PROCEEDINGS OF SCIENCE

Accurate Bottom-Quark Mass from Borel QCD Sum Rules for the Decay Constants of B and B_s Mesons

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We show that in the context of QCD sum rules a strong (anti)correlation between the b-quark mass m_b and the B-meson's decay constant f_B emerges: $\delta f_B/f_B \approx -8\,\delta m_b/m_b$. This observation allows us to derive a precise value of m_b from a Borel sum rule for the two-point correlator of heavy–light currents exploiting accurate f_B results from lattice QCD as input: $\overline{m}_b(\overline{m}_b) = (4.247 \pm 0.034)$ GeV.

The XXIst International Workshop High Energy Physics and Quantum Field Theory
June 23 – June 30, 2013
Saint Petersburg Area, Russia

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PoS(QFTHEP 2013)048

1. Introduction

The standard "model" of elementary particle physics involves, at least, 26 free parameters or 28 if neutrinos are not Dirac but Majorana fermions, most of them related to the fermion-mass sector of the theory. One of these basic parameters is the mass of the bottom quark. Its actual numerical value depends on the choice made for its rigorous definition; results for this quantity are usually presented in terms of either a merely perturbatively given pole mass or, in the $\overline{\mathrm{MS}}$ renormalization scheme, the running mass $\overline{m}_b(\nu)$ at renormalization scale ν or the latter's specific value $m_b \equiv \overline{m}_b(\overline{m}_b)$ at $\nu = \overline{m}_b$.

In principle, *lattice QCD* offers a possibility to infer the b-quark mass from first principles, *i.e.*, directly from QCD. Unfortunately, the b quark is too heavy for current lattice setups: some loophole of one kind or the other has to be found. Moreover, lattice evaluations of the b-quark's running mass involve the calculation of a nonperturbative renormalization constant; this limits the precision of the mass extraction. Accordingly, the accuracy of present lattice findings for m_b is not particularly high.

Table 1 summarizes some recent predictions for the b-quark mass found from lattice QCD with unquenched gauge configurations and two dynamical quarks in the sea by extrapolating from lighter simulated masses [1, 2] or adopting "heavy-quark effective theory" (HQET) [3 – 5] or from moment sum rules for two-point correlators of *heavy–heavy* quark currents that take advantage of three-loop $O(\alpha_s^2)$ [6] or four-loop $O(\alpha_s^3)$ [7][1] fixed-order perturbative-QCD results combined with experiment or renormalization-group-improved next-to-next-to-leading-logarithmic-order results plus data [9].

Table 1: Bottom-quark mass $m_b \equiv \overline{m}_b(\overline{m}_b)$ in $\overline{\mathrm{MS}}$ renormalization scheme: selection of previous evaluations.

Approach	Collective of authors	m_b (GeV)
Lattice QCD	ETM Collaboration [1]	4.29 ± 0.14
	ETM Collaboration [2]	4.35 ± 0.12
	Gimenez *et al.* [3]	4.26 ± 0.09
	UKQCD Collaboration [4]	4.25 ± 0.11
	ALPHA Collaboration [5]	4.22 ± 0.11
Moment sum rules	Kühn and Steinhauser [6]	4.191 ± 0.051
	Chetyrkin *et al.* [7]	4.163 ± 0.016
	Hoang *et al.* [9]	$4.235 \pm 0.055_{(\mathrm{pert})} \pm 0.03_{(\mathrm{exp})}$

In the recent study reported here, we used precise values of the $B_{(s)}$-meson decay constants $f_{B_{(s)}}$ as hadronic input to *heavy–light* Borel QCD sum rules to predict m_b with comparable accuracy [10].

2. Lesson from Quantum Mechanics: Expect Clear-cut Anticorrelation of f_B and m_b

Our present intention is to perform a precision determination of the heavy-quark mass $m_Q = m_b$ from knowledge of the decay constants $f_{B_{(s)}}$. Within QCD, the question arises: how sensitive are the numerical values of these two quantities to each other, what kind and amount of correlation between them should we expect? To answer this question, before addressing the real-life problem let us have a look at the corresponding situation in quantum mechanics. There, *nonrelativistic potential models* are utilized since long for describing (sufficiently heavy) hadrons as bound states of quarks [11, 12].

