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A property of fermions at finite density by a reduction formula of fermion determinant

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We report our recent studies on the zero temperature limit of Wilson fermion determinant at nonzero quark chemical potential. We show that the fermion determinant is independent of quark chemical potential and the quark number operator is zero for $\mu < m_\pi/2$ in lattice QCD. A reduction formula of the fermion determinant, which is a formula to calculate the temporal part of the fermion determinant analytically, is a key to taking the zero temperature limit.

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1. Introduction

Lattice QCD simulations for nonzero quark chemical potential μ are challenging topics. The γ_5-hermiticity ensures the fermion determinant $\det\Delta$ to be real at $\mu = 0$. The nonzero quark chemical potential breaks the γ_5-hermiticity, and allows the fermion determinant to be complex. The complex determinant causes the breakdown of Monte Carlo simulations, so called sign problem.

Recently, there are advances in lattice QCD simulations for finite density QCD especially for high temperature and small density regions of the QCD phase diagram. The sign problem is mild in the QGP phase. In addition, several approaches have been developed, such as Taylor expansion, analytic continuation with imaginary chemical potential, canonical approaches, etc, which allow to test the validity of an approach by comparing it to others. It was shown that several approaches are consistent for $\mu/T < 1$. Lattice simulations are to some extent tractable for this region.

In contrast, lattice QCD simulations are difficult for low-T and finite μ regions. Several quantities, such as the fermion determinant, quark number density, Polyakov loop decrease with decreasing temperature. The calculation of them suffers from small signal-to-noise ratio. The fermion determinant becomes less sensitive to μ at lower temperatures, which also makes it difficult to investigate the μ-dependence of quantities.

In such a situation, it is useful to consider the zero temperature limit. In previous studies [1, 2, 3], we have developed an approach to take the zero temperature limit of the fermion determinant, where we considered $T = 0$ by taking $N_t \to \infty$. The approach is based on the reduction formula of the fermion determinant [4, 5, 6, 7, 8, 9, 10]. It offers a way to calculate the temporal part of the fermion determinant analytically and transforms the determinant into an analytic function of quark chemical potential μ. The formula incorporates a reduced matrix (propagator matrix), which controls the μ-dependence of the fermion determinant. Investigating the eigenvalues of the reduced matrix, we found that they obey a scaling law with regard to the temporal lattice size N_t. Using the N_t-scaling law, it turned out that the fermion determinant is independent of μ for small μ. The eigenvalues of the reduced matrix are also related to meson masses [4, 11]. We showed that the fermion determinant is independent of μ for $\mu < m_\pi/2$ for $T = 0$. The same result was previously obtained for continuum case by Cohen[12]. Our study provides a complementary understanding of the previous studies from the viewpoint of the zero temperature limit of lattice QCD.

In this article, we report our recent studies on the properties of lattice fermions at zero temperature and small quark chemical potential.

2. Reduction Formula and Reduced Matrix

2.1 Formulation

We employ the clover-improved Wilson fermions with two flavor and renormalization group-improved gauge action. The Wilson fermion matrix Δ is given by

$$\Delta(\mu) = B - \kappa\left[e^{+\mu a}(1-\gamma_4)U_4(x)\delta_{x',x+\hat{4}} + e^{-\mu a}(1+\gamma_4)U_4^\dagger(x')\delta_{x',x-\hat{4}}\right], \tag{2.1}$$

where κ is the hopping parameter. μ and a are quark chemical potential and lattice spacing. B is the spatial part of the Wilson fermion matrix,

$$B = \delta_{x,x'} - \kappa \sum_{i=1}^{3} \Big[(1-\gamma_i) U_i(x) \delta_{x',x+\hat{i}} + (1+\gamma_i) U_i^{\dagger}(x') \delta_{x',x-\hat{i}} \Big] - \kappa\, C_{SW}\ \delta_{x,x'} \sum_{\mu \le \nu} \sigma_{\mu\nu} F_{\mu\nu}, \tag{2.2}$$

