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The gradient flow in a twisted box

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We study the perturbative behavior of the gradient flow in a twisted box. We apply this information to define a running coupling using the energy density of the flow field. We study the step-scaling function and the size of cutoff effects in SU(2) pure gauge theory. We conclude that the twisted gradient flow running coupling scheme is a valid strategy for step-scaling purposes due to the relatively mild cutoff effects and high precision.

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1. Introduction

The Yang-Mills gradient flow [1, 2] is a new powerful tool to investigate non-perturbative aspects of non abelian quantum field theories. In the context of $SU(N)$ Yang-Mills theories one introduces an extra coordinate t (called flow time, not the same as Euclidean time, denoted x_4), and defines a flow gauge field $B_\mu(x,t)$ according to the equation

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t)\,, \tag{1.1}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \tag{1.2}$$

with initial condition

$$B_\mu(x,0) = A_\mu(x)\,. \tag{1.3}$$

Since $D_\nu G_{\nu\mu}(x,t) \sim -\delta S_{YM}/\delta B_\mu$ the flow drives the gauge field towards a classical solution (i.e. the flow smooths the field). Due to this smoothing property ultra-violet fluctuations are suppressed and composite operators do not need any renormalization at positive flow time. In particular the energy density

$$\langle E(t)\rangle = \frac{1}{4}\langle G_{\mu\nu}G_{\mu\nu}\rangle \tag{1.4}$$

has the following perturbative expansion in $\mathbb{R}^4$[1]

$$\langle E(t)\rangle = \frac{3g^2_{\overline{MS}}}{16\pi^2 t^2}(1 + c_1 g^2_{\overline{MS}} + \mathscr{O}(g^4_{\overline{MS}})) \tag{1.5}$$

and therefore can be used for a non-perturbative definition of the coupling at a scale $\mu = 1/\sqrt{8t}$

$$\alpha(\mu) = \frac{4\pi}{3}t^2\langle E(t)\rangle = \alpha_{\overline{MS}}(\mu) + \ldots\,. \tag{1.6}$$

Moreover, in a finite volume one can identify the renormalization scale μ with the size of the system and use the energy density to define a running coupling.

In these proceedings we would like to explore an alternative to the previously studied running coupling schemes based on the gradient flow that use either periodic [3] or Schrödinger Functional [4] boundary conditions. It is based on twisted boundary conditions for the gauge fields and has several practical advantages. The twisted boundary conditions, if chosen appropriately, guarantees that the only zero modes of the action are the gauge modes, and therefore this coupling definition is analytic in $g^2_{\overline{\mathrm{MS}}}$ and has an universal 2-loop beta function. Moreover in this setup the fields still live on a torus and therefore there are no boundary counterterms: $\mathscr{O}(a)$ improvement is guaranteed without any tuning. The weak point of this running coupling scheme shows up when one considers Yang-Mills fields coupled to matter. Fermions in the fundamental representation are naively incompatible with the twisted boundary conditions. One can overcome this obstruction in $SU(N)$ YM theories only when N_f is an integer multiple of N. On the other hand fermions in multi-index representations (like adjoint fermions) do not suffer from this obstruction.

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2. Twisted boundary conditions

We have no space here to review the basics of twisted boundary conditions. Instead we will say a few basic things to clarify the notation and refer the reader to the literature. In [5] the reader can find a nice summary on twisted boundary conditions with a similar notation and in a similar setup than the ones used here. This setup is basically the same as the one used in the Twisted Polyakov Loop (TPL) scheme [6]. Finally the review [7] contains more in-depth information and proofs about some of the results used here.

We are going to work in a four dimensional torus $\mathscr{T}^4$ of sides $L \times L \times L \times L$. The basic idea is that on a torus physical quantities have to be periodic, but this does not imply that the gauge potential $A_\mu(x)$ has to be periodic. It is enough if it is periodic modulo a gauge transformation

$$A_\mu(x+L\hat{\nu}) = \Omega_\nu(x) A_\mu(x) \Omega_\nu^+(x) + \Omega_\nu(x) \partial_\mu \Omega_\nu^+(x). \tag{2.1}$$