[1] These findings get support when combining perturbative QCD and lattice QCD with $2+1$ dynamical sea quarks [8].

Now, if the potential involves just one coupling constant, for instance, if it is a pure Coulomb or pure harmonic-oscillator potential, for a ground state its wave function at the origin, $\psi(0)$, is related to its binding energy ε by $|\psi(0)| \propto \varepsilon^{3/2}$; for sums of confining and Coulomb potentials, this relation holds approximately [13]. Realizing that $|\psi(0)|$ assumes the rôle of a decay constant and exploiting the scaling behaviour of a heavy-meson decay constant in the heavy-quark limit then relates the pole mass m_Q of a heavy quark Q to the B-meson mass M_B, approximately by $f_B\sqrt{M_B} = \kappa\,(M_B - m_Q)^{3/2}$. Upon accepting this, it is straightforward to obtain the variation δf_B of f_B as consequence of a small variation δm_Q around some chosen value of m_Q. From the experimental finding $M_B = 5.27$ GeV and for $f_B \approx 200$ MeV near $m_Q \approx 4.6$–4.7 GeV, we get $\kappa \approx 0.9$–1.0 and $\delta f_B \approx -0.5\,\delta m_Q$, which entails

$$\frac{\delta f_B}{f_B} \approx -(11\text{–}12)\,\frac{\delta m_Q}{m_Q}\ .$$

For instance, $\delta m_Q = +100$ MeV implies $\delta f_B \approx -50$ MeV. Hence, we feel entitled to expect a rather high and negative correlation of m_b and $f_{B_{(s)}}$ manifesting also in QCD sum-rule predictions [14, 15].

3. Earlier Predictions for $B_{(s)}$-Meson Decay Constants by QCD Sum-Rule Approach

Relying on, essentially, one and the same expression for the heavy–light correlation function at three-loop accuracy [16], in the last years several QCD sum-rule extractions of beauty-meson decay constants have been performed [17 – 20]; their results for f_B are compiled in Table 2. At first glance, all these findings appear to be consistent and reliable but they are not, as they do not comply with the quantum-mechanical expectations for the relationship between f_B and m_b. The crucial issues are the definition of heavy-quark masses in use and a proper incorporation of the *effective* continuum onset.

Table 2: B-meson decay constant f_B: some predictions by QCD sum rule for heavy–light two-point function.

	Reference [17]	Reference [18]	Reference [19]	Reference [20]
m_b (GeV)	4.05 ± 0.06	4.21 ± 0.05	4.245 ± 0.025	4.236 ± 0.069
f_B (MeV)	203 ± 23	210 ± 19	193 ± 15	206 ± 7

After rather successful application [19, 21] of QCD sum rules arising from the correlator of two heavy–light pseudoscalar quark currents to an extraction of the decay constants of charmed mesons, we recently revisited, *mutatis mutandis* by the same formalism, the beauty-meson system. There, in contrast to the charmed-meson case, we indeed observe the presumed *pronounced* anticorrelation of heavy-quark mass and heavy-meson decay constant [10]. Formulating our correlator in terms of the $\overline{\text{MS}}$ running instead of the pole b-quark mass and applying consistent extraction procedures, we find for the QCD-sum rule prediction of f_B a linear dependence on m_b with negative slope, if keeping the input values of all other OPE quantities, such as renormalization scales, α_s, quark condensate, fixed:

$$f_B(m_b) = \left(192.0 - 37\,\frac{m_b - 4.247\text{ GeV}}{0.1\text{ GeV}} \pm 3_{(\text{syst})}\right)\text{MeV}\ . \tag{3.1}$$

This observation suggests to invert, in the $B_{(s)}$-meson case, our line of reasoning: using, as hadronic input, our average $f_B^{\text{LQCD}} = (191.5 \pm 7.3)$ MeV of recent lattice-QCD results for f_B [1, 2, 5, 22 – 24] in our QCD sum rule deriving from the heavy–light correlator at $O(\alpha_s^2)$ accuracy yields the accurate estimate $m_b = (4.247 \pm 0.034)$ GeV. In the following, we present some relevant details of this study.

