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Four-loop on-shell integrals: $\overline{\mathrm{MS}}$–on-shell relation and g-2

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We present results for the heavy-lepton contributions to the anomalous magnetic moment of both electron and muon. Furthermore, we present first results towards a full four-loop calculation $\overline{\mathrm{MS}}$–on-shell relation and of the universal and light-lepton contributions to the anomalous magnetic moment of the muon.

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1. Introduction

The anomalous magnetic moments of electron and muon have been measured with unrivaled precision. In the case of the muon the experimental value

$$a_\mu^{\exp} = 1.16592080(54)(33)[63] \cdot 10^{-3} \tag{1.1}$$

has to be compared with the theory prediction

$$a_\mu^{theo} = 1.16591790(65) \cdot 10^{-3} \,. \tag{1.2}$$

The discrepancy between the two values of about 3 standard deviations is approximately of the same order as the four-loop QED corrections. The four- and five-loop corrections have been calculated in Ref. [2] and have not been verified by an independent calculation. In this paper we will present first steps to such an independent calculation to verify the results in Ref. [2]. An technically related object is the $\overline{\text{MS}}$–on-shell relation for quark masses in QCD. One of the main motivations, why it is of importance to know the $\overline{\text{MS}}$–on-shell relation with four-loop accuracy, is the planned determination of the top-quark mass at a future linear collider. The precision reached at such an experiment requires an equally precise knowledge of the $\overline{\text{MS}}$–on-shell relation.

In the following we will discuss recent results for the calculation of heavy-lepton induced contributions to the anomalous magnetic moment of electron and muon. We will also briefly review the status of calculations for the $\overline{\text{MS}}$–on-shell relation and electronic contributions to the anomalous magnetic moment of the muon.

2. Calculation and Results

The calculation is set up as follows, the Feynman diagrams are generated using `QGRAF` [12], its output is then converted into `FORM` [15] input using `q2e` and `exp` [13, 4]. Suitable projectors are applied, if necessary, an asymptotic expansion is performed, and the resulting scalar integrals are reduced to master integrals using integration-by-parts identities implemented in `CRUSHER` [10] and `FIRE` [14].

2.1 Heavy lepton contributions to the anomalous magnetic moment

The key to the calculation of the heavy-lepton contributions to the anomalous magnetic moment of electron and muon at four-loop order is the method of asymptotic expansion. The strong hierarchy of the lepton masses allows for the expansion of the contributing diagrams in the mass ratios m_e/m_μ and m_e/m_τ, m_μ/m_τ, respectively, the calculation can be much simplified. After the expansion, instead of the calculation of complicated four-loop propagator diagrams only the calculation of relatively simple four-loop vacuum diagrams is required. The method is illustrated in Fig. 1.

Using an asymptotic expansion in Ref. [6] the four-loop corrections due to heavy-leptons were obtained. In Tab.1 we show the separate contributions from several classes of diagrams for the case of the muon anomalous magnetic moment. As can be seen a better accuracy has been achieved for all diagram classes. Our improved result for the anomalous magnetic moment of the muon

$$A_{2,\mu}^{(8)}(M_\mu/M_\tau) = 0.0421670 + 0.0003257 + 0.0000015 = 0.0424941(2)(53) \tag{2.1}$$

