WZW models in extended superspace*, *Phys. Lett.* **B 265** (1991) 303; M. Roček, C. H. Ahn, K. Schoutens and A. Sevrin, *Superspace WZW models and black holes*, `hep-th/9110035`.

[18] Martin Roček: private communication.

[19] A. Sevrin, W. Staessens and A. Wijns, *An N=2 worldsheet approach to D-branes in bihermitian geometries: I. Chiral and twisted chiral fields*, *J. High Energy Phys.* **0810** (2008) 108, `[arXiv:0809.3659 [hep-th]]`.

[20] C. M. Hull, *A Geometry for non-geometric string backgrounds*, JHEP **0510** (2005) 065 `[hep-th/0406102]`; C .M .Hull, *Doubled Geometry and T-Folds*, JHEP **0707** (2007) 080 `[hep-th/0605149]`.

[21] O. Hohm and B. Zwiebach, *Towards an invariant geometry of double field theory*, `arXiv:1212.1736 [hep-th]` and references therein.

[22] A. Sevrin and D. C. Thompson, work in progress.