4. (Borel-Transformed) QCD Sum Rule from Heavy–Light Two-Current Correlator

Arising from an evaluation of correlation functions of appropriate interpolating currents at both the QCD level (with quarks and gluons as basic degrees of freedom) and the hadron level, *QCD sum rules* relate the fundamental parameters of the theory (such as quark masses and strong coupling α_s) to experimentally observable features of hadronic bound states of the QCD degrees of freedom. Our goal is to adopt this QCD sum-rule approach in order to arrive at a prediction of the b-quark mass m_b from the decay constants $f_{B_{(s)}}$ of the $B_{(s)}$ mesons. To this end, we start from the correlator [14, 15] of two pseudoscalar currents of a b quark and a light quark q of mass m, $j_5(x) \equiv (m_b+m)\,\bar{q}(x)\,\mathrm{i}\,\gamma_5\,b(x)$:

$$\Pi\big(p^2\big) \equiv \mathrm{i}\int \mathrm{d}^4x \exp(\mathrm{i}\,p\,x)\left\langle 0\left|\mathrm{T}\Big(j_5(x)\,j_5^\dagger(0)\Big)\right|0\right\rangle .$$

At QCD level, Wilson's operator product expansion (OPE) substitutes nonlocal products of currents by series of local operators composed of the QCD degrees of freedom, at the price of introducing — in addition to perturbative contributions given in form of integrals of spectral densities $\rho_{\rm pert}(s,\mu)$ — power corrections of nonperturbative origin, $\Pi_{\rm power}(\tau,\mu)$, involving so-called vacuum condensates. Applying to both QCD and hadronic expressions for a correlator under study a Borel transformation $\Pi\big(p^2\big)\to\Pi(\tau)$ to a Borel variable τ suppresses at hadron level both higher excitations and hadronic continuum. The hadronic states above the ground state are subsumed by integrals of hadron spectral densities $\rho_{\rm hadr}(s)$ with *physical thresholds* $s_{\rm phys}$ as lower endpoints; in our case, $s_{\rm phys} = (M_{B^*}+M_P)^2$ is given by the beauty vector meson's mass M_{B^*} and the mass M_P of the lightest pseudoscalar meson with appropriate quantum numbers, *i.e.*, π or K. In this way, we get for the QCD sum rule sought, in terms of the $B_{(s)}$ meson's mass M_B and decay constant f_B defined by $(m_b+m)\,\langle 0|\bar{q}\,\mathrm{i}\,\gamma_5\,b|B\rangle = f_B\,M_B^2$,

$$\begin{aligned}\Pi(\tau) &= f_B^2\,M_B^4\exp\big(-M_B^2\,\tau\big) + \int\limits_{s_{\rm phys}}^{\infty}\mathrm{d}s\exp(-s\,\tau)\,\rho_{\rm hadr}(s)\\ &= \int\limits_{(m_b+m)^2}^{\infty}\mathrm{d}s\exp(-s\,\tau)\,\rho_{\rm pert}(s,\mu)+\Pi_{\rm power}(\tau,\mu)\ .\end{aligned}$$

Quark–hadron duality serves to banish all contributions of higher hadronic states by assuming them to be counterbalanced by perturbative contributions beyond an *effective continuum threshold* $s_{\rm eff}(\tau)$ that is an object intrinsic to the QCD sum-rule framework with interesting and nontrivial facets [25], depends on the Borel variable τ if requiring rigour in the description of ground-state properties [26], but must not be confused with $s_{\rm phys}$. We end up with a QCD sum rule relating ground state and OPE:

$$f_B^2\,M_B^4\exp\big(-M_B^2\,\tau\big) = \int\limits_{(m_b+m)^2}^{s_{\rm eff}(\tau)}\mathrm{d}s\exp(-s\,\tau)\,\rho_{\rm pert}(s,\mu)+\Pi_{\rm power}(\tau,\mu)\ . \tag{4.1}$$