where C_{SW} is the clover coefficient. We introduce two block-matrices

$$\alpha_i = B^{ab,\mu\sigma}(\vec{x},\vec{y},t_i)\ r_-^{\sigma\nu} - 2\kappa\, r_+^{\mu\nu} \delta^{ab} \delta(\vec{x}-\vec{y}), \tag{2.3a}$$

$$\beta_i = \Big[B^{ac,\mu\sigma}(\vec{x},\vec{y},t_i)\ r_+^{\sigma\nu} - 2\kappa\, r_-^{\mu\nu} \delta(\vec{x}-\vec{y}) \Big] U_4^{ab}(\vec{y},t_i), \tag{2.3b}$$

where indices a, b, and c are color indices, and μ, ν and σ are Dirac indices. $r_\pm = (1 \pm \gamma_4)/2$. α_i describes a spatial hop of a quark at $t = t_i$, while β_i describes a spatial hop at $t = t_i$ as well as a temporal hop to the next time slice.

Using a reduction formula, $\det\Delta(\mu)$ is transformed into [9]

$$\det\Delta(\mu) = \xi^{-N_{\rm red}/2} C_0 \det(\xi + Q), \tag{2.4a}$$

$$Q = (\alpha_1^{-1}\beta_1)\cdots(\alpha_{N_t}^{-1}\beta_{N_t}), \tag{2.4b}$$

$$C_0 = \prod_{i=1}^{N_t} \det(\alpha_i), \tag{2.4c}$$

where $\xi = e^{-\mu/T}$, and $N_{\rm red} = 4N_c N_s^3$. N_c and N_s are the number of colors and spatial lattice size. Q, which we refer to as the reduced matrix, describes the propagation of a quark from the initial to final time slices, see Fig. 1.

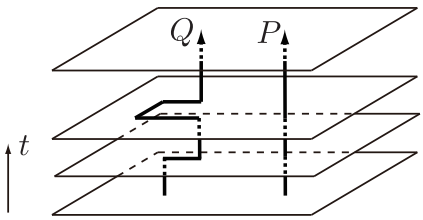

Figure 1: A diagram of the propagation of a quark described by the matrix Q. It is analogous to a Polyakov line P(straight line). Each plane denotes a spatial plane at a fixed time.

With $\lambda_n (n = 1, 2, \cdots N_{\rm red})$ denoting eigenvalues of Q, $\det\Delta$ is given by

$$\det\Delta(\mu) = C_0 \xi^{-N_{\rm red}/2} \prod_{n=1}^{N_{\rm red}} (\lambda_n + \xi). \tag{2.5}$$

Q and C_0 depend only on link variables and not on μ for a given configuration. The effect of μ is included only in ξ in Eqs. (2.4a) and (2.5). Once C_0 and λ_n are obtained, values of $\det\Delta(\mu)$ can be obtained for arbitrary μ.

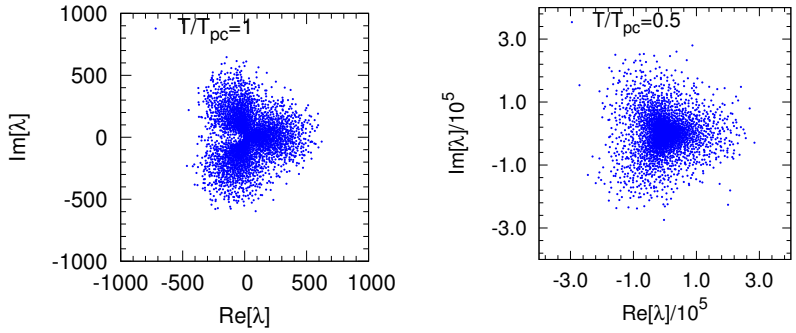

Figure 2: The scatter plot of eigenvalues on the complex λ plane for $N_t = 4$ and $N_t = 8$. The same parameters were used except for N_t; κ was fixed with a LCP of $m_{\rm PS}/m_{\rm V} = 0.8$ [13], and $\beta = 1.86$ corresponding to $T = T_c$ for $N_t = 4$ and $T = T_c/2$ for $N_t = 8$ at $\mu = 0$.

