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1st or 2nd; the order of finite temperature phase transition of $N_f = 2$ QCD from effective theory analysis

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In the previous work, we have shown that the SU(2) chiral symmetry recovered above the critical temperature gives a strong constraint on the Dirac eigenvalue spectrum and this constraint is strong enough for a set of anomalous U(1) chiral symmetry breaking operators to vanish in the thermodynamical and chiral limits. We use this condition as an input and impose a constraint on the Landau low energy effective theory of QCD. The only constraint we can set is that the mass splitting term between the pion and eta meson should vanish. All the singlet/non-singlet scalar/pseudo-scalar mesons contribute to the effective theory. We evaluate the renormalization group β-function for the effective theory using the ε-expansion at one loop level, but find no stable infra-red fixed point except for the trivial Gaussian one. The chiral phase transition seems to be of first order.

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1. Introduction

The chiral symmetry $U(N_f)_R \times U(N_f)_L$ is one of the most important symmetry of the QCD Lagrangian. This chiral symmetry is known to be broken through two different mechanisms. The non-singlet $SU(N_f)_R \times SU(N_f)_L$ part is broken spontaneously into $SU(N_f)_V$. On the other hand the singlet $U(1)_R \times U(1)_L$ part is broken explicitly by the anomaly to $U(1)_V$.

The spontaneously broken non-singlet symmetry is believed to be recovered above a critical temperature T_c. On the other hand, as the anomaly originates from the cut-off of the theory, we cannot expect any relation between the $U(1)_A$ and the restored $SU(N_f)_L \times SU(N_f)_L$ symmetries. It is natural to assume that the anomalous $U(1)_A$ symmetry remains broken above T_c.

However, in a recent work [1], we have shown that in the two-flavor lattice QCD, the $SU(2)_L \times SU(2)_R$ chiral symmetry gives a set of strong constraints on the Dirac eigenvalue distribution, which has a tight connection to the $U(1)_A$ symmetry. In fact, the obtained constraints are strong enough for a set of $U(1)_A$ breaking operators to vanish in the thermodynamical limit [1]. We have also shown that the $U(1)_A$ breaking $Q = \pm 1$ topological sector [4] cannot survive in the infinite volume limit, and a lattice simulation should have lattice artifacts in the $U(1)_A$ breaking operators (if exists) unless it employs the exact chiral symmetric Dirac operator [5].

The presence/absence of the $U(1)_A$ symmetry at, and above the critical temperature is important, since it should largely affect the phase transition [6]. If the $U(1)_A$ symmetry remains to be broken, the chiral phase transition is likely to be the second order and its critical behavior should be in the same universality class as the $O(4)$ sigma-model. If the $U(1)_A$ is restored at the same critical temperature, the phase transition is likely to be the first-order [6], or at least, be in the different universality class from $O(4)$, even if it is the second-order [7].

In this work, we investigate the effect of our previous work [1] on the chiral phase transition. We construct the general effective theory, and give the constraints from our previous results in the $U(1)_A$ order parameters. Then, we calculate the β functions of the parameters, and study if there is any infra-red fixed points. Our result turns out that there is no infra-red fixed point at one-loop with the ε-expansion. If this one-loop observation is correct even at the non-perturbative level, the chiral phase transition is likely to be the first-order.

2. Restoration of $U(1)_A$ symmetry

We briefly review the restoration of the $U(1)_A$ symmetry shown in Ref. [1]. We consider the $N_f = 2$ QCD and start from an assumption that the non-singlet $SU(2)_R \times SU(2)_L$ chiral symmetry is fully recovered above the critical temperature T_c. So any order parameters of the non-singlet chiral symmetry should vanish. We consider the following type of the order parameter

$$\frac{1}{V^k}\langle \delta^a O_{n_1,n_2,n_3,n_4} \rangle \tag{2.1}$$

given by performing the $SU(2)_A$ transformation δ^a on a non-singlet parity odd operator

$$O_{n_1,n_2,n_3,n_4} = (P^a)^{n_1} (S^a)^{n_2} \left(P^0\right)^{n_3} \left(S^0\right)^{n_4}, \tag{2.2}$$

[1] A similar observation was first reported by Cohen [2, 3]. Our work has confirmed that the chiral zero-mode's effect, as well as the effect from the UV-cutoff, he had neglected, do not change his result.

