We investigate interactions between Omega baryons in lattice QCD. Employing the HAL QCD method, we extract the Omega-Omega potential from the Nambu-Bethe-Salpeter (NBS) wave function, which are calculated on 2+1 flavor full QCD gauge configurations generated by the CP-PACS/JLQCD Collaboration at $m_{\pi} = 875$ MeV and $m_{\Omega} = 2108$ MeV. Both a shape of the potential and the phase shift of the Omega-Omega scattering calculated with it indicate that the interaction is strongly attractive. We finally discuss a possibility for an existence of a weak bound state in this system.
1. Introduction

While hyperon interactions become important in high density matters such as the core of the neutron star [1], investigations so far have been mainly focused on the octet sector. A main reason is, of course, that all decuplet baryons except Omega are unstable due to strong decays. Furthermore, even for the Omega-Omega case, it is still difficult to investigate their interaction experimentally due to their short-life time via weak decays. Theoretically, there exists two model calculations for the omega-omega interaction in the $J=0$ channel [2, 3] leading to controversial results: while one reported a weakly repulsive interaction, the other indicated a strong attraction. The recent lattice QCD calculation using the standard finite volume method [Luescher] concluded that the Omega-Omega interaction is weakly repulsive in this channel: it’s scattering length $a = -0.16 \pm 0.22$ fm [4].

Recently, a new but first-principle method was proposed to investigate nucleon-nucleon interactions in QCD on the lattice [5–7], where the potential can be extracted from the Nambu-Bethe-Salpeter (NBS) wave function. This method has been generalized to derive potentials including hyperons (YN and YY) [8–13] and the three-nucleon force [14–16]. In this report, we therefore employ the HAL QCD method to study the Omega-Omega interaction, by calculating the corresponding potential in lattice QCD. Our results suggest an attractive Omega-Omega interaction in the $J=0$ channel, which seems rather strong.

2. Extraction of potentials

The potential method was originally introduced by the HAL QCD collaboration [5], where a non-local potential which is defined from the equal time Nambu-Bethe-Salpeter (NBS) wave functions. For the two Omega system, the NBS wave function is defined as

$$\psi_n(\vec{r}) \equiv \langle 0 | \Omega(\vec{r}, 0) \Omega(0, 0) | \Omega(k_n) \Omega(-k_n); in \rangle$$

(2.1)

where $| \Omega(k_n) \Omega(-k_n); in \rangle$ is an eigenstate of two-omega in QCD with the energy $2 \sqrt{m_\Omega^2 + k_n^2}$, $\Omega(x)$ and $\bar{\Omega}(x)$ are local operators for Omega, whose explicit definition will be given in section 3.1. One of the most important properties here is that the NBS wave function at large $r = |\vec{r}|$ in QCD has the same asymptotic form to that of the scattering wave in quantum mechanics. From this fact, one can define a non-local but energy independent potential from the NBS wave function as

$$(E_n - H_0) \psi_n(\vec{r}) = \int d^3 r' U(\vec{r}, \vec{r}') \psi_n(\vec{r}')$$

(2.2)

where $H_0 = -\frac{1}{2 \mu_\Omega} \nabla^2$ is the non-interaction part of the Hamiltonian, $\mu_\Omega = m_\Omega^2 / 2$ is reduced mass, $E_n = k_n^2 / (2 \mu_\Omega)$ is the kinetic energy in the center-of-mass frame, in two Omega system. The non-local potential $U(\vec{r}, \vec{r}')$ can be made to be energy-independent. Although we use the non-relativistic Schrödinger equation to define the potential, no non-relativistic approximation is made here [8]. Of course, potentials are not physical observables, but the potential defined above can reproduce physical observables such as phases of the S-matrix correctly by construction.
In lattice QCD simulations, NBS wave functions can be extracted from the two-Omega correlation function as

$$C_{\Omega\Omega}(\vec{r},t,t_0) = \frac{1}{V} \sum_x \langle \Omega(t,\vec{x}+\vec{r})\Omega(t,\vec{x})|\Omega(t_0)\Omega(t_0)\rangle = \sum_n \psi_n(\vec{r})a_n e^{-E_n(t-t_0)} + \ldots$$

(2.3)

$$\simeq a_0\psi(\vec{r})a_n e^{-E_0(t-t_0)}, \quad t-t_0 \to \infty$$

(2.4)

with $a_n = \langle \Omega(k_n)\Omega(-k_n)|\Omega(t_0)\Omega(t_0)\rangle$, where $E_0$ is the smallest energy of the system, and ellipses represent inelastic contributions.

