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# Large-$N$ mesons

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We present an update of our project of computing the meson spectrum and decay constants in large-$N$ QCD. The results are obtained in the quenched approximation with the Wilson fermion action for $N = 2,3,4,5,6,7$ and 17 and extrapolated to $N = \infty$. We non-perturbatively determine the renormalization factors for local quark bilinears that are needed to compute the decay constants. We extrapolate our SU(7) results to the continuum limit, employing four different lattice spacings.

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## 1. Introduction

Quantum Chromodynamics (QCD), the theory of strong interactions, is characterized by local SU($N$) gauge invariance where $N = 3$ denotes the number of "colours". The adjoint gauge bosons (gluons) couple $n_f$ "flavours" of fermionic matter fields in the fundamental representation (quarks). QCD dynamically generates a mass gap. Moreover, at low temperatures, the (approximate) chiral symmetry is broken. These and other non-perturbative low energy features can be addressed systematically by lattice QCD simulations.

A different non-perturbative approach to QCD is based on an expansion in powers of $1/N$ of the inverse number of colours [1]. In the 't Hooft limit where $N$ is sent to infinity, keeping the 't Hooft coupling $\lambda = Ng^2$ ($g$ denotes the gauge coupling) as well as $n_f$ fixed, the theory simplifies considerably, see ref. [2] for a recent review. For instance, all amplitudes of physical processes are determined by a particular subset of Feynman diagrams (planar diagrams), the low-energy spectrum consists of stable meson and glueball states and the scattering matrix becomes trivial. One may study the physical $N = 3$ case, expanding around the large-$N$ limit in terms of $1/N$. Interestingly, the non-flavour-singlet spectra of QCD with sea quarks and quenched QCD agree within 10 % [3]. This may indicate that both $n_f/N$ and $1/N^2$ corrections are small in these channels.

Another non-perturbative approach to low-energy properties of non-Abelian gauge theories is based on the conjectured correspondence between gauge theories and classical gravity in an anti-de-Sitter spacetime (AdS/CFT correspondence) [4]. Unlike lattice regularization, in this case the continuous spacetime symmetry is retained but the large-$N$ limit (as well as a large 't Hooft coupling) is implied. During the last decade techniques based on this correspondence have been employed to construct models which reproduce the main features of the meson spectrum of QCD, see, e.g., ref. [5].

The large-$N$ limit also plays a central role in the chiral effective theory approach where the $N$-dependence of low-energy constants is known [6] and, within this framework, in studies of properties of unstable resonances, see, e.g., refs. [7 – 9]. Clearly, it is important to determine the meson spectrum of large-$N$ QCD to constrain effective field theory parameters and also to enable a comparison with AdS/CFT and AdS/QCD predictions.

Large-$N$ QCD still remains far from trivial and requires lattice simulation. The quenched theory becomes unitary and identical to full QCD in the large-$N$ limit where quark loop effects are suppressed. Neglecting the fermion determinant does not only save computer time but the quenched theory should converge more rapidly (with leading $1/N^2$ rather than with $n_f/N$ corrections) towards the limit $N \to \infty$. Recently, the dependence of various quantities on $N$ was studied in quenched lattice simulations. For instance, pseudoscalar and vector meson masses (among other observables) were determined in refs. [10 – 14].

In ref. [14] we chose to normalize the spectrum with respect to the pion decay constant $F$. However, the renormalization of $F$ was only done perturbatively, resulting in an estimated uncertainty of about 8 %. Here we determine the renormalization constants non-perturbatively. For instance, $Z_A$ turns out almost 10 % smaller than our previous estimate. We also perform the continuum limit for $N = 7$. Other values of $N$ are in progress which will then enable a joint large-$N$ and continuum limit extrapolation.