where P^0 and $P^{a=1,2,3}$ is the singlet/non-singlet pseudo-scalar operator integrated in the space-time

$$P^a = \int d^4x \bar{q}\gamma_5\tau^a q(x), \quad P^0 = \int d^4x \bar{q}\gamma_5 q(x) \tag{2.3}$$

and $S^{a=1,2,3}$ and S^0 the scalar operator

$$S^a = \int d^4x \bar{q}\tau^a q(x), \quad S^0 = \int d^4x \bar{q}q(x). \tag{2.4}$$

A summation is not taken in a. Here we notice that the order parameter should be normalized by an appropriate power k of the volume since we adopted the integrated operator. The value of k is chosen so that the order parameter takes finite value at zero temperature.

We require all the order parameters to vanish above T_c when the thermodynamical and then chiral limit is taken in a right way

$$\lim_{m\to 0}\lim_{V\to\infty} \frac{1}{V^k}\langle \delta^a O_{n_1,n_2,n_3,n_4}\rangle = 0. \tag{2.5}$$

We convert this requirement to a constraint on the eigenvalue density (eigenvalue distribution function) $\rho^A(\lambda)$ of the Dirac operator, where the superscript A intends the gauge configuration dependence. We impose a few assumptions on a behavior of the VEV of a mass independent operator as a function of the quark mass introduced through the Boltzmann factor. The eigenvalue density is also assumed to be analytic around $\lambda = 0$

$$\rho^A(\lambda) = \sum_{n=0}^{\infty} \rho_n^A \frac{\lambda^n}{n!}. \tag{2.6}$$

Then we have shown that the restoration of the $SU(2)_R \times SU(2)_L$ chiral symmetry (2.5) requires the VEV of the first three coefficients should vanish

$$\lim_{m\to 0}\lim_{V\to\infty} \langle \rho_0^A \rangle = \lim_{m\to 0}\lim_{V\to\infty} \langle \rho_1^A \rangle = \lim_{m\to 0}\lim_{V\to\infty} \langle \rho_2^A \rangle = 0. \tag{2.7}$$

Contributions from the exact zero mode were treated independently from $\rho^A(\lambda)$. Defining $N_{R+L}(A) = N_R(A) + N_L(A)$ and $Q(A) = N_R(A) - N_L(A)$, where $N_{L/R}$ is the number of left/right-handed chiral zero-modes, it is shown that the VEV of these operators should vanish satisfying (2.5)

$$\lim_{V\to\infty} \frac{1}{V^k}\left\langle N_{R+L}(A)^k \right\rangle = \lim_{V\to\infty} \frac{1}{V^k}\left\langle Q(A)^{2k} \right\rangle = 0. \tag{2.8}$$

These constraints are enough to show the $U(1)_A$ order parameters of the similar type to vanish above T_c

$$\lim_{m\to 0}\lim_{V\to\infty} \frac{1}{V^k}\langle \delta^0 O'_{n_1,n_2,n_3,n_4}\rangle = 0, \tag{2.9}$$

where δ^0 is the singlet $U(1)_A$ transformation and O'_{n_1,n_2,n_3,n_4} is an iso-spin singlet parity odd operator given as a product in (2.2). This means that the anomaly is invisible in the set of VEV with operators O'_{n_1,n_2,n_3,n_4}.

3. Low energy effective theory

Let us construct the low energy effective theory of $N_f = 2$ QCD above T_c which reproduce the $U(1)_A$ restoration condition (2.9). The corresponding Landau effective theory is given in terms of meson field

$$\Phi(x) = \frac{1}{2}\left(\sigma(x) + i\eta(x)\right) + \frac{\tau^a}{2}\left(\delta^a(x) + i\pi^a(x)\right), \tag{3.1}$$

where σ and δ^a denote the scalar singlet and triplet field respectively, while η and π^a are pseudoscalar counterparts. τ^a is the Pauli matrix.