Even with $\rho_{\rm pert}(s,\mu)$ and $\Pi_{\rm power}(\tau,\mu)$ known up to a certain accuracy, the evaluation of this relation requires us to formulate both criterion and resulting prescription for determining the function $s_{\rm eff}(\tau)$ and to assure reasonable convergence of the OPE. We accomplish the latter by expanding $\rho_{\rm pert}(s,\mu)$ perturbatively not in terms of the pole mass [16] but in terms of the $\overline{\rm MS}$ mass of the b quark. Explicit results for $\rho_{\rm pert}(s,\mu)$ at three-loop level and $\Pi_{\rm power}(\tau,\mu)$ have been given by Refs. [16, 18]. Table 3 presents the numerical values of all OPE quantities adopted as input to our extraction of m_b [27, 28].

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Table 3: Operator product expansion inputs: QCD parameters and lowest-dimensional vacuum condensates.

OPE quantity	Symbol	Numerical input value
Light-quark mass	$\overline{m}_d(2\ \text{GeV})$	(3.5 ± 0.5) MeV
Strange-quark mass	$\overline{m}_s(2\ \text{GeV})$	(95 ± 5) MeV
Strong coupling constant	$\alpha_s(M_Z)$	0.1184 ± 0.0007
Light-quark condensate	$\langle\bar{q}q\rangle(2\ \text{GeV})$	$-[(269 \pm 17)\ \text{MeV}]^3$
Strange-quark condensate	$\langle\bar{s}s\rangle(2\ \text{GeV})$	$(0.8 \pm 0.3) \times \langle\bar{q}q\rangle(2\ \text{GeV})$
Two-gluon condensate	$\left\langle\frac{\alpha_s}{\pi}GG\right\rangle$	$(0.024 \pm 0.012)\ \text{GeV}^4$

5. Effective Continuum Threshold: Allowing for Dependence on Borel Parameter(s)

Entering in the course of the evaluation of QCD sum rules at the level of the basic QCD degrees of freedom, the effective continuum threshold s_{eff} constitutes, indisputably, one of the key quantities of the entire formalism: to a large extent, it determines the numerical value of any hadron parameter extracted from some QCD sum rule. In order to improve the output of this QCD sum-rule technique and to acquire, in a systematic manner, an idea of the *intrinsic* uncertainties of the approach [25], we collected arguments for a dependence of this effective continuum threshold on the Borel parameters introduced, as new variables, into this framework upon performing Borel transformations [26], here summarized by the generic label τ: $s_{\text{eff}} = s_{\text{eff}}(\tau)$. Surprisingly, the authors of Ref. [29] question this τ dependence; by providing a few clarifying remarks on this issue, let us try to avoid misconception:

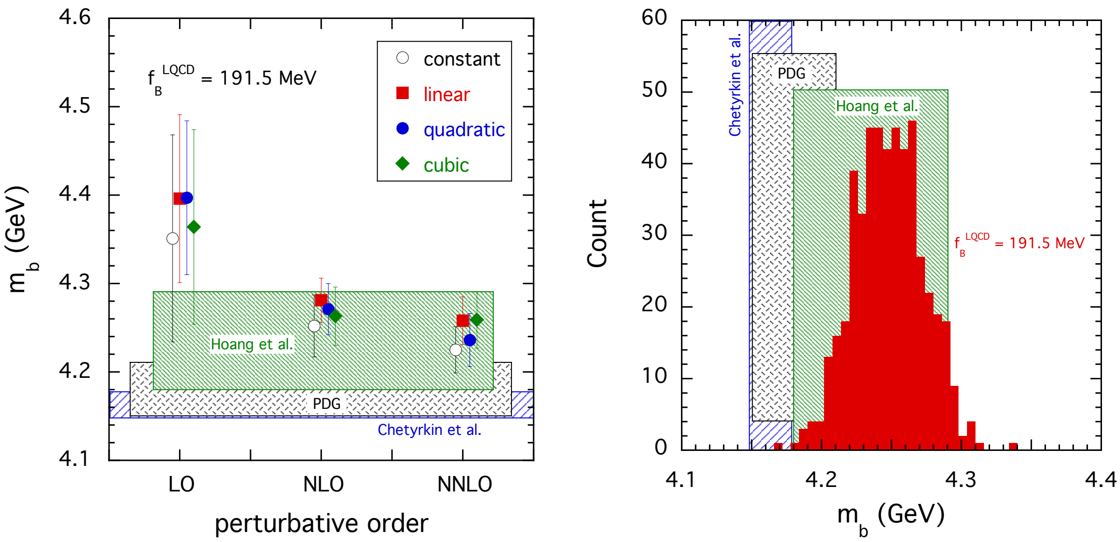
- The τ dependence of the effective continuum threshold is just a trivial and direct consequence of requiring QCD sum rules such as Eq. (4.1) to be *rigorous* relations; from this point of view, $s_{\text{eff}}(\tau)$ is a convenient tool to realize exact quark–hadron duality and as such *not* questionable.