2.2 Properties of reduced matrix

Figure 2 shows the eigenvalue distribution on the complex plane for $N_t = 4$ ($T/T_c \sim 1$) and $N_t = 8$ ($T/T_c \sim 0.5$). Here, we use the data of λ_n obtained in the previous lattice QCD simulation for two-flavor clover-improved Wilson fermions [1]. Eigenvalues are generally complex, and its distribution is of Z(3) approximately. This is naturally understood from the analogy between the reduced matrix and Polyakov line.

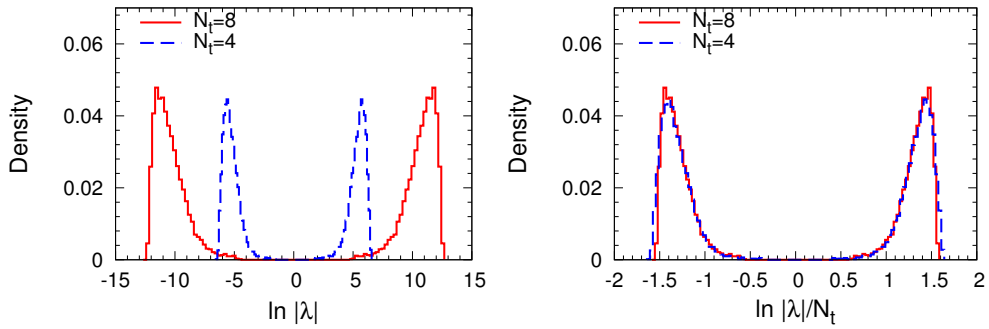

Figure 3: The histogram of the magnitude of the eigenvalues for $N_t = 4$ and $N_t = 8$. Left panel: original, right panel : scaled. Left and right peaks correspond to S and L, respectively.

Figure 3 shows the histogram of $\ln|\lambda_n|$(left panel). The eigenvalue distribution is symmetric with regard to $|\lambda| = 1$. This follows from the fact that the eigenvalues of the reduced matrix form pairs; if there is λ, then it has a partner $1/\lambda^*$. Then, they are divided into two groups $S = \{\lambda_n | |\lambda_n| < 1, n = 1,2,\cdots,N_{\rm red}/2\}$ and $L = \{\lambda_n | |\lambda_n| > 1, n = 1,2,\cdots,N_{\rm red}/2\}$.

No eigenvalue is observed near $|\lambda_n| = 1$. The size of the gap is given by the eigenvalue nearest to the unit circle, $\max_{\lambda_n \in S} \lambda_n$ or its pair. Gibbs pointed out that this eigenvalue is related to the pion mass for $N_t \to \infty$. Hence, the size of this gap is expected to be nonzero for nonzero quark mass.

The eigenvalues follow a scaling law with regard to the temporal lattice size N_t, although the magnitude of the eigenvalues strongly depends on N_t, as shown in Figs. 2 and 3. The histogram of $\ln|\lambda_n|$ depends on N_t, while the scaled histogram of $(\ln|\lambda_n|)/N_t$ is almost independent of N_t, see