In this report, we have employed the improved extraction, called the time dependence method [7], which is given by

$$\left(\frac{1}{4m} \frac{\partial^2}{\partial t^2} + \frac{1}{m} \nabla^2 - \frac{\partial}{\partial t}\right)R(\vec{r},t,t_0) = \int d^3r'U(\vec{r},\vec{r}')R(\vec{r}',t,t_0)$$

(2.5)

where

$$R(\vec{r},t,t_0) = \frac{C_{\Omega\Omega}(\vec{r},t,t_0)}{e^{-2m(t-t_0)}} = \sum_n a_n\psi_n(\vec{r})e^{-\Delta W_n(t-t_0)} + \ldots$$

(2.6)

with $\Delta W_n \equiv E_n - 2m_\Omega$. Since all two-Omega elastic scattering states with different $n$ satisfy the same Schrödinger equation with the same energy-independent non-local potential, the large $t-t_0$ limit necessary for the ground state saturation is no longer required. A condition necessary for this method to work is that $t-t_0$ should be large enough to suppress both inelastic contributions in the two-Omega system and excited states in the single-Omega correlation function.

To extract potentials in practice, we employ the derivative expansion of the non-local potential [5]. We consider the leading-order term at low energies as

$$U(\vec{r},\vec{r}') = V_{\alpha'\beta'\alpha\beta}(\vec{r})\delta(\vec{r}-\vec{r}') + O(\vec{r})$$

(2.7)

where $\alpha',\beta',\alpha,\beta = 0 \sim 3$ are spin indices. In this report, we consider a case that the total spin of two Omega baryons is zero, whose potential depend on $r = |\vec{r}|$ only.

3. Symmetry of the Omega-Omega system

In this section we discuss a symmetry of the Omega-Omega system.

3.1 Symmetry

A single Omega baryon can not decay to a pair of an octet baryon and a pseudo-scalar meson in QCD, since the lowest energy pair, $\Sigma$ and $K$, has larger energy than the Omega baryon mass. On the lattice, we simply define local operators Omega and anti-Omega baryons as

$$\Omega_{\alpha,k}(x) \equiv \epsilon^{abc}x^\mu(x)C\gamma_5s_b(x)s_{\alpha}(x), \quad \Omega_{\alpha,k}(x) \equiv \Omega^\dagger_{\alpha,k}(x)\gamma^0 = \epsilon^{abc}\gamma_{ua}(x)s^\mu_b(x)y_{\alpha}C\gamma_e(x)$$

(3.1)

where $a,b,c$ are color indices, $\epsilon^{abc}$ is the totally anti-symmetric tensor, $\gamma_5$ is the gamma matrix, $\alpha$ is the spinor index, and $C \equiv \gamma_4\gamma_5$ is the charge conjugation matrix.

If a distance between two Omega baryons becomes large, we can neglect interactions between them. Such asymptotic Omega-Omega states can be classified by the orbital angular momentum
we decompose states with conserved quantum numbers $J$ and $P$ in terms of asymptotic Omega-Omega states with given $L$ and $S$. 

<table>
<thead>
<tr>
<th>$J$</th>
<th>$P = +$</th>
<th>$P = -$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J = 0$</td>
<td>$S = 0, L = 0 ; S = 2, L = 2$</td>
<td>$S = 1, L = 1 ; S = 3, L = 3$</td>
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<tr>
<td>$J = 1$</td>
<td>$S = 2, L = 2$</td>
<td>$S = 1, L = 1 ; S = 3, L = 3$</td>
</tr>
<tr>
<td>$J = 2$</td>
<td>$S = 2, L = 0 ; S = 0, L = 2 ; S = 2, L = 2 ; S = 2, L = 4$</td>
<td>$S = 1, L = 1 ; S = 3, L = 1 ; S = 1, L = 3 ; S = 3, L = 3 ; S = 3, L = 5$</td>
</tr>
<tr>
<td>$J = 3$</td>
<td>$S = 2, L = 2 ; S = 0, L = 4 ; S = 2, L = 4 ; S = 2, L = 6$</td>
<td>$S = 3, L = 1 ; S = 1, L = 3 ; S = 3, L = 3 ; S = 3, L = 5$</td>
</tr>
<tr>
<td>$J = 4$</td>
<td>$S = 2, L = 2 ; S = 0, L = 4 ; S = 2, L = 4 ; S = 2, L = 6$</td>
<td>$S = 3, L = 1 ; S = 1, L = 3 ; S = 3, L = 3 ; S = 3, L = 7$</td>
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Table 1: Decomposition of the Omega-Omega system with conserved quantum numbers $J$ and $P$ in terms of states with a given $(L, S)$.