The matrices $\Omega_\mu(x)$ are called twist matrices, and they have to obey the consistency relation

$$\Omega_\mu(x+L\hat{\nu})\Omega_\nu(x) = z_{\mu\nu}\Omega_\nu(x+L\hat{\mu})\Omega_\mu(x) \tag{2.2}$$

where $z_{\mu\nu}$ are elements of the center of $SU(N)$. The particular choice of twist matrices is irrelevant, since they change under gauge transformations, but it is easy to check that $z_{\mu\nu}$ is gauge invariant, and therefore it encodes the physical part of the twisted boundary conditions. We are going to use a particular setup of this general scheme: we twist the plane x_1, x_2, while the gauge potential will be periodic in the directions x_3 and x_4. Moreover we are going to choose

$$z_{12} = z = e^{2\pi\imath/N}, \tag{2.3}$$

and the twist matrices to be constant $\Omega_{1,2}(x) = \Omega_{1,2}$ obeying the relation

$$\Omega_1\Omega_2 = e^{2\pi\imath/N}\Omega_2\Omega_1. \tag{2.4}$$

We will define the usual space momentum

$$p_\mu = \frac{2\pi n_\mu}{L}, \quad \mu = 1,2,3,4;\ n_\mu = 0,\ldots \tag{2.5}$$

and what is usually called the color momentum

$$\tilde{p}_i = \frac{2\pi\tilde{n}_i}{NL}, \quad i = 1,2;\ \tilde{n}_i = 0,\ldots,N-1. \tag{2.6}$$

The total momentum is the sum of both

$$P_i = p_i + \tilde{p}_i \quad (i = 1,2), \tag{2.7}$$

$$P_{3,4} = p_{3,4}. \tag{2.8}$$

It can be proved [7] that any gauge connection compatible with these particular boundary conditions can be written as

$$A_\mu^a(x)T^a = \frac{1}{L^4}\sum_{p,\tilde{p}\neq 0} \tilde{A}_\mu(P) e^{\imath Px}\hat{\Gamma}(P), \tag{2.9}$$

where $\tilde{A}_\mu(P)$ are complex coefficients (*not* matrices) and $\hat{\Gamma}(P)$ are matrices given by

$$\hat{\Gamma}(P) = \Omega_1^{-k\tilde{n}_2}\Omega_2^{k\tilde{n}_1}. \tag{2.10}$$

Note that the only constant gauge connection compatible with our choice of boundary conditions is $A_\mu = 0$, and this is the only configuration (up to gauge transformations) with zero action.

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3. $\langle E(t)\rangle$ to leading order in a twisted box and running coupling definition

In perturbation theory one scales the gauge potential with the bare coupling $A_\mu \to g_0 A_\mu$. The flow field of equation (1.1) becomes a function of g_0 with an asymptotic expansion

$$B_\mu(x,t) = \sum_{n=1}^{\infty} B_{\mu,n}(x,t) g_0^n. \tag{3.1}$$

After gauge fixing and inserting this expansion in (1.1), we find that to leading order the gradient flow equation reads

$$\frac{\mathrm{d}B_{\mu,1}(x,t)}{\mathrm{d}t} = \partial_\nu^2 B_{\mu,1}(x,t)\,, \quad B_{\mu,1}(x,0) = A_\mu(x). \tag{3.2}$$

The solution to this linear equation compatible with our twisted boundary conditions can be written as

$$B_{\mu,1}(x,t) = \frac{1}{L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t} \tilde{A}_\mu(P) e^{\imath P x} \hat{\Gamma}(P). \tag{3.3}$$

We will expand our observable of interest $\langle E(t)\rangle$ in powers of g_0

$$\langle E(t)\rangle = \frac{1}{4}\langle G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\rangle = \mathscr{E}(t) + \mathscr{O}(g_0^4)\,, \tag{3.4}$$

where the leading order contribution is given by

$$\mathscr{E}(t) = \frac{g_0^2}{2}\langle \partial_\mu B_{\nu,1}\partial_\mu B_{\nu,1} - \partial_\mu B_{\nu,1}\partial_\nu B_{\mu,1}\rangle. \tag{3.5}$$

A short computation gives as result

$$\mathscr{E}(t) = \frac{3g_0^2}{2L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t}. \tag{3.6}$$

