

Quantum ideal hydrodynamics on the lattice

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After discussing the problem of defining the hydrodynamic limit from microscopic scales, we give an introduction to ideal hydrodynamics in the Lagrange picture, and show that it can be viewed as a field theory, which can be quantized using the usual Feynman sum-over-paths prescription. We then argue that this picture can be connected to the usually neglected thermal microscopic scale in the hydrodynamic expansion. After showing that this expansion is generally non-perturbative, we show how the lattice can be used to understand the impact quantum and thermal fluctuations can have on the fluid behavior.

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1. Introduction

At first sight, “quantizing hydrodynamics” appears nonsensical: Hydrodynamics is usually defined as a classical theory, because it is an infrared effective description of “many” microscopic degrees of freedom which have equilibrated. Nevertheless, as shown in [1, 2, 3], ideal hydrodynamics *can* be rewritten in quantum form. Can we learn something useful from this exercise?

A lot has been written, recently, about the hydrodynamic limit of “infinitely strongly coupled” theories, and whether a “quantum lower limit” of viscosity over entropy density (η/s) exists. While famous results in both the Boltzmann limit [4] and the strongly coupled planar limit of a conformal field theory [5] suggest the existence of such a lower bound at $\eta/s \sim \mathcal{O}(10^{-2})$, a physical argument for such a limit’s existence beyond such idealized setups is still lacking.

As argued in [6] and [7], such an argument might come from the possibility, at “infinitely small mean free path”, for *thermal* fluctuations to excite hydrodynamic degrees of freedom: When viscosity is so low that “typical” sound waves, of frequency $\sim T$ and amplitude comparable to a thermal fluctuation, $\Delta\rho/\langle\rho\rangle \sim C_V/T^3$ (where ρ is the energy density, C_V the heat capacity and T the temperature), survive for a time much larger than the thermal scale, $\sim 1/T$, Kubo’s formula needs to be renormalized to account for the energy-momentum carried by the sound waves.

While in the planar limit [5] this contribution is negligible as it is $\mathcal{O}(N_c^0)$, at any finite degeneracy it will alter both the “viscosity” and the “entropy density”, and, because of turbulence, such perturbations are not guaranteed to stay small without a cutoff.

In the absence of a sizeable microscopic cutoff, quantum mechanics becomes relevant: a typical turbulent evolution in four dimensions involves a cascade from high amplitude low frequency perturbations to low amplitude high frequency ones [8]. This tacitly assumes that one can conserve energy by, simultaneously, decreasing the amplitude and increasing the frequency indefinitely. Classically, this is indeed possible. However, by Planck’s law, a sound wave can not have a transverse energy $\sim \omega c_s^{-2} \Delta\rho$ smaller than its frequency ω in natural units. Quantum mechanics and energy conservation, therefore, cut the Kolmogorov cascade at frequencies $\sim T$ in natural units. Since this cutoff is usually provided by viscosity, the existence of a quantum limit on viscosity is plausible from hydrodynamic arguments alone.

More generally [9], it is often neglected that hydrodynamics is an expansion not in one small parameter, but two: The most explored one is the dissipative Knudsen number, $\sim l_{mfp}/R \sim \eta/(sTR)$, the mean free path ($l_{mfp} \sim \eta/(sT) \sim$ the sound wave dissipation length) over “system size” R . The other parameter, the “distance between microscopic degrees of freedom” $\sim 1/(gT)$, where g is the microscopic degeneracy, has to be $\ll l_{mfp}$ for the BBGKY hierarchy to converge (in [5], $N_c \rightarrow \infty$ gives the same effect). Thus, hydrodynamics requires, in terms of the microscopic degeneracy g , the entropy s , and the viscosity η , that

$$\underbrace{l_{micro}}_{s^{-1/3} \sim 1/g^{1/3} T} \ll \underbrace{l_{mfp}}_{\eta/(sT)} \ll \underbrace{l_{macro}}_{R \sim \rho/\partial_\mu \rho} \quad (1.1)$$

Keeping the mean free path small but *comparable* to $1/(gT)$ is precisely the limit where microscopic thermal fluctuations can excite sound-waves. This is an explored limit, yet it is relevant for systems such as heavy ion collisions and ultracold atoms, where the number of degrees of freedom is clearly nowhere near infinite, and $(T^3 V)^{-1}$ is not far from $(sTR)/\eta$.