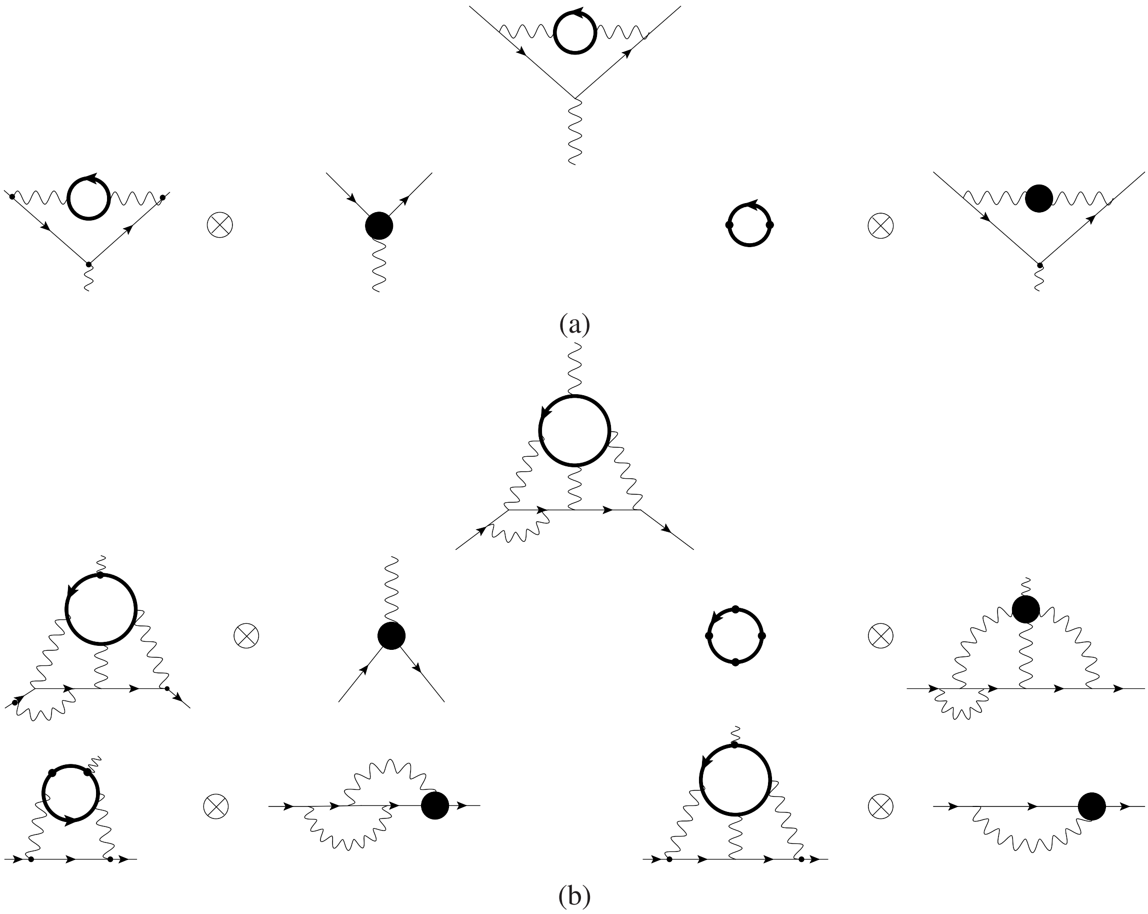

Figure 1: Graphical description of the method of asymptotic expansions.

is more precise than the results previously available and in full agreement with previous evaluations [2]. The situation is similar for the electron case. Here our improved result reads

$$\begin{aligned} A_{2,e}^{(8)}(M_e/M_\mu) &\approx (9.161259603 + 0.000711078 + 2.2\cdot 10^{-8})\cdot 10^{-4} \\ &\approx 9.161970703(2)(372)\cdot 10^{-4}\,, \\ A_{2,e}^{(8)}(M_e/M_\tau) &\approx (7.42923268609971 + 2.75209424\cdot 10^{-6} + 3.2\cdot 10^{-13})\cdot 10^{-6} \\ &\approx 7.42924(0)(118)\cdot 10^{-6}\,, \end{aligned} \tag{2.2}$$

which is again in agreement with previous evaluations [3].

2.2 $\overline{\text{MS}}$–on-shell relation

The $\overline{\text{MS}}$–on-shell relation has only been calculated up to three-loop order in Refs. [11, 9] and comprises a fundamental quantity of QCD. To take full advantage of the experimental precision reached at a future linear collider it is mandatory to calculate it at four-loop order. Let us briefly review the status of this calculation. It can be written as a power series in the strong coupling

group	$10^2 \cdot A_{2,\mu}^{(8)}(M_\mu/M_\tau)$	
	[6]	[2]
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	−0.6292808(6)	−0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	−2.316756(5)	−2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)	0.612661(5)	0.6106(31)
IV(c)	−1.83010(1)	−1.823(11)

Table 1: Mass-dependent corrections to a_μ at four-loop order as obtained in this paper and the comparison with Refs. [2]. The uncertainties assigned to our numbers correspond to 10% of the highest available expansion terms, i.e., the ones of order $(M_\mu/M_\tau)^6$ and $(M_\mu/M_\tau)^7$. Uncertainties from the muon and tau lepton mass are not shown.