The non-singlet $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral rotation is given by $\Phi \to g_L \Phi g_R^{-1}$ where $g_{L/R} \in \mathrm{SU}(2)$. The singlet $\mathrm{U}(1)_A$ transformation is given by $\Phi \to e^{i\alpha}\Phi$. The general form of the renormalizable Lagrangian which is invariant under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_A$ transformation is given by

$$\mathscr{L}_0 = \mathrm{tr}\left(\partial_\mu \Phi^\dagger \partial_\mu \Phi\right) + m_\Phi^2 \mathrm{tr}\Phi^\dagger\Phi + \frac{\lambda_1}{2}\left(\mathrm{tr}\Phi^\dagger\Phi\right)^2 + \frac{\lambda_2}{2}\mathrm{tr}\left(\Phi^\dagger\Phi\right)^2. \tag{3.2}$$

An explicit breaking effect due to the anomaly is taken into account by adding the $\mathrm{U}(1)_A$ breaking terms

$$\mathscr{L}_{\mathrm{U}(1)_A} = \mathscr{L}_c + \mathscr{L}_x + \mathscr{L}_y, \tag{3.3}$$

$$\mathscr{L}_c = c'\left(\det\Phi + \det\Phi^\dagger\right), \tag{3.4}$$

$$\mathscr{L}_x = \frac{x}{4}\left(\mathrm{tr}\Phi^\dagger\Phi\right)\left(\det\Phi + \det\Phi^\dagger\right), \quad \mathscr{L}_y = \frac{y}{4}\left((\det\Phi)^2 + \left(\det\Phi^\dagger\right)^2\right). \tag{3.5}$$

The external source term corresponding to the quark mass may be given by

$$\mathscr{L}_M = \mathrm{tr}\left(M\Phi\right) + \mathrm{tr}\left(M^\dagger\Phi^\dagger\right), \tag{3.6}$$

where M is a quark mass matrix.

The total Lagrangian

$$\mathscr{L}_{\mathrm{tot}} = \mathscr{L}_0 + \mathscr{L}_{\mathrm{U}(1)_A} \tag{3.7}$$

is invariant under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, parity and charge conjugation. $\mathscr{L}_{\mathrm{U}(1)_A}$ does not break the $\mathrm{U}(1)_A$ symmetry completely. $\mathscr{L}_c$ and $\mathscr{L}_x$ are still invariant under Z_2 rotation. $\mathscr{L}_y$ has a Z_4 symmetry given by $\alpha = \pi/2$.

4. Constraints from vanishing order parameters

In this section, we consider the constraints on the $\mathscr{L}_{\mathrm{U}(1)_A}$ under the condition (2.9). The $U(1)_A$ transformation of the operator is given by

$$\begin{aligned}\delta^0 O_{n_1,n_2,n_3,n_4} &= -2n_1 O_{n_1-1,n_2+1,n_3,n_4} + 2n_2 O_{n_1+1,n_2-1,n_3,n_4} \\ &\quad -2n_3 O_{n_1,n_2,n_3-1,n_4+1} + 2n_4 O_{n_1,n_2,n_3+1,n_4-1}.\end{aligned} \tag{4.1}$$

For $n_1 + n_2 + n_3 + n_4 =$odd operator the condition (2.9) is automatically satisfied if we require the restoration of the non-singlet symmetry by (2.5). This is because two equations (2.5) and (2.9) are related by a linear transformation for $n_1 + n_2 + n_3 + n_4 =$odd. We shall consider $n_1 + n_2 + n_3 + n_4 =$even operators in the following.

4.1 A constraint from $n_1 + n_2 + n_3 + n_4 = 2$ order parameters

There are two kinds of order parameters for $n_1 + n_2 + n_3 + n_4 = 2$ case given as a difference between the chiral susceptibilities

$$\chi^{\eta-\sigma} = \frac{1}{2V^2} \left\langle \delta^0 O_{0011} \right\rangle = \frac{1}{V^2} \left(\langle O_{0020} \rangle - \langle O_{0002} \rangle \right), \tag{4.2}$$

$$\chi^{\pi-\delta} = \frac{1}{2V} \left\langle \delta^0 O_{1100} \right\rangle = \frac{1}{V} \left(\langle O_{2000} \rangle - \langle O_{0200} \rangle \right). \tag{4.3}$$