2. The theory

3D Ideal hydrodynamics with no conserved charges¹ can be rewritten [1, 2] in terms of three fields $\phi^{I=1,2,3}$, which physically correspond to the x, y, z coordinates of the comoving frame w.r.t. the lab frame. The choice of $\phi^{I=1,2,3}$ is of course not unique, as a perfect fluid is homogeneous, and in its comoving frame, invariant under rotations and rescalings. This ambiguity can be represented as a symmetry, restricting the Lagrangian to the form

$$L = F(B) = T_0^4 F(\det(B_{IJ})) \quad , \quad B_{IJ} = \partial^\mu \phi_I \partial_\mu \phi_J \quad (2.1)$$

The function $F(B)$ is left arbitrary, as it corresponds to different equations of state for the fluid.

Dimensional analysis makes it apparent that the $F(B)$ should be defined, in terms of an energy scale T_0 . T_0 , in this context, must be the “temperature” of the microscopic degrees of freedom. Equivalently, T_0^{-1} means the distance at which such microscopic degrees of freedom become relevant. Note that this tells us *only* about the density (and equilibrium/quantum fluctuations of it), and is in general different from the mean free path of the interacting theory, which in the ideal hydrodynamic limit goes to zero.

Thus, if Eq. 2.1 is used to build a partition function, the effective “Planck’s constant” becomes dimensional; It is natural to identify T_0^{-1} with the l_{micro} parameter in Eq. 1.1, since as $T_0 \rightarrow \infty$ the classical picture of hydrodynamics should emerge. Eq. 2.1, however, makes it clear the expansion in T_0 might be strongly non-perturbative, as can be expected from a “turbulent” thermally fluctuating ideal fluid. It is straight-forward to show that the energy-momentum tensor corresponding to the Lagrangian in Eq. 2.1 is that of ideal hydrodynamics

$$T_{\mu\nu} = (p + \rho)u^\mu u^\nu - pg^{\mu\nu} \quad (2.2)$$

and hence this is simply an unusual reparametrization of ideal hydrodynamics. The energy density and pressure in this notation are

$$\rho = -F(B) \quad , \quad p = F(B) - 2B \frac{dF}{dB} \quad (2.3)$$

hydrodynamic flow is defined in terms of energy flow as

$$u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K \quad (2.4)$$

We can also show that $\partial_\mu(\sqrt{B}u^\mu) = 0$. By inspection, without any conserved charges (those are examined in [3]) we can identify

$$s = gT_0^3 \sqrt{B} \quad (2.5)$$

with the microscopic entropy. Using the Gibbs-Duhem relation, then, the temperature will be

$$T = \frac{e + p}{s} = T_0 \frac{\sqrt{B}(dF/dB)}{g} \quad (2.6)$$

¹We assume, for simplicity there are no conserved charges, so all “density” is energy density (any “particles” are balanced by antiparticles”). Dense systems can be described by an extension of the approach described here [3]

The non-perturbative nature of the theory is confirmed by examining the “vortex” degrees of freedom [2]: In a hydrostatic background, vortices *do not propagate*, yet carry arbitrarily small amounts of energy and momentum. Thus, an S-matrix cannot be defined, since quantum vortices can survive for an arbitrarily long time, and dominate the vacuum (in the $T_0 \rightarrow \infty$ limit *all* such quantum fluctuations are suppressed). Thus, at finite T_0 , the quantum expectation value of $T_{\mu\nu}$ could very well be different from the classical one, due to the backreaction of sound waves and vortices. On the lattice, one can investigate this independently of any perturbative expansion, and without deforming the theory in the infrared (as was done in [1, 7]).