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## 2. Simulation details

We employ the standard Wilson action for the gauge fields and for the fermions. In our main data set we tune the lattice coupling, keeping the square root of the string tension $a\sqrt{\sigma} \approx 0.2093$ fixed, in lattice units $a$. In addition to this main infinite-$N$ extrapolation trajectory, we now also realize one finer lattice spacing $a = 0.1500/\sqrt{\sigma}$ and two coarser spacings $a = 0.2512/\sqrt{\sigma}$ and $a = 0.3140/\sqrt{\sigma}$, to enable a controlled continuum limit extrapolation.

The lattice 't Hooft coupling $\lambda = Ng^2 = 2N^2/\beta$ varies along the above trajectories of *constant physics*, i.e., constant lattice spacing in units of the string tension, by terms of $\mathcal{O}(1/N^2)$. Other strategies, e.g., keeping the pion decay constant in the chiral limit $F$, the critical temperature $T_c$, the gradient flow scale $t_0$ or $\lambda$ fixed, are admissible, with specific advantages and disadvantages. Fixing $\lambda$ for instance would be much more expensive in terms of computer time as two phase transitions have to be avoided along the extrapolation to $N = \infty$: if $\lambda$ is taken too large the system will undergo a strong coupling phase transition once $a\sqrt{\sigma} \gtrsim 0.4$ while for small volumes $\ell^3 \times 2\ell$, i.e. for $\ell < \ell_c \approx 2/\sqrt{\sigma} \gtrsim 1/T_c$, a transition (similar to the finite temperature transition) into a deconfined phase will occur. We find $\lambda$ to reduce by about 8 % at constant $a\sqrt{\sigma}$, when increasing $N$ from $N = 3$ to $N = 17$. This means setting $\lambda$ sufficiently small to avoid crossing into the strong coupling phase at large $N$ implies tiny values of $a\sqrt{\sigma} \sim \exp[-1/(2b_0\lambda)]$ at small $N$, and hence of the lattice spacing $a$. This in turn would necessitate a large number of lattice points $N_s^3 \times 2N_s \propto 1/a^4$ to remain in the confined phase $\ell = N_s a > \ell_c$.

We remark that, as long as $\ell > \ell_c$, finite volume effects are irrelevant for the large-$N$ extrapolation since these are suppressed by factors $1/N^2$ [15]. Nevertheless, due to the unitarity violations of the quenched model, at small values of $N$ the volume needs to be taken much bigger than this limit to enable simulating light pion masses down to $m_\pi \approx \sqrt{\sigma}/2$.

We cancel the leading $N$-dependence of meson decay constants by defining

$$\hat{F}_\pi = \sqrt{\frac{3}{N}} F_\pi \,, \quad \hat{f}_\rho = \sqrt{\frac{3}{N}} f_\rho \,. \tag{2.1}$$

The normalization is chosen such that $\hat{F}_X = F_X$ for $N = 3$. $F_X = f_X/\sqrt{2}$ as usual. We denote the (appropriately normalized) pion decay constant in the combined chiral and large-$N$ limit as

$$\hat{F}_\infty = \lim_{N\to\infty} \hat{F}_\pi(m_q = 0) = \lim_{N\to\infty} \sqrt{\frac{3}{N}} F := 85.9(1.2)\,\text{MeV}\,, \tag{2.2}$$

where we impose the phenomenological QCD value [16]. This gives a lattice spacing $a \approx 0.095$ fm for $N \to \infty$, along our main $a\sqrt{\sigma} = 0.2093$ trajectory. Of course we can only determine ratios of dimensionful quantities and — in the absence of experimental input from a $N = \infty$ world — any scale-setting in physical units will be arbitrary and is just meant as a guide. Nevertheless, we remark that other ways of setting the scale appear to give similar results. For instance, using the *ad hoc* value $\sigma = 1$ GeV/fm $\approx (444\,\text{MeV})^2$, our lattice spacing reads $a \approx 0.093$ fm, instead.