To define a running coupling we simply identify the renormalization scale $\mu = 1/\sqrt{8t}$ with the linear size of the finite volume box

$$\sqrt{8t} = cL. \tag{3.7}$$

The parameter c identify the scheme. The definition of the twisted gradient flow running coupling reads

$$g^2_{TGF}(L) = \mathscr{N}_T^{-1}(c) t^2 \langle E(t)\rangle\Big|_{t=c^2L^2/8} = g^2_{\overline{\rm MS}} + \mathscr{O}(g^4_{\overline{\rm MS}}) \tag{3.8}$$

with

$$\mathscr{N}_T(c) = \frac{3g_0^2 c^4}{128} \sum_{p,\tilde{p}\neq 0} e^{-\frac{c^2L^2}{4}P^2} = \frac{3g_0^2 c^4}{128} \sum_{n_\mu=-\infty}^{\infty} \sum_{\tilde{n}_i=0}^{N-1\,\prime} e^{-\pi^2 c^2 (n^2 + \tilde{n}^2/N^2 + 2\tilde{n}_i n_i/N)}. \tag{3.9}$$

The prime over the sum recalls that the term $\tilde{n}_1 = \tilde{n}_2 = 0$ has to be dropped.

β	$g^2_{\rm TGF}(L)$			
	$L/a=20$	$L/a=24$	$L/a=30$	$L/a=36$
12.0	0.40162(60)	0.40514(62)	0.40772(64)	0.41078(64)
10.0	0.50340(78)	0.50786(79)	0.51280(81)	0.51809(83)
8.0	0.6742(11)	0.6809(11)	0.6900(11)	0.6987(11)
7.0	0.8125(13)	0.8240(13)	0.8369(13)	0.8497(13)
6.0	1.0222(17)	1.0379(17)	1.0608(17)	1.0819(18)
5.0	1.3766(23)	1.4091(23)	1.4562(25)	1.4968(25)
4.0	2.1439(38)	2.2260(41)	2.3453(42)	2.4465(44)
3.75	2.5047(47)	2.6107(47)	2.7636(51)	2.9277(54)
3.5	3.0037(57)	3.1720(60)	3.4170(65)	3.6494(69)
3.25	3.7581(72)	4.0223(76)	4.4397(87)	4.8568(99)
3.0	5.088(11)	5.630(12)	6.573(15)	7.587(20)
2.9	5.989(13)	6.835(17)	8.518(26)	-
2.8	7.439(20)	9.116(27)	-	-
2.75	8.699(25)	-	-	-

Table 1: Values of the twisted gradient flow coupling for different values of β in lattices of sizes $L/a = 20,24,30,36$. These values were determined with 2048 measurements well spaced in Monte Carlo time so that autocorrelations were negligible.

4. $SU(2)$ running coupling

As a test we have performed a non-perturbative running of the $SU(2)$ coupling. Here we will only sketch the simulations and results. A detailed description of the simulations, algorithms and parameters will be presented later [8]. We have performed simulations with the Wilson action for $\beta \in [2.75,12.0]$ on lattices of sizes $L/a = 10,12,15,18$, and to perform the step-scaling, also on lattices of sizes $L/a = 20,24,30,36$. We collect 2048 measurements of twisted gradient flow coupling for $c = 0.3$. The measurements are well spaced in simulation time and autocorrelations are negligible. With this statistics we achieve a precision between a $0.15-0.3\%$, independently of the value of L/a (see table 1 for some representative values of the coupling).

For each value of L/a we fit the data using a Padé ansatz of the form

$$g^2_{TGF}(a/L,\beta) = \frac{4}{\beta}\frac{\sum_{n=0}^{N-1} a_n(a/L)\beta^n + \beta^N}{\sum_{n=0}^{N-1} b_n(a/L)\beta^n + \beta^N}. \tag{4.1}$$

We obtain fits with good quality ($\chi^2/\text{ndof} \sim 1$) with 4 parameters in all our cases. An example of such a fit, for the case of the $L/a = 36$ lattice can be seen in the figure 2a.

We apply the usual step-scaling technique, starting the recursion in a volume $L_{\max}$ where $g^2_{\rm TGF}(L_{\max}) = 7.5$. The continuum extrapolations of the step-scaling function are very flat, as can be seen in figure 1. The non-perturbative running of the coupling is recursively carried over a factor 2^{25} change in the scale down to a volume $L_{\min}$ where $g^2_{\rm TGF}(L_{\min}) = 0.5323(83)$. Figure 2b shows the scale dependence of the coupling, and a comparison with the universal 2-loop $\beta-$function.