- Beyond doubt, one may stick to assuming s_{eff} to be a τ-independent constant. QCD sum rules of the kind (4.1) then remain truly *approximate* relations; one can then merely try to minimize the discrepancy between QCD and hadron sides of one's sum rule in suitably chosen τ ranges, to derive in this way some "best" s_{eff} value. In actual extractions, one simultaneously fits both effective continuum threshold on the QCD side and bound-state features on the hadronic side.

- Anyway, we should keep in mind one fact: whatever one does, any bound-state parameter can be extracted from QCD sum rules only with limited accuracy reflected by its *systematic* error, even if the OPE for the correlator is known with arbitrarily high accuracy in a limited τ range, the Borel window. Thus, in principle *any* algorithm for fixing s_{eff} can be used if it enables one to get a realistic estimate of this systematic error. Explicit examples from quantum mechanics (where the "exact" bound-state observables may be found by solving a Schrödinger equation) show that procedures based on τ-independent s_{eff} entail *uncontrollable* errors of the extracted bound-state properties; we did not succeed in identifying any example where such a treatment yields a realistic estimate of its systematic uncertainty [25]. In contrast to this, our procedure, based on τ-dependent s_{eff} [26], provides realistic systematic-error estimates and more precise estimates of the central values of extracted bound-state parameters compared to the outcomes if forcing effective continuum thresholds by arbitrary decision to be τ-independent constants.

6. Reverting the Line of Thought: Calculating the $\overline{\rm MS}$ Mass m_b of the Bottom Quark

Even if the rapid variation (3.1) of f_B with m_b renders difficult to determine f_B from knowledge of m_b, it offers a possibility to arrive at a precision prediction for m_b by taking advantage of accurate evaluations of $f_{B_{(s)}}$ provided by lattice QCD. We seize this opportunity by implementing in the QCD sum rule (4.1) the τ dependence of the effective continuum threshold $s_{\rm eff}(\tau)$ in form of a polynomial *Ansatz* for $s_{\rm eff}(\tau)$ up to third order. Figure 1 presents a pictorial overview of our findings. Following the evolution of our m_b results with increasing perturbative accuracy (*cf.* Table 4) from $O(1)$ leading order (LO) via $O(\alpha_s)$ next-to-leading order (NLO) to $O(\alpha_s^2)$ next-to-next-to-leading order (NNLO), we find for m_b a nice perturbative convergence, *viz.*, a decrease of its central value and its OPE error.



(a) (b)

Figure 1: Extraction of the mass of the bottom quark in $\overline{\rm MS}$ renormalization scheme, $m_b \equiv \overline{m}_b(\overline{m}_b)$, from our heavy–light QCD sum rule (4.1) by a bootstrap analysis of the errors of all OPE parameters for a central value of the beauty-meson decay constant f_B of $f_B = 191.5$ MeV: (a) Our predictions for m_b calculated for different perturbative accuracy of the correlator (identified by the labels "LO," "NLO," and "NNLO," respectively) and different order of our polynomial *Ansatz* employed for the effective continuum threshold $s_{\rm eff}(\tau)$ (indicated by "constant," "linear," "quadratic," and "cubic," respectively). For comparison, the ranges corresponding to the $(\pm 1\,\sigma)$ errors of the m_b values reported, for instance, by Chetyrkin *et al.* [7], Hoang *et al.* [9], and the Particle Data Group (PDG) [28] are represented by the differently shaded rectangles. (b) Bootstrapping results for the distribution of masses m_b obtained by assuming Gaussian distributions for the OPE parameters except for the renormalization scales μ and ν and, for the latter, uniform distributions in the interval $3\,{\rm GeV} < \mu, \nu < 6\,{\rm GeV}$.