We realize spatial extents $N_s a = 24a \approx 2.3\,\text{fm} \gg \ell_c$ at $a\sqrt{\sigma} = 0.2093$. To investigate finite size effects, we also simulate $N_s = 16$ and $N_s = 32$ for SU(2) and SU(3). For SU(3) no significant effects are found and we conclude that our $N \geq 3$ results effectively agree with the infinite volume limit. At the two coarser spacings we simulate $N_s = 16$ and $N_s = 20$, keeping the volume approximately

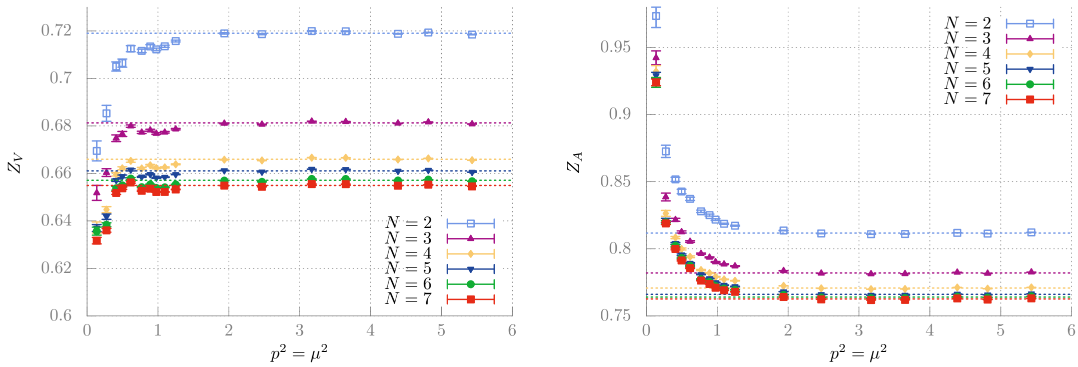


**Figure 1:** Non-perturbative determination of the vector and axial-vector renormalization constants.

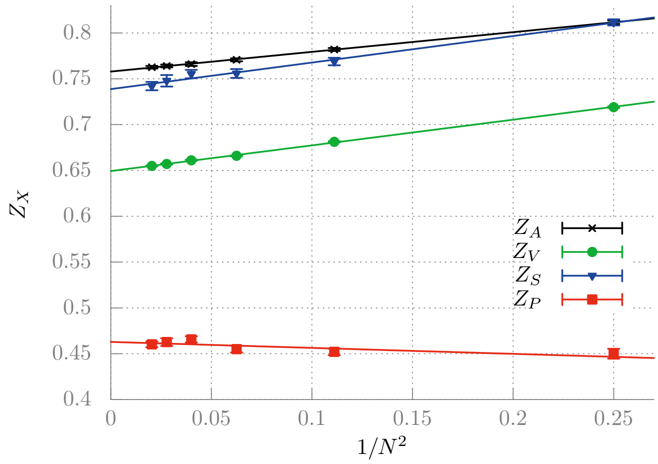


**Figure 2:** Renormalization factors versus $1/N^2$. The scalar and pseudoscalar factors $Z_S$ and $Z_P$ translate the lattice results obtained at $a^{-1} \approx 2.13$ GeV to the $\overline{\mathrm{MS}}$ scheme at a scale $\mu = 2$ GeV $\approx 23.3\hat{F}_\infty$.

constant in physical units, while we employ $N_s = 24$ and $N_s = 32$ on the finest lattice. The largest $N = 17$ we only simulate at our main lattice spacing and restrict ourselves to $N_s = 12$. These small volume SU(17) results are found to be consistent with the large-$N$ extrapolations of the $7 \geq N \geq 3$ data [14], confirming finite volume effects to become irrelevant at large $N$ and also adding credibility to our extrapolation. To enable chiral extra- and interpolations, at each $N$ we realize at least six quark masses, tuned to keep one set of pion masses approximately constant across the different SU($N$) theories and lattice spacings. These correspond to pseudoscalar masses ranging from $m_\pi \approx 2.7\sqrt{\sigma}$ down to $m_\pi \approx 0.5\sqrt{\sigma}$ for $N \geq 5$ and $m_\pi \approx 0.75\sqrt{\sigma}$ for $N \leq 4$.