The restoration of the $U(1)_A$ symmetry is discussed in terms of the leading contribution in the thermodynamical limit [1]. From a diagrammatic point of view the leading term in $\chi^{\eta-\sigma}$ comes from the disconnected diagram of the singlet scalar field

$$\chi^{\eta-\sigma} = -\left(\frac{1}{V} \langle S_0 \rangle \right)^2 + \mathscr{O}\left(\frac{1}{V} \right), \tag{4.4}$$

where we used the parity symmetry to eliminate $\langle P_0 \rangle$. The $\mathscr{O}(1/V)$ terms are contributions from the connected diagrams. $\chi^{\eta-\sigma}$ is trivially zero since the simplest order parameter vanishes above T_c

$$\lim_{M\to 0} \lim_{V\to\infty} \frac{1}{V} \langle S_0 \rangle = 0. \tag{4.5}$$

The second order parameter $\chi^{\pi-\delta}$ is written as a difference between the pion and the delta meson propagators

$$\chi^{\pi-\delta} = \left\langle \pi^b(-p)\pi^b(p) \right\rangle \Big|_{p=0} - \left\langle \delta^b(-p)\delta^b(p) \right\rangle \Big|_{p=0}. \tag{4.6}$$

The $\mathrm{U}(1)_A$ breaking $\mathscr{L}_c$ plays a role of the mass term to give different masses for each meson

$$\mathscr{L}_c = \frac{c'}{2} \left(\sigma^2 - \eta^2 - \delta^a \delta^a + \pi^a \pi^a \right). \tag{4.7}$$

Tree level propagators of the non-singlet and singlet mesons are given by

$$\left\langle \pi^b(-p)\pi^b(p) \right\rangle^{(0)} \Big|_{p=0} = \frac{1}{m_\Phi^2 + c'}, \quad \left\langle \delta^b(-p)\delta^b(p) \right\rangle^{(0)} \Big|_{p=0} = \frac{1}{m_\Phi^2 - c'}. \tag{4.8}$$

The condition $\chi^{\pi-\delta} = 0$ is satisfied at tree level if we set a constraint $c' = 0$.

A general form of the two point function including the quantum correction is given by

$$\left\langle \pi^b \pi^b \right\rangle \Big|_{p=0} = \frac{1}{m_\Phi^2 + \delta m_\Phi^2 + c' + \delta c'}, \quad \left\langle \delta^b \delta^b \right\rangle \Big|_{p=0} = \frac{1}{m_\Phi^2 + \delta m_\Phi^2 - (c' + \delta c')}. \tag{4.9}$$

The condition $\chi^{\pi-\delta} = 0$ is satisfied by setting a constraint on the renormalized parameter $c'_{\mathrm{R}} \propto c' + \delta c' = 0$. We may consider two possibilities to accomplish this condition. One is to fine tune the bare parameter c'. The other is to protect $c'_{\mathrm{R}} = 0$ by a symmetry. If we set $c' = x = 0$ the remaining Lagrangian $\mathscr{L}_0 + \mathscr{L}_y$ acquires a $Z_4 \subset \mathrm{U}(1)_A$ symmetry. $c'_{\mathrm{R}} = 0$ is guaranteed at any loop level and the order parameter vanishes automatically.

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4.2 Constraint from general $n_1 + n_2 + n_3 + n_4$ order parameter

We consider order parameters with general $n_1 + n_2 + n_3 + n_4$ =even operators. However as we shall see shortly the restoration condition with them does not give any further constraint than $c'_{\rm R} = 0$. This is mainly due to a fact that our constraint (2.9) is given with the appropriate volume normalization factor V^k and disconnected diagrams tend to dominate.

The general order parameter is given by four terms in (4.1). The leading terms in the order parameter can be categorized into three types from a diagrammatic point of view.

1. All the leading terms contain the singlet scalar operator S_0. In this case all the leading term should be proportional to the vev of singlet scalar field and vanish automatically above T_c.
2. Only a part of the leading terms contain the singlet scalar operator S_0. This happens for the operator $O_{2k_1+1,2k_2+1,2k_3+2,0}$. One can easily see that the leading term is proportional to either the singlet scalar S_0 or a difference of the operator $P^aP^a - S^aS^a$. The disconnected contribution from the latter should vanish as $\chi^{\pi-\delta} = 0$ is satisfied by $c'_R = 0$.
3. Any leading term does not contain the singlet scalar operator. This is the case for $O_{2k_1+1,2k_2+1,0,0}$. The leading term of this operator is proportional to $P^aP^a - S^aS^a$ and vanishes as in the case of 2 above.