Table 4: Bottom-quark mass $m_b \equiv \overline{m}_b(\overline{m}_b)$ in $\overline{\rm MS}$ renormalization scheme: tracing perturbative convergence.

Perturbative order	m_b (GeV)
Leading order (LO)	$4.38 \pm 0.1_{\rm (OPE)} \pm 0.020_{\rm (syst)}$
Next-to-leading order (NLO)	$4.27 \pm 0.04_{\rm (OPE)} \pm 0.015_{\rm (syst)}$
Next-to-next-to-leading order (NNLO)	$4.247 \pm 0.027_{\rm (OPE)} \pm 0.011_{\rm (syst)}$

The *OPE uncertainty* of our QCD sum-rule extraction of m_b arises from the uncertainties of the OPE parameters listed in Table 3 and from allowing the two renormalization scales μ [demanded by the strong coupling $\alpha_s(\mu)$] and ν [introduced when expressing the b-quark pole mass in terms of the $\overline{\rm MS}$ mass $\overline{m}_b(\nu)$] to vary independently in the interval $3\,\text{GeV} < \mu, \nu < 6\,\text{GeV}$; we estimate this error by a bootstrap analysis. Table 5 discloses all individual contributions to our NNLO-level prediction; adding these in quadrature gives 27 MeV as total OPE error. The *systematic uncertainty* of the QCD sum-rule formalism is estimated from the spread of results obtained for different *Ansätze* for $s_{\rm eff}(\tau)$. Here, it amounts to 11 MeV. Moreover, the certainly limited accuracy of all hadronic input forces us to take into account an additional uncertainty labelled as *experimental*, even if it derives from lattice QCD but not from experimental observation. In our case, $f_B^{\rm LQCD}$ adds a (Gaussian) error of 18 MeV.

Table 5: Composition of OPE uncertainty: contributions by uncertainties of all parameters entering the OPE.

OPE quantity	Individual contribution (MeV)
Light-quark mass	4
Strong coupling constant	8
Quark condensate	20
Gluon condensate	7
Renormalization scales	14

To make a long story short, our findings for the bottom-quark $\overline{\rm MS}$ mass $m_b \equiv \overline{m}_b(\overline{m}_b)$, extracted from a Borel QCD sum rule for the correlator of two heavy–light quark currents known up to $O(\alpha_s^2)$ accuracy by adopting precise lattice-QCD evaluations of the B-meson decay constant as input, reads

$$m_b = (4.247 \pm 0.027_{\rm (OPE)} \pm 0.018_{\rm (exp)} \pm 0.011_{\rm (syst)})\ \text{GeV}\ . \tag{6.1}$$

Evidently, the systematic error is under control. Adding all uncertainties in quadrature finally yields

$$m_b = (4.247 \pm 0.034)\ \text{GeV}\ . \tag{6.2}$$

7. Summary of Main Results and Conclusions

The observation of the unexpected scale (3.1) of *negative correlation* between m_b and the QCD sum-rule prediction for f_B forms both basis and starting point of our entire subsequent investigation:

$$\frac{\delta f_B}{f_B} \approx -8\,\frac{\delta m_b}{m_b}\ .$$

Given this behaviour, feeding sufficiently accurate lattice-QCD values of f_B into our QCD sum-rule machinery renders possible a precise evaluation of the b-quark mass, culminating in our predictions (6.1) and (6.2) [10]. Confronted with other published predictions (see Table 1), our m_b result enjoys excellent agreement with Ref. [9], acceptable agreement with Ref. [6], and agreement at the level of two standard deviations with the Particle Data Group average $m_b = (4.18 \pm 0.03)$ GeV [28]; there is, however, undeniable tension with the finding of Ref. [7] and the value $m_b = (4.171 \pm 0.009)$ GeV by Ref. [30]. For completeness, with our m_b result (6.2) Eq. (4.1) predicts, for the $B_{(s)}$ decay constants,

$$f_B = \left(192.0 \pm 14.3_{\rm (OPE)} \pm 3.0_{\rm (syst)}\right)\ \text{MeV}\ , \qquad f_{B_s} = \left(228.0 \pm 19.4_{\rm (OPE)} \pm 4_{\rm (syst)}\right)\ \text{MeV}\ .$$

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Acknowledgments. D.M. was supported by the Austrian Science Fund (FWF), project no. P22843.