## 3. Renormalization constants

The hopping parameter $\kappa$ is related to the vector and axial quark Ward identity lattice quark masses $m_q$ and $m_{\text{PCAC}}$, respectively, via

$$am_q = \frac{1}{2}\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right) = \frac{Z_A Z_S}{Z_P}(1 - b\,am)\,am_{\text{PCAC}}\,, \tag{3.1}$$

where $\kappa = \kappa_c$ corresponds to a massless quark. The improvement parameter $b$ is, for our calculation with unimproved Wilson quarks, redundant. Fitting $am_{\rm PCAC}$ for each $N$ as a function of $1/\kappa$ according to the above parametrization, we obtain the critical hopping parameters $\kappa_c(N)$ and the (scale-independent) combination of renormalization factors $Z_A Z_S/Z_P$, as described in ref. [14]. We use this to determine $Z_P(a\mu)$, once $Z_A(a)$ and $Z_S(a\mu)$ have been computed.

We determine the renormalization constants $Z_A(a)$ (required for the pion decay constant) and $Z_V(a)$ (for the vector decay constant) via the Roma-Southampton non-perturbative matching [17] to the RI'MOM scheme. In the case of $Z_S(a\mu)$ (needed for the chiral condensate and quark mass renormalization, not presented here) this is then perturbatively matched to the $\overline{\rm MS}$ scheme. To remove lattice artefacts we parameterize (see, e.g., ref. [18]):

$$Z_X(a) = Z_X^{\rm latt}(p,a) - z_0(a)S^2(ap) - z_1(a)\frac{S^4(ap)}{S^2(ap)}, \tag{3.2}$$

where $S^n(ap) := \sum_\mu (ap_\mu)^n$. $Z_X$, $z_0$ and $z_1$ are fit parameters. In the case of $Z_S$ which has an anomalous dimension, we take $S^2(ap)$ as the argument of the leading log as indicated by lattice perturbation theory. The lattice artefact subtracted data for $Z_V$ and $Z_A$ are displayed for the various $N$-values in figure 1 and the four renormalization factors are shown in figure 2.

## 4. Spectrum and decay constants

We compute correlation matrices between differently smeared interpolators. This gives us access to excited states in many channels, in addition to the ground states. We then perform joint large-$N$ and chiral extrapolations. As demonstrated in ref. [14], the $N \geq 3$ data are consistent with purely quadratic dependencies on $1/N$, with small slopes. A notable exception is the scalar particle $a_0$. The chiral extrapolations are performed as polynomials in the quark mass $m_{\rm PCAC}$ that can be determined more precisely than $m_\pi^2$. For $N \leq 5$, we detect the expected chiral log.

We interpolate and extrapolate the spectrum to three values of the quark mass, $m = 0$, $m = m_{ud}$ and $m = m_s$, where $m_\pi(m_{ud}) = 138\,\text{MeV} \approx 1.6\hat{F}_\infty$ and $m_\pi(m_s) = \left(m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^\pm}^2\right)^{1/2} = 686.9\,\text{MeV} \approx 8.0\hat{F}_\infty$. We display the resulting $m = m_{ud}$ spectrum for the lattice spacing $a \approx 0.095$ fm in figure 3. On the scale of the plot this is indistinguishable from the $m = 0$ spectrum. The value $\sqrt{\sigma} = 433$ MeV corresponds to setting the scale with $\hat{F}_\infty$ as the input. Due to the non-perturbative renormalization of the pion decay constant, this differs somewhat from our previous results [14]. Interestingly, the ground states, including the $a_0$, are close to the experimental $N = 3$ QCD values. However, the continuum limit still needs to be taken.