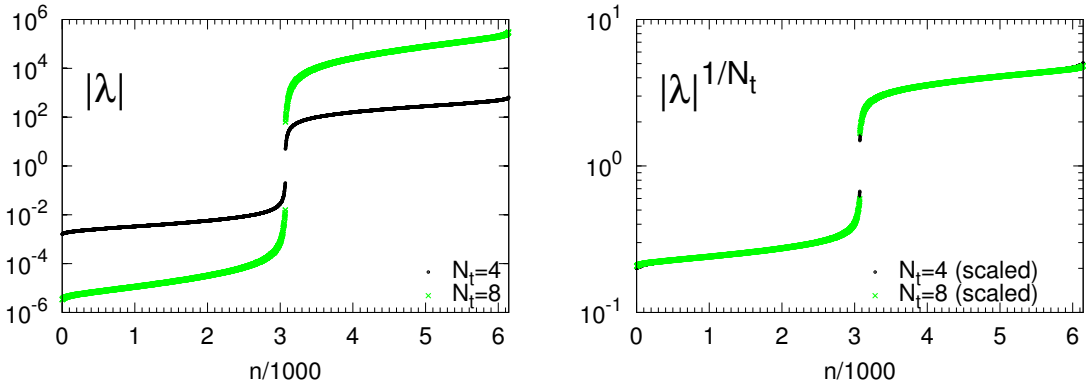

Figure 4: The magnitude of the eigenvalues in descending order for $N_t = 4$ and $N_t = 8$ for $\beta = 1.86$. There are $N_{\rm red} = 8^3 \times 12 = 6144$ eigenvalues. The left panel shows the spectrum for $|\lambda_n|$, while the right panel shows the scaled eigenvalues $|\lambda_n|^{1/N_t}$.

right panel of Fig. 3. The spectrum shown in Fig. 4 also manifests this behavior, see the spectrum of $|\lambda_n|$ (left panel) and $|\lambda_n|^{1/N_t}$ (right panel). These results mean that the magnitude of λ_n is scaled as $|\lambda_n| \sim l_n^{N_t}$, where $l_n (\in R)$ is independent of N_t. We consider the small eigenvalues S below, and they can be denoted as $\lambda_n = \exp(-\varepsilon_n a N_t + i\theta_n)$, with $l_n = \exp(-\varepsilon_n a)$. Here ε_n is also independent of N_t, but depends on a. Using $T = (aN_t)^{-1}$, the small eigenvalues are rewritten as

$$\lambda_n = \exp(-\varepsilon_n/T + i\theta_n). \tag{2.6}$$

This scaling is physically natural according to the analogy between the reduced matrix and Polyakov line.

Note that $|\lambda_n|$ approaches to unity as ε_n decreases. Because there is the gap in the eigenvalue distribution, ε_n has the minimum value given by $\varepsilon_{\rm min} = -T \max_{\lambda_n \in S} |\lambda_n|$,

3. Fermion Determinant and Quark Number at $T = 0$

Now, we consider the zero temperature limit of the fermion determinant. We rewrite Eq. (2.5) using the pair nature of the eigenvalues as

$$\det\Delta(\mu) = C_0 \prod_{n=1}^{N_{\rm red}/2} (\lambda_n^*)^{-1}(1+\lambda_n \xi^{-1})(1+\lambda_n^* \xi), \tag{3.1}$$

where the index n runs over the small eigenvalues $\lambda_n \in S$. Using the N_t-scaling law Eq. (2.6), we obtain

$$\det\Delta(\mu) = C_0 \prod_{n=1}^{N_{\rm red}/2} (\lambda_n^*)^{-1}(1+e^{-(\varepsilon_n-\mu)/T+i\theta_n})(1+e^{-(\varepsilon_n+\mu)/T-i\theta_n}). \tag{3.2}$$

We consider the zero temperature limit by taking $N_t \to \infty$ with a fixed lattice spacing a and fixed spatial volume N_s^3. Equation (3.2) is described in terms of the small eigenvalues which have positive $\varepsilon_n (= -T\ln|\lambda_n|)$. Hence, $e^{-(\varepsilon_n+\mu)/T}$ vanishes for $N_t \to \infty$. On the other hand, $e^{-(\varepsilon_n-\mu)/T}$ goes to

0 or ∞, depending on the sign of $\varepsilon_n - \mu$. Because there is an inequality $e^{-(\varepsilon_{\min}-\mu)/T} > e^{-(\varepsilon_n-\mu)/T}$, $e^{-(\varepsilon_n-\mu)}$ vanishes for all the eigenvalues at $T=0$ for $\mu < \varepsilon_{\min}$. In this case, we obtain

$$\det\Delta(\mu) \overset{N_t\to\infty}{=} C_0 \prod_{n=1}^{N_{\rm red}/2} (\lambda_n^*)^{-1}, (\mu < \varepsilon_{\min}). \tag{3.3}$$

Thus, the fermion determinant is independent of quark chemical potential μ at $T=0$ for $\mu < \varepsilon_{\min}$.