Conversely, a lattice calculation could show the theory *is* deformed in the IR in a way that breaks some, if not most, of the lagrangian’s symmetries. An example for this in 2d classical hydrodynamics is the famous regular patterns of vortices that form in 2d fluids [10]. If quantum fluctuations generate solutions like this, the low energy theory will contain effective terms which break most of the symmetries of Eq. 2.1, but will be missing in the perturbative expansion.

The theory formulated via Eq. 2.1 can be put on the lattice in the usual way, via

$$\ln \mathcal{Z} = \int \mathcal{D}\phi_I \exp\left(i \int d^4x L + J\phi_I\right) \xrightarrow{\text{lattice+Wick}} \int d\phi_I^i \exp\left[-(aT_0)^4 \sum_i F(\phi_i) + J\phi_I\right] \quad (2.7)$$

care needs to be taken since this theory is non-renormalizable and, once $F(B)$ has been defined, has no free parameters. The existence of a well-defined continuous limit is therefore not guaranteed. The scaling of observables with T_0 as one approaches the limit of $T_0 \rightarrow \infty$ can however still be explored. As we will investigate non-trivial quantum structures in configuration space, such as quantum-seeded vortices, the large lattice limit will be as important as the continuum limit.

3. Lattice implementation

In order to ensure that the fields on all lattice sites participate in the same updating procedure, we use one-sided finite differences for derivatives and average over the eight per hypercube. The derivatives $\partial_\mu \phi^I$ and all quantities derived from them (e.g., B_{IJ} , u_μ , $T_{\mu\nu}$) are thereby envisioned as occupying the centers of the hypercubes. Since the fields represent the comoving coordinates of the fluid, it is better to use “shifted” variables to avoid problems with the periodic boundaries (i.e., one subtracts the hydrostatic background):

$$\pi^I = \phi^I - x^I \rightarrow \partial_\alpha \phi^I = \partial_\alpha \pi^I + 1\delta_\alpha^I. \quad (3.1)$$

We expect this theory to describe quite extended structures (e.g., vortices) which are easily created from the “vacuum” and we therefore use HMC updates in an attempt to learn something about their amenability to change in the Markov process. Thus, we require the variation of the action with respect to the local field values:

$$\begin{aligned} \frac{\delta S}{\delta \phi^I(x)} &= \frac{\delta S}{\delta \sqrt{B}} \frac{\delta \sqrt{B}}{\delta (\partial_\alpha \phi^J)} \frac{\delta (\partial_\alpha \phi^J)}{\delta \phi^I(x)} \\ &= \sum_{y,\mu,\nu,\sigma} \frac{dF}{d\sqrt{B}} \delta^{IJ} \delta(y-x \pm \hat{\mu}/2 \pm \hat{\nu}/2 \pm \hat{\sigma}/2) \frac{\sqrt{B}}{8} B_{JK}^{-1} |\epsilon_{\mu\nu\sigma\alpha}| \partial_\alpha \phi^K \Big|_{y+\hat{\alpha}/2}^{y-\hat{\alpha}/2}. \end{aligned} \quad (3.2)$$

L^4	$C(= aT_0)$	traj	$d\tau_{MD}$	accept
20^4	0.8	4000	0.001 / 0.0005	49% / 85%
16^4	1	10000	0.001	52%
12^4	1.33333	10000	0.0005	61%
10^4	1.6	10000	0.0005	41%
8^4	2	10000	0.00025	72%
6^4	2.66667	10000	0.00025	56%

Table 1: Lattice parameters for runs with constant volume.

We have implemented the updating algorithm and the calculation of observables in C code, with parallelization via OpenMP. Table 1 displays a subset of our runs performed with the aim of preserving the physical volume: $L^4 = (16/(aT_0))^4$. To achieve reasonable acceptance rates, quite small values of the molecular-dynamics time step ($d\tau_{MD}$) are required (even at such small volumes). Correspondingly, convergence to a plateau (say, in $\langle\sqrt{B}\rangle$) is slow, typically requiring the omission of the first few thousand trajectories from observables and blocking of data to avoid autocorrelations. In future runs at larger volumes and lattice spacings (ideally, $L^{-1} \ll l_{macro}^{-1} \ll a^{-1} \ll T_0$), we may need to add mass terms to the scalar fields (or perhaps work at finite density: see Ref. [3]) and extrapolate to the massless (zero chemical potential) limit.

