

$\mathcal{O}(a^2)$ -improved actions for heavy quarks and scaling studies on quenched lattices

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We investigate a new class of improved relativistic fermion action on the lattice with a criterion to give excellent energy-momentum dispersion relation as well as to be consistent with tree-level $\mathcal{O}(a^2)$ -improvement. Main application in mind is that for heavy quark for which $ma \simeq O(0.5)$. We present tree-level results and a scaling study on quenched lattices.

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1. Introduction

Precise non-perturbative calculation in heavy quark physics is one of the long-standing goals of lattice QCD. For quantities involving heavy quark, the discretization effect may become more significant than that in the light quark sector. With a naive order counting it appears as a power of am , the heavy quark mass in unit of the lattice cutoff $1/a$, which is not much smaller than one. While effective theories for heavy quarks in non-relativistic kinematics have been developed and used on the lattice, a brute-force approach of taking a as small as possible would also be powerful since there is no need of additional matching of parameters. This may be combined with the Symanzik improvement of the lattice fermion action to eliminate leading discretization effects.

The JLQCD collaboration is currently generating 2+1-flavor gauge configurations at fine lattice spacings of $a^{-1} = 2.4\text{--}4.8$ GeV using a chirally symmetric fermion formulation for light quarks [1]. For the valence heavy quarks, we plan to use other fermion formulations that may have better scaling towards the continuum limit.

In this work we investigate some choices of the lattice fermion action to be used for valence quarks, focusing on their discretization effect and continuum scaling for heavy quarks. At this initial study, we mainly consider the charm quark mass region, and leave an extension towards heavier masses for future study. The quantities to be studied are the energy-momentum dispersion relation, hyperfine splitting and decay constants of the heavy-heavy mesons. For this purpose, we are generating a series of quenched gauge configurations that have a roughly matched physical volume (at about 1.6 fm) and cover a range of lattice spacings between $1/a = 2$ and 4 GeV. Since these lattices do not contain sea quarks and have small physical volume, we do not expect precise agreement with the corresponding experimental data for the charm quark, but rather we are interested in their scaling towards the continuum limit.

The gauge configurations are generated with the tree-level $\mathcal{O}(a^2)$ -improved Symanzik action, so far at $\beta = 4.41$ and 4.66 on $16^3 \times 32$ and $24^3 \times 48$ lattices, respectively. Using the energy density expectation value after the Wilson flow, we determine the lattice spacing with an input $w_0 = 0.176(2)$ fm [2] as $1/a = 1.97(2)$ and 2.81(3) GeV for the two lattices. (Note that this input value is given through the Ω baryon mass in 2+1-flavor QCD in [2].) For each β value we have analysed 100 independent gauge configurations. In the following we mainly discuss the newly developed $\mathcal{O}(a^2)$ -improved Brillouin fermion action, and present our preliminary studies of the dispersion relation and hyperfine splitting. We also analyse the heavy-heavy decay constant calculated with the domain-wall fermion action in the valence sector.

2. $\mathcal{O}(a^2)$ -improved Brillouin fermions

We develop a new class of lattice fermion action which is free from $\mathcal{O}(a)$ and $\mathcal{O}(a^2)$ discretization effects at the tree-level. The action is based on the Isotropic-derivative and the Brillouin Laplacian studied in [3, 4]. The Dirac-operator is defined as

$$D^{bri}(x, y) = \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{iso}(x, y) - \frac{a}{2} \Delta^{bri}(x, y) + m_0 \delta_{x, y}, \quad (2.1)$$

where $\nabla_{\mu}^{iso}(x, y)$ and $\Delta^{bri}(x, y)$ include 1-, 2-, 3- and 4-hop terms in a 3^4 hypercube defined by $|x_{\mu} - y_{\mu}| \leq 1$, and the resulting Dirac operator is ultralocal. The leading discretization effect contained

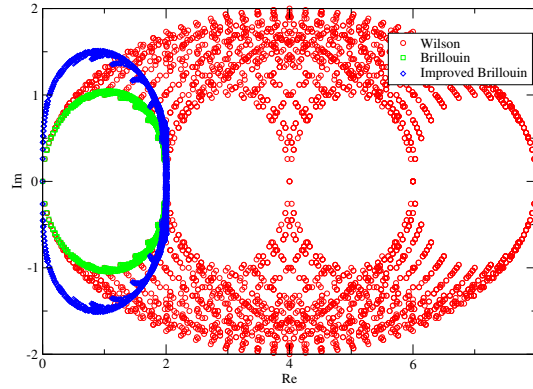


Figure 1: Eigenvalue distributions of the Wilson (open circles), the Brillouin (open squares) and the improved Brillouin (open diamonds).