Next, we need to determine $\varepsilon_{\min}$. As we have mentioned, the magnitude of the gap is related to the pion mass. Gibbs found a relation [4]

$$am_\pi = -\frac{1}{N_t} \max_{\lambda_n \in S} \ln |\lambda_n|^2, \tag{3.4}$$

for $N_t \to \infty$. Later, Fodor, Szabó and Tóth proposed thermodynamic considerations for hadron spectroscopy, and derived a relation [11]

$$am_\pi = -\frac{1}{N_t} \ln \left\langle \left| \sum_{\lambda_n \in S} \lambda_n \right|^2 \right\rangle. \tag{3.5}$$

The two expressions are expected to become consistent at $T=0$, because $\max_{\lambda_n\in S}\lambda_n$ dominates Q for $N_t \to \infty$. $\varepsilon_{\min}$ is identified with $m_\pi/2$ at $\mu = 0$. Since the fermion determinant is independent of μ for $\mu < \varepsilon_{\min}$, then $\varepsilon_{\min}$ and m_π remain unchanged for $\mu < m_\pi/2$. Hence, the relation $\varepsilon_{\min} = m_\pi/2$ is valid for $\mu < \varepsilon_{\min}$.

In this way, the μ-independence of the fermion determinant in lattice QCD is obtained by using the reduction formula, and N_t-scaling law of the eigenvalues of the reduced matrix, and the relation between the pion mass and the reduced matrix.

3.1 Quark number operator

The approach can be applied to the quark number operator

$$\hat{n} = \frac{T}{V_s} (\det\Delta(\mu))^{-1} \frac{\partial \det\Delta(\mu)}{\partial\mu}, \tag{3.6}$$

where V_s is the spatial lattice volume $V_s = N_s^3$. Using Eq. (3.1), we obtain

$$\hat{n} = \frac{1}{V_s} \sum_{n=1}^{N_{\rm red}/2} \left(\frac{\lambda_n \xi^{-1}}{1+\lambda_n\xi^{-1}} - \frac{\lambda_n^*\xi}{1+\lambda_n^*\xi} \right). \tag{3.7}$$

Substituting Eq. (2.6) and $\xi = e^{-\mu/T}$, we can rewrite $\hat{n}$ as

$$\hat{n} = \frac{1}{V_s} \sum_{n=1}^{N_{\rm red}/2} \left(\frac{1}{1+e^{(\varepsilon_n-\mu)/T - i\theta_n}} - \frac{1}{1+e^{(\varepsilon_n+\mu)/T+i\theta_n}} \right). \tag{3.8}$$

This is analogous to a Fermi-distribution function. Because Q describes a temporal quark line and has winding number one, ε_n is considered as an energy level of single quark. Here Eq. (3.8) is the operator, and therefore $\hat{n}$ and ε_n describe the Fermi-distribution function and energy levels of single quark for each configuration, respectively. In this sense, they are microscopic quantities, and different from physical quantities defined for ensemble average.

For $N_t \to \infty$, $e^{(\varepsilon_n-\mu)/T}$ and $e^{(\varepsilon_n+\mu)/T}$ go to ∞ if $\mu < \varepsilon_{\min}$. We get

$$\hat{n}(\mu) = 0 \quad \text{for} \quad \mu < \varepsilon_{\min}. \tag{3.9}$$

Thus, we obtained that the quark number density operator is zero at $T = 0$ for small μ in the viewpoint of lattice QCD.

4. Summary

We showed that the fermion determinant is independent of quark chemical potential for $\mu < m_\pi/2$. The reduction formula provides the analytic dependence of the fermion determinant on μ, and the eigenvalues of the reduced matrix characterizes its μ-dependence. Using the N_t-scaling law of the reduced matrix, the fermion determinant is reduced to the μ-independent form in the zero temperature limit. The relation between the pion mass and reduced matrix was used to determine the threshold of the μ-independence. Applying the same approach, we showed that the quark number operator can be rewritten in the form considered as the microscopic Fermi distribution function, and that the quark number operator is zero for $\mu < m_\pi/2$ at $T = 0$.

We have employed the Wilson fermion. The reduction formula is also derived for staggered fermions. The present technique can be applied to staggered fermions.

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