4. Some preliminary results

A good initial example is the ideal gas EOS,

$$F(B) = T_0^4 B^{2/3}. \quad (4.1)$$

This can be easily generalized to any monotonic EoS (without phase transitions), for example an EoS, say, fitting the QCD cross-over [7]. The ideal gas, however, is a good testing laboratory as all of its parameters are very simple

$$\langle e \rangle = T_0^4 B^{2/3} = \frac{g\pi^2}{60} T^4, \quad \langle p \rangle = \frac{\langle e \rangle}{3}, \quad \langle s \rangle = T_0^3 \sqrt{B} = \frac{g\pi^2}{45} T^3, \quad T = \frac{4}{3g} T_0 B^{1/6} \quad (4.2)$$

This allows us to fill in all members of $T_{\mu\nu}$ in the static frame, $\langle T_{\mu\nu} \rangle = \text{diag}[\langle e \rangle, \langle e \rangle/3, \langle e \rangle/3, \langle e \rangle/3]$. Correlations should be localized around standard thermodynamic fluctuations

$$\langle e(x')e(x) \rangle - \langle e(x) \rangle \langle e(x') \rangle = \delta^3(\vec{x} - \vec{x}') C_V T^2 \sim \frac{4\pi^2}{15} g T_0^8 \quad (4.3)$$

Fig. 1 shows average and fluctuations of the s observable. We have checked that the other observables scale with entropy as expected from Eq. 4.2.

The left panel of Fig. 1 shows the entropy density as a function of the lattice spacing. Allowing for finite-size effects, no appreciable dependence is seen. The same goes for the rather sizeable, relative fluctuations in the entropy density (middle plot). There also appears to be little dependence upon the ‘‘macroscopic temperature’’ $(aN_i)^{-1}$, i.e., the temperature of the phonon / vortex gas. It

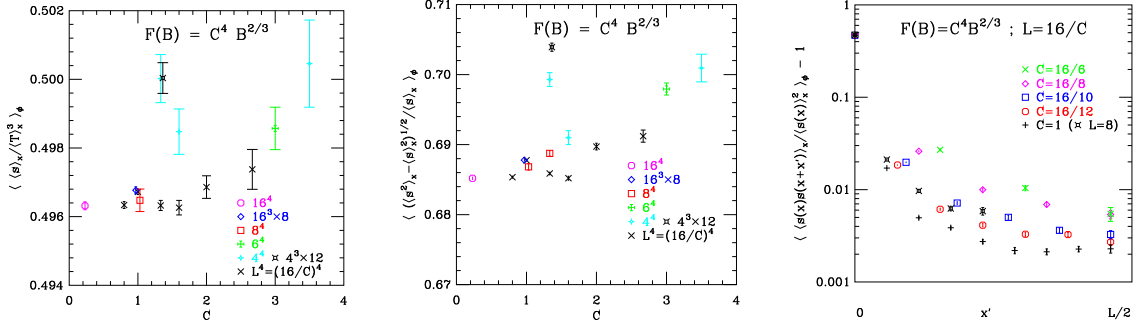


Figure 1: The expectation value of the entropy density (left), the corresponding fluctuation (middle) and the relative spatial correlation along \hat{x} (right).

will be especially interesting to see if this trend (or lack thereof) continues as we push to larger lattice extents at larger aT_0 values: i.e., as we push from the microscopic-dominated entropy region to one where macroscopic structures carry a large share of entropy. The fluctuations do indeed lead to a “correction” to the input equation of state (EoS): the EoS dictates $s/T^3 = 27/64 \sim 0.422$, whereas averaging first over the lattice leads to $\langle s \rangle_x / \langle T \rangle_x^3 \sim 0.496$. One can see in the rightmost plot that the spatial correlations in the entropy density persist across the lattice (the same can be seen for T , p , and ρ). We cannot comment more on this result as yet, beyond remarking that it might show that quantum fluctuations are non-negligible for observables at *all* scales.