References

[1] P. Dimopoulos *et al.* (ETM Collaboration), JHEP **1201** (2012) 046.

[2] N. Carrasco *et al.* (ETM Collaboration), PoS(**Lattice 2012**)104 (2012); PoS(**ICHEP2012**)428 (2012).

[3] V. Gimenez, L. Giusti, G. Martinelli, and F. Rapuano, JHEP **03** (2000) 018.

[4] C. McNeile, C. Michael, and G. Thompson (UKQCD Collaboration), Phys. Lett. B **600** (2004) 77.

[5] F. Bernardoni *et al.* (ALPHA Collaboration), Nucl. Phys. Proc. Suppl. **234** (2013) 181.

[6] J. H. Kühn and M. Steinhauser, Nucl. Phys. B **619** (2001) 588; **640** (2002) 415.

[7] K. G. Chetyrkin *et al.*, Phys. Rev. D **80** (2009) 074010.

[8] C. McNeile *et al.* (HPQCD Collaboration), Phys. Rev. D **82** (2010) 034512.

[9] A. Hoang, P. Ruiz-Femenia, and M. Stahlhofen, JHEP **1210** (2012) 188.

[10] W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D **88** (2013) 056011.

[11] W. Lucha, F. F. Schöberl, and D. Gromes, Phys. Rep. **200** (1991) 127.

[12] W. Lucha and F. F. Schöberl, Int. J. Mod. Phys. A **14** (1999) 2309.

[13] W. Lucha, D. Melikhov, and S. Simula, Phys. Lett. B **687** (2010) 48; Phys. Atom. Nucl. **73** (2010) 1770.

[14] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147** (1979) 385.

[15] T. M. Aliev and V. L. Eletsky, Yad. Fiz. **38** (1983) 1537.

[16] K. G. Chetyrkin and M. Steinhauser, Phys. Lett. B **502** (2001) 104; Eur. Phys. J. C **21** (2001) 319.

[17] S. Narison, Phys. Lett. B **520** (2001) 115.

[18] M. Jamin and B. O. Lange, Phys. Rev. D **65** (2002) 056005.

[19] W. Lucha, D. Melikhov, and S. Simula, J. Phys. G **38** (2011) 105002.

[20] S. Narison, Phys. Lett. B **718** (2013) 1321.

[21] W. Lucha, D. Melikhov, and S. Simula, Phys. Lett. B **701** (2011) 82.

[22] H. Na *et al.* (HPQCD Collaboration), Phys. Rev. D **86** (2012) 034506.

[23] C. McNeile *et al.* (HPQCD Collaboration), Phys. Rev. D **85** (2012) 031503(R).

[24] A. Bazavov *et al.* (Fermilab Lattice and MILC Collaborations), Phys. Rev. D **85** (2012) 114506.

[25] W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D **76** (2007) 036002; Phys. Lett. B **657** (2007) 148; Phys. Atom. Nucl. **71** (2008) 1461; Phys. Lett. B **671** (2009) 445; D. Melikhov, Phys. Lett. B **671** (2009) 450.

[26] W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D **79** (2009) 096011; J. Phys. G **37** (2010) 035003; W. Lucha, D. Melikhov, H. Sazdjian, and S. Simula, Phys. Rev. D **80** (2009) 114028.

[27] G. Colangelo *et al.*, Eur. Phys. J. C **71** (2011) 1695.

[28] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86** (2012) 010001.

[29] P. Gelhausen, A. Khodjamirian, A. A. Pivovarov, and D. Rosenthal, Phys. Rev. D **88** (2013) 014015.

[30] S. Bodenstein *et al.*, Phys. Rev. D **85** (2012) 034003.