The restoration of the non-singlet $SU(2)_L \times SU(2)_R$ symmetry and the constraint $c'_{\rm R} = 0$ is enough to eliminate all the $U(1)_A$ order parameters of the form (2.9). We notice that the $U(1)_A$ breaking interaction term with parameters x, y cannot be constrained at all.

5. Renormalization group analysis of phase transition

The last step is the renormalization group analysis of the low energy effective theory according to Ref. [6]. Our effective Lagrangian above T_c is given by

$$\mathscr{L}_{\rm tot} = \mathscr{L}_0 + \mathscr{L}_x + \mathscr{L}_y. \tag{5.1}$$

We start by deriving the renormalization group function of the four couplings $g_i = \{\lambda_1, \lambda_2, x, y\}$. The β-function is defined in a standard manner by taking a derivative of the renormalized coupling

$$\beta_{g_i}(\lambda_1, \lambda_2, x, y) = \mu \frac{\partial}{\partial \mu} g_i(\mu), \tag{5.2}$$

where we adopted the MS scheme with the dimensional regularization $d = 4 - \varepsilon$.

We shall evaluate the β-function using the ε-expansion at one loop level in this proceedings. The β function for each coupling turns out to be

$$\beta_{\lambda_1} = -\varepsilon\lambda_1 + \frac{1}{8\pi^2}\left(8\lambda_1^2 + 8\lambda_1\lambda_2 + 3\lambda_2^2 + \frac{3}{2}x^2 + \frac{5}{4}y^2\right), \tag{5.3}$$

$$\beta_{\lambda_2} = -\varepsilon\lambda_2 + \frac{1}{8\pi^2}\left(4\lambda_2^2 + 6\lambda_1\lambda_2 - \frac{3}{4}x^2 - y^2\right), \tag{5.4}$$

$$\beta_x = -\varepsilon x + \frac{1}{8\pi^2}x\left(12\lambda_1 + 6\lambda_2 + 3y\right), \tag{5.5}$$

$$\beta_y = -\varepsilon y + \frac{1}{8\pi^2}\left(6\lambda_1 y + \frac{3}{2}x^2\right). \tag{5.6}$$

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We search for the fixed points to satisfy $\beta_{\lambda_1} = \beta_{\lambda_2} = \beta_x = \beta_y = 0$ staying on the critical surface $m_{\Phi R} = c'_R = 0$, which is accomplished by fine tuning the bare parameters m_Φ and c'. We found six fixed points. Only one of them at $\lambda_1 = \lambda_2 = x = y = 0$ is the stable ultra violet (Gaussian) fixed point. Remaining five fixed points are saddle point and are not stable. We do not find any stable infra red fixed points for our low energy effective theory at one loop. Our perturbative analysis shows that the chiral restoration phase transition is likely to be of first order.

6. Discussion

In the previous work we have shown that the $U(1)_A$ order parameters written in terms of the scalar/pseudo-scalar operators should vanish above T_c for the $N_f = 2$ QCD as the thermodynamical and the chiral limit is taken in the right way. We use this condition as an input and impose a constraint on the low energy effective theory. The only constraint we can set is that the mass splitting term between the pion and eta meson should vanish as $c'_R = 0$. All the singlet/non-singlet scalar/pseudo-scalar mesons π, η, σ, δ contribute to the effective theory. However the $U(1)_A$ symmetry breaking interaction term still remains with the coefficient x and y.

We evaluate the β-function of the four couplings in the effective theory at one loop level and search for the stable infra-red fixed point on the critical surface $m_{\Phi R} = c'_R = 0$. We cannot find any such fixed points except for the trivial Gaussian fixed point. The chiral phase transition seems to be of first order.

However the ε-expansion may not be sufficient to have a rigid conclusion. We need to set $\varepsilon \to 1$ in the end to discuss the second order phase transition. Non-trivial fixed points, which are proportional to ε, tend to be in a strong coupling region. A higher loop [7] or non-perturbative analysis may be required, with which one may find a stable IR fixed point [7]. In this case a position of the fixed point would be important. If a stable IRFP was on $x = 0$ the low energy theory would acquire higher symmetry $SU(2)_L \times SU(2)_R \times Z_4$ on the fixed point. The universality class would be different from the widely believed $O(4)$.

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