The ideal hydrostatic background further requires any space component of a vector or tensors observable’s average should be zero. These can be parametrized into the flow tensor

$$\langle \Omega_{\mu\nu} \rangle = \langle u_\mu u_\nu + g_{\mu\nu} \rangle = \langle B_{IJ}^{-1} \partial_\mu \phi^I \partial_\nu \phi^J \rangle, \quad (4.4)$$

which gives insight to the turbulence seeded by quantum fluctuations. While, unlike the locally-defined scalar perturbation, the average $\Omega_{\mu\nu}$ is set by symmetries, its correlation function can reveal interesting structures. In particular, a non-vanishing value at large distances can signal the vacuum generally contains “quantum turbulence”, and can break some of the original symmetries.

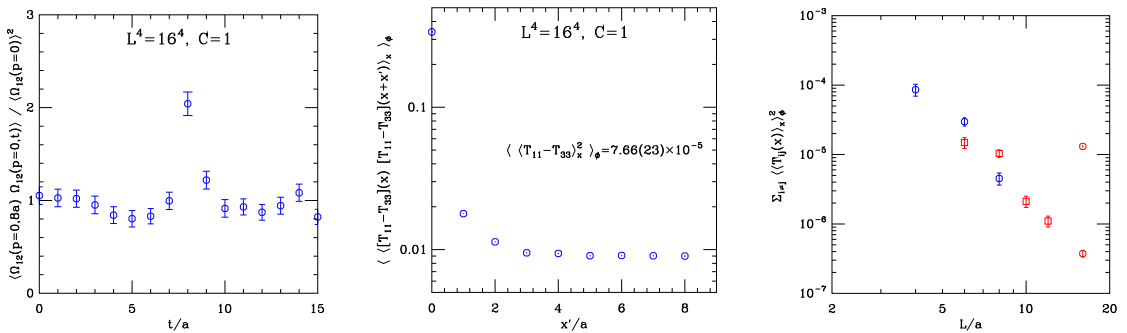


Figure 2: Left: zero-momentum-projected xy flow-tensor correlators. Middle: Spatial correlations of $xx - zz$ stress-energy tensor. Right: the squared-average of T_{ij} as a function of lattice extent L (red symbols at constant physical size $L \propto 1/C$, circles at $C = 1$; the lowest lying data fit to $\sim L^{-3.85(14)}$).

The left plot in Fig. 2 shows a zero-momentum-projected flow-tensor correlator (source at $t = 8a$). While there appears to be some structure, suggesting the presence of a quantum “sound mode”, the same is not seen at all lattice spacings (outside of the “enhancement” at $t = t_{source}$). Like with the entropy density, some components (or combinations thereof) of the flow and stress-energy tensors show non-zero, persistent spatial correlations (middle plot). Indeed, the off-diagonal spatial components (T_{ij} and Ω_{ij}) give non-zero averages (indicating non-zero flow) which only vanish in the “thermodynamic” limit ($L \rightarrow \infty$; see the rightmost plot). Even then, some ensembles appear to be rather stubborn and find a different minimum of the action, perhaps due to macroscopic, topological features (e.g., vortex rings wrapped around the torus). It would be interesting to search these lattices for such structures in order to determine to what extent such ensembles should contribute to an overall average (i.e., whether they are indeed due to the boundary conditions or not). That these elements are non-zero may also be a hint of existence of structures analogous to well-studied calorons and instantons in QCD. In a hydrodynamic context, these structures can be interpreted as quantum vortices, perhaps triggering turbulence. Observables such as the relativistic circulation $C_P = \oint_P (p + e) u_\mu dx^\mu$ across a closed path P can be used to investigate the existence and relevance of such phenomena.

In conclusion, we have argued that a lattice implementation of ideal quantum hydrodynamics can give insight into a hitherto unexplored limit of strongly interacting matter, one where the dissipation vanishes but the microscopic and macroscopic length scales may not be well separated. We have discussed the technical details of this implementation and presented some preliminary results. None of the latter should be taken as anything other than a feasibility demonstration, this is the beginning of what can only be a very involved research project.

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