(L), the total spin ($S$), the total angular momentum ($J$) and parity($P$), even though conserved quantum numbers in QCD are $J$ and $P$ only. The two fermion state must change a sign under an exchange of them, while the asymptotic Omega-Omega state with given $L$ and $S$ has a factor $(-1)^{J+S+L+1}$ by the exchange, so that we should have $S + L = even$. In table 1 we decompose states with conserved quantum numbers $J$ and $P$ in terms of asymptotic Omega-Omega states with given $L$ and $S$.

3.2 Spin Projection

Since we employ wall sources, which has $L = 0$ and $P = +$, in our lattice calculations, the $J$ is determined by the total spin $S$ generated by the sources.

To construct the Omega-Omega operator which creates the state with the definite total spin $S$, let us first define a single spin 3/2 Omega baryon operator with a given $S_z$ as

$$\Omega_{1/2} Z^\pm \equiv - (\psi \Gamma_{\pm} \psi)^{1/2}$$

$$\Omega_{1/2} Z^\pm \equiv \frac{1}{\sqrt{2}} \left[ (\psi \Gamma_+ \psi) \psi_{1/2} + (\psi \Gamma_- \psi) \psi_{-1/2} \right]$$

$$\Omega_{1/2} Z^\pm \equiv \frac{1}{\sqrt{3}} \left[ (\psi \Gamma_{\pm} \psi_{1/2} \right] + (\psi \Gamma_{\pm} \psi_{-1/2} \right]$$

$$\Omega_{1/2} Z^\pm \equiv (\psi \Gamma_{\pm} \psi_{1/2}$$

where $\Gamma_{\pm} \equiv \frac{1}{2} (C \gamma^2 \pm i C \gamma^3)$, $\Gamma_\pm \equiv \frac{1}{2} C \gamma^2$, so that spin 1 di-quark operators, $\psi \Gamma_{\pm} \psi$, $\psi C \Gamma_\pm \psi$ and $\psi \Gamma_{\pm} \psi$, have $S_z = 1, 0, -1$, respectively, in non-relativistic limit.

Combining these operators, we can construct spin 3,spin 2 , spin 1,spin 0 states of Omega-Omega. The spin 0 state, used in this research, is given by

$$(\Omega\Omega)_{0,0} \equiv \frac{1}{2} \left( \Omega_{1/2} \Omega_{-1/2} - \Omega_{3/2} \Omega_{-3/2} + \Omega_{1/2} \Omega_{-1/2} \right)$$

4. Results

4.1 Lattice set up

In our calculation we employ 700 gauge configurations generated by CP-PACS and JLQCD Collaborations [17] with the renormalization group improved gauge action and the non-perturbatively
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\( \mathcal{O}(a) \) improved Wilson quark action [17] at \( \beta = 1.83 \) \( (a \simeq 0.12 \text{fm}) \) on the 16\(^3\) \times 32 lattice, whose physical extension becomes \( L = 1.92 \text{ fm} \). The hopping parameter is \((\kappa_{ud}, \kappa_s) = (0.13760, 0.13710)\), which gives \( m_\pi = 875(1) \text{ MeV} \) and \( m_K = 916(1) \text{ MeV} \). The periodic boundary condition is imposed on the quark fields along the spatial direction, while the Dirichlet boundary condition is employed along the temporal direction on the time-slice \( t = \frac{T}{2} \). To improve statistics, we calculate nine sources on different time slices per config, where the Dirichlet boundary is always separated from the source by \( \frac{T}{2} \). Statistical errors are estimated by the Jackknife method where the bin-size is taken to be 1 configuration.