constant α_s

$$z_m^{\rm OS}(\mu) = \frac{\bar{m}_q(\mu)}{M_q} = \frac{Z_m^{\rm OS}}{Z_m^{\overline{\rm MS}}} = 1+\frac{\alpha_s(\mu)}{\pi}\delta z_m^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \delta z_m^{(2)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \delta z_m^{(3)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^4 \delta z_m^{(4)} + \mathscr{O}\left(\alpha_s^5\right) \tag{2.3}$$

and labeling contributions from massless and massive quark loops by n_l and n_h, respectively, we obtain the result for contributions from diagrams with at least two massless quark loops [8]

$$z_m^{\rm OS} = 1 - A_s 1.333 + A_s^2\left(-14.229 - 0.104\,n_h + 1.041\,n_l\right) + A_s^3\left(-197.816 - 0.827\,n_h - 0.064\,n_h^2 + 26.946\,n_l - 0.022\,n_h n_l - 0.653\,n_l^2\right) + A_s^4\left(-43.465\,n_l^2 - 0.017\,n_h n_l^2 + 0.678\,n_l^3 + \ldots\right) + \mathscr{O}\left(A_s^5\right), \tag{2.4}$$

with $A_s = \alpha_s(\mu)/\pi$.

2.3 Light-lepton contributions to the anomalous magnetic moment

In the approximation of a massless electron only the leading term including the logarithms can be obtained. For the sub-leading contributions a proper asymptotic expansion has to be performed. Expanding a_μ in a power series in the fine structure constant α

$$a_\mu = 1 + \frac{\alpha}{\pi} a_\mu^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 a_\mu^{(2)} + \left(\frac{\alpha}{\pi}\right)^3 a_\mu^{(3)} + \left(\frac{\alpha}{\pi}\right)^4 a_\mu^{(4)} + \mathscr{O}\left(\alpha^5\right) \tag{2.5}$$

and marking contributions from electron loops by n_l

$$a_\mu^{(4)} = n_l^3 a_\mu^{(43)} + n_l^2 a_\mu^{(42)} + \cdots \tag{2.6}$$

we obtain the result for contributions with three electron loops

$$\begin{aligned} a_\mu^{(43)} &= \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324}+\frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \\ &\approx 7.196\,66\,, \end{aligned} \tag{2.7}$$

where $L_{\mu e} = \ln(M_\mu^2/M_e^2)$. The result for diagrams with two electron loops can be further split into a contribution with and without an additional muon loop, $a_\mu^{(42)b}$ and $a_\mu^{(42)a}$, respectively,

$$a_\mu^{(42)} = a_\mu^{(42)a} + a_\mu^{(42)b}\,,$$

with

$$\begin{aligned} a_\mu^{(42)a} &= L_{\mu e}^2\left[\pi^2\left(\frac{5}{36}-\frac{a_1}{6}\right)+\frac{\zeta_3}{4}-\frac{13}{24}\right] + L_{\mu e}\left[-\frac{a_1^4}{9}+\pi^2\left(-\frac{2a_1^2}{9}+\frac{5a_1}{3}-\frac{79}{54}\right)\right. \\ &\quad \left. -\frac{8a_4}{3}-3\zeta_3+\frac{11\pi^4}{216}+\frac{23}{6}\right] - \frac{2a_1^5}{45}+\frac{5a_1^4}{9}+\pi^2\left(-\frac{4a_1^3}{27}+\frac{10a_1^2}{9}\right. \\ &\quad \left. -\frac{235a_1}{54}-\frac{\zeta_3}{8}+\frac{595}{162}\right)+\pi^4\left(-\frac{31a_1}{540}-\frac{403}{3240}\right)+\frac{40a_4}{3}+\frac{16a_5}{3}-\frac{37\zeta_5}{6} \\ &\quad +\frac{11167\zeta_3}{1152}-\frac{6833}{864} \\ &\approx -3.624\,27\,, \end{aligned} \tag{2.8}$$

$$\begin{aligned} a_\mu^{(42)b} &= \left(\frac{119}{108}-\frac{\pi^2}{9}\right)L_{\mu e}^2 + \left(\frac{\pi^2}{27}-\frac{61}{162}\right)L_{\mu e} - \frac{4\pi^4}{45}+\frac{13\pi^2}{27}+\frac{7627}{1944} \\ &\approx 0.494\,05\,. \end{aligned} \tag{2.9}$$

Our results for $a_\mu^{(43)}$ and $a_\mu^{(42)b}$ agree with the results given in Refs. [7, 1]. The result for $a_\mu^{(42)a}$ can be compared with the result from Refs. [5, 2]

$$a_\mu = -3.642\,04(1\,12)\,. \tag{2.10}$$

Our new result confirms the previously obtained results, the small discrepancy is due to missing terms in the expansion in m_e/m_μ.

3. Conclusions

We presented results for both the $\overline{\mathrm{MS}}$–on-shell relation and the anomalous magnetic moment of the muon at four-loop order. These result comprise a first step towards the full four-loop calculation and confirm the results known in the literature.

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