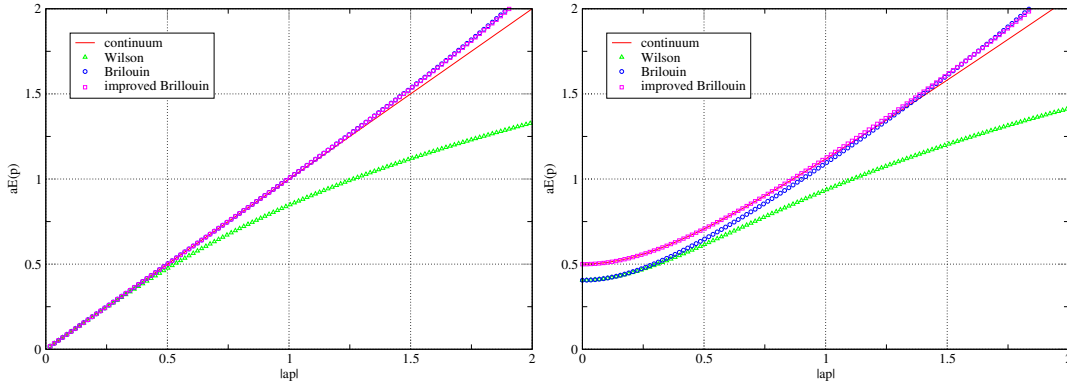


Figure 2: Dispersion relation calculated at the tree-level for different fermion formulations, *i.e.* Wilson (green), Brillouin (blue), improved Brillouin (magenta). The results at $ma = 0.0$ (left) and $ma = 0.5$ (right) are plotted.

in $\nabla_{\mu}^{iso}(x,y)$ is $\mathcal{O}(a^2)$ and is isotropic. The Brillouin Laplacian $\Delta^{bri}(x,y)$ is designed such that all the fifteen doublers have the same mass $2/a$ at the tree-level. This can be seen from the eigenvalue spectrum on the complex plane as shown in Figure 1 for the free case. For the Wilson fermion, the spectrum has five branches, *i.e.* one corresponding to the physical (real part = 0) and the others to doublers ($2/a$, $4/a$, $6/a$ and $8/a$). The Brillouin operator approximately gives an unit circle centered at $(1,0)$, which resembles the Ginsparg-Wilson-type fermions. This suggests that the Brillouin operator approximately satisfies the Ginsparg-Wilson relation.

With the Brillouin operator, it is found that the energy-momentum dispersion relation calculated at the tree-level follows very precisely that of continuum theory [4], which is demonstrated in Figure 2 (left: massless; right massive $ma = 0.5$). With the massless Wilson fermion, the deviation from the continuum is already seen at $ap \sim 0.5$ in the massless case, while with the Brillouin fermion it does not start until around $ap \sim 1.5$. This is confirmed also nonperturbatively using the dispersion relation of mesons and baryons calculated on quenched lattices [5].

In this work, we further improve the Brillouin fermion by eliminating its leading discretization

effects. For instance, with the Brillouin operator, the relation between the static energy E , defined through a pole of the free propagator, and the bare mass m is given as

$$(Ea)^2 = (ma)^2 - (ma)^3 + \frac{11}{12}(ma)^4 - \frac{5}{6}(ma)^5 + \mathcal{O}((ma)^6) \quad (2.2)$$

at finite lattice spacing a , and the error starts from the term of $(ma)^3$, which represents a relative $O(a)$ effect. Such deviation from the continuum is seen in the plot of Figure 2 (right) at $ap = 0$, where the case of $am = 0.5$ is plotted. Here, the Brillouin operator has a similar discretization effect as the Wilson fermion, which gives $Ea = \ln(1 + ma)$.

In order to make the Brillouin fermion consistent with the Symanzik improvement, we eliminate the leading discretization errors by modifying the action as

$$D^{imp} = \sum_{\mu} \gamma_{\mu} \left(1 - \frac{a^2}{12} \Delta^{bri} \right) \nabla_{\mu}^{iso} \left(1 - \frac{a^2}{12} \Delta^{bri} \right) + c_{imp} a^3 (\Delta^{bri})^2 + m_0. \quad (2.3)$$

The terms $(1 - a^2 \Delta^{bri}/12)$ sandwiching ∇_{μ}^{iso} are introduced to eliminate the a^2 errors while keeping the γ_{μ} hermiticity property. The Wilson-like term is simply squared so that its effect starts from $O(a^3)$. The relation between the energy and the bare mass becomes

$$(Ea)^2 = (ma)^2 + c_{imp}(ma)^5 + \mathcal{O}((ma)^6), \quad (2.4)$$

and the leading error starts from $O(a^3)$ as expected.

For this improved Brillouin fermion action, we observe good dispersion relation also for the massive case (see a plot on the right panel of Figure 2). The difference from the continuum is invisible below $ap \sim 1.5$. The eigenvalues of the improved operator D^{imp} no longer lie on the unit circle as shown in Figure 1 (blue dots), because it approaches the continuum limit which is in this case the imaginary axis. The blue points indeed touch the imaginary axis more closely.

Numerical implementation of the Brillouin operator is complicated once the gauge-link is introduced, because one has to preserve symmetries under cubic rotations for 2-, 3- and 4-hop terms. We explicitly average over all possible paths of minimal lengths. Computational code is implemented on the IroIro++ package [6].

3. Nonperturbative studies on quenched lattices

Our scaling studies on the quenched configurations are ongoing. In the following we show the results for the dispersion relation and hyperfine splitting of heavy-heavy mesons obtained with the improved Brillouin fermion, as well as a study of heavy-heavy decay constant using the domain-wall fermion.

For a heavy-heavy meson, we calculate an effective speed-of-light extracted from the energy at finite momentum \vec{p} as

$$c_{\text{eff}}^2(\vec{p}) = \frac{E^2(\vec{p}) - E^2(\vec{0})}{\vec{p}^2}. \quad (3.1)$$

The heavy quark mass is tuned until the spin-averaged 1S mass becomes 3 GeV, and c_{eff}^2 is calculated for the pseudo-scalar channel. The Wilson and improved Brillouin fermions are used on the