4.2 Omega-Omega Potential

We show the central potential between Omega-Omega in the \( ^1S_0 \) channel at \( t - t_0 = 7, 8, 9 \) in Fig. 1, where we use the notation \( ^{2S+1}L_J \) to specify quantum numbers of the channel. Overall structures of potentials are similar to those of NN potentials previously obtained in the lattice QCD [5]. This potential have repulsive core and deep attractive pocket, whose depth is 70–80 MeV. We observe that time dependence is small except for a long distance part at \( t = 9 \), which seems to be affected by finite volume effect. In future studies, we should investigate a volume dependence of the potential.

![Figure 1: The central potential \( V_c(r) \) between Omega-Omega in the \( ^1S_0 \) channel at \( t - t_0 = 7, 8, 9 \).](image)

4.3 Phase shift and Scattering length

To calculate phase shift and scattering length, we first fit the potential with the form that

\[
V(r) = a_1 e^{-ar^2} + a_3 e^{-ar^2} + a_5 e^{-ar^2},
\]

which gives, for example, \( a_1 = -1.5(0.5) \times 10^2 \text{MeV} \), \( a_2 = 2.0(0.6) \text{fm}^{-2} \), \( a_3 = 2.0(0.9) \times 10^3 \text{MeV} \), \( a_4 = 1.2(1.1) \times 10^3 \text{MeV} \), \( a_5 = 1.0(0.1) \times 10^3 \text{MeV} \), \( a_6 = 6.9(1.1) \times 10^3 \text{MeV} \) at \( t - t_0 = 7 \).
Using this fit result, we solve the Schrödinger equation in the infinite volume, and we obtain the phase shift $\delta(k)$ in the $^1S_0$ channel. The scattering phase shift $\delta(k)$ at a given momentum $k$ can be obtained as ratio

$$\tan \delta(k) = \lim_{x_1,x_2 \to \infty} \frac{\psi_k(x_2) \sin(kx_1) - \psi_k(x_1) \sin(kx_2)}{\psi_k(x_1) \cos(kx_2) - \psi_k(x_2) \cos(kx_1)}.$$  

We show a center of mass energy dependence of the scattering phase shift in Fig. 2, where $E_{CM} = k^2/(2\mu_\Omega)$. As shown in the figure, while phase shift calculated from the potential at $t=7$ suggests an existence for a bound state, phase shifts from data at $t=8$ and 9 indicate that the Omega-Omega interaction is strongly attractive but it may not be strong enough to form a bound state at this quark mass.

The scattering length $a$ and effective range $r_e$ are defined by

$$k \cot \delta(k) = \frac{1}{a} + \frac{1}{2} r_e k^2 + O(k^4).$$  

The fit of $k \cot \delta(k)$ near $k=0$ by the above formula gives $1/a = -2.4(2.5) \times 10^{-1}$, $9.0(6.9) \times 10^{-2}$, $2.9(1.2) \times 10^{-1}$ fm$^{-1}$, and $r_e = 5.0(5.1) \times 10^{-3}$, $5.1(1.0) \times 10^{-3}$, $5.2(1.8) \times 10^{-3}$ fm at $t - t_0 = 7, 8, 9$, respectively. Unfortunately, errors of both $1/a$ and $r_e$ are quite large, as expected from Fig. 2.

Figure 2: Scattering phase shift at $t - t_0 = 7$(blue) , 8(green) , 9(red), as a function of the center of mass energy $E_{CM}$.

5. Summary

In this paper, we have investigated the Omega-Omega interaction in the channel with the orbital angular momentum $L=0$ and the total spin $S=0$ in 2+1 flavor lattice QCD, using the method developed by the HAL QCD collaboration. We find that the central potential obtained from the
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NBS wave function for the $^1S_0$ channel shows attractions at long distance and the repulsive core at short distance. The phase shift derived from this potential shows that the Omega-Omega interaction is strongly attractive, while an existence for an Omega-Omega bound state is unfortunately inconclusive at the pion mass in this study. In future we will examine the Omega-Omega interaction at more lighter pion masses in larger volumes.

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References