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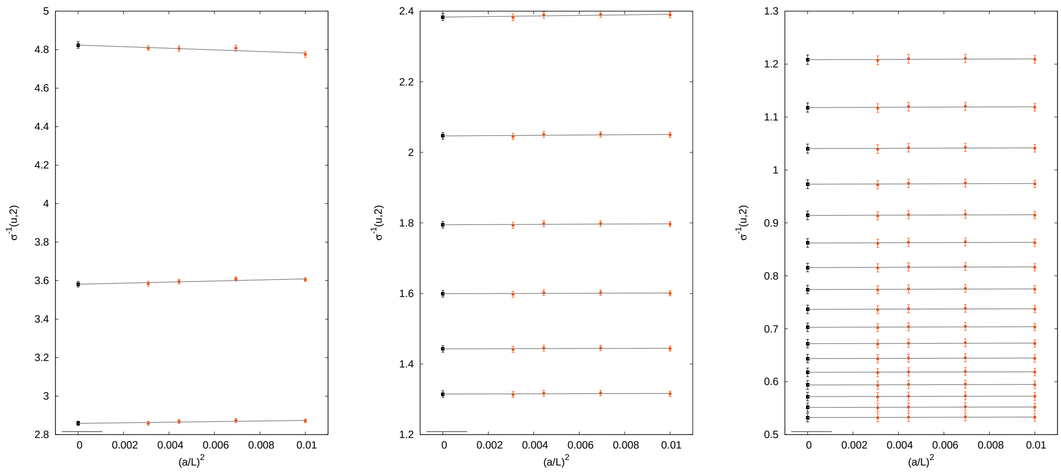
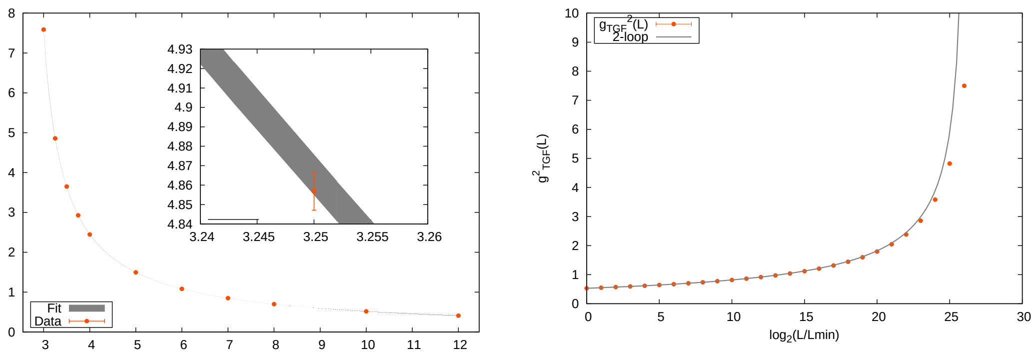


(a) (b) (c)

Figure 1: Continuum extrapolation of the step-scaling function. We start the recursion in a volume $L_{\rm max}$ where $g^2_{\rm TGF}(L_{\rm max}) = 7.5$.



(a) (b)

Figure 2: Left: The twisted gradient flow coupling for $L/a = 36$ as a function of β. The points are fitted to a Padé functional form (grey band in the figure) with 4 parameters and the $\chi^2/\text{ndof} = 5.9/7$. Right: Non-perturbative running of the twisted gradient flow coupling.

5. Conclusions and comments

In this proceedings we have studied perturbatively the gradient flow in a four dimensional torus with twisted boundary conditions. The energy density of the flow field can be used to define a running coupling. We have presented some preliminary results on a $SU(2)$ running coupling where we have shown that the twisted gradient flow coupling is a convenient choice. The observable is precise and 2048 independent measurements are enough to reach a per mile precision. Cutoff effects of the step-scaling functions are mild.

Some comments about the inclusion of matter fields are in order. In principle our coupling definition is perfectly valid with any kind of matter fields, but fermions in the fundamental representation are incompatible with the twisted boundary conditions. A fermion in the fundamental representation transforms as

$$\psi(x+L\hat{\mu}) = \Omega_\mu \psi(x)\,, \tag{5.1}$$

but this transformation is not consistent due to

$$\psi(x+L\hat{1}+L\hat{2}) = \Omega_1\Omega_2\psi(x) \tag{5.2}$$

$$\psi(x+L\hat{2}+L\hat{1}) = \Omega_2\Omega_1\psi(x) = e^{2\pi\imath/N}\Omega_1\Omega_2\psi(x)\,. \tag{5.3}$$

This obstruction can be overcome if the number of fermions is an integer multiple of the degree of the gauge group N [9].

On the other hand fermions in multi-index representations (like adjoint fermions) do not suffer from any kind of restriction.

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