We are in the process of performing a combined large-$N$ and continuum limit. Within errors the finite-$a$ SU(7) results agree with our $N \to \infty$ extrapolations. In figure 4 we display the continuum limit extrapolation of some SU(7) masses and decay constants. Quantitatively, the slopes are very similar to results obtained previously in the SU(3) theory [19, 20]. Therefore, we do not anticipate complications when the combined limit will be performed. It is clear from the extrapolation that the finite-$a$ masses displayed in figure 3 are subject to systematics of order 10 %. In particular, the ratio $m_{a_0}/m_\rho$ will move closer to unity than that figure suggests.

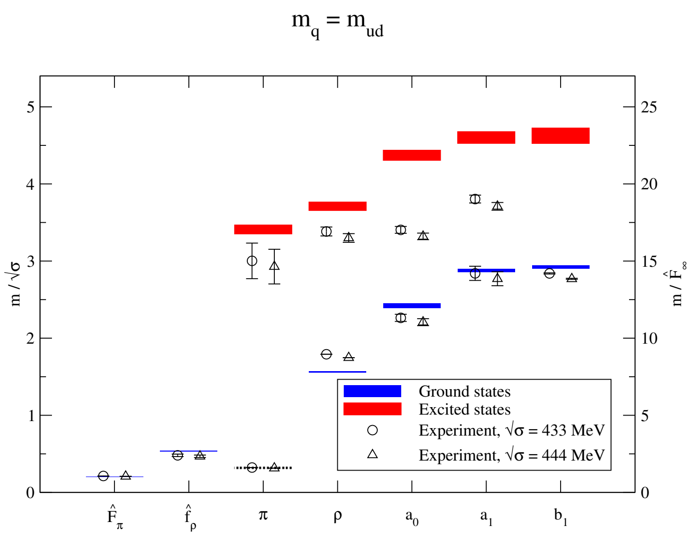


**Figure 3:** The SU($\infty$) spectrum at $a = 0.2093/\sqrt{\sigma} \approx 0.095$ fm at the physical light quark mass ($m_\pi \approx 1.6\hat{F}_\infty$).

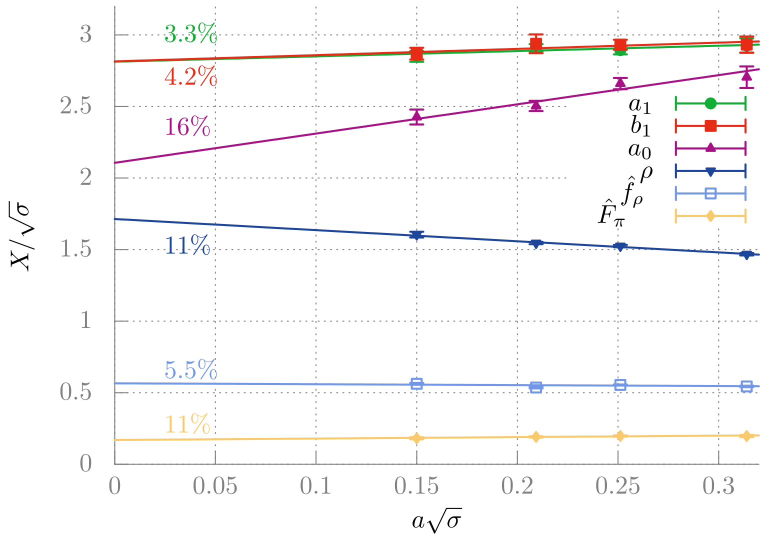


**Figure 4:** Continuum limit extrapolation of the SU(7) results. The percentage numbers indicate changes relative to the results obtained at $a\sqrt{\sigma} = 0.2093$ (second data points from the left).

## 5. Summary

We have determined the decay constants as well as the ground and first excited state masses of mesons in the large-$N$ limit of QCD. A continuum limit extrapolation is in progress. This will then allow the results to be used as input, e.g., to effective field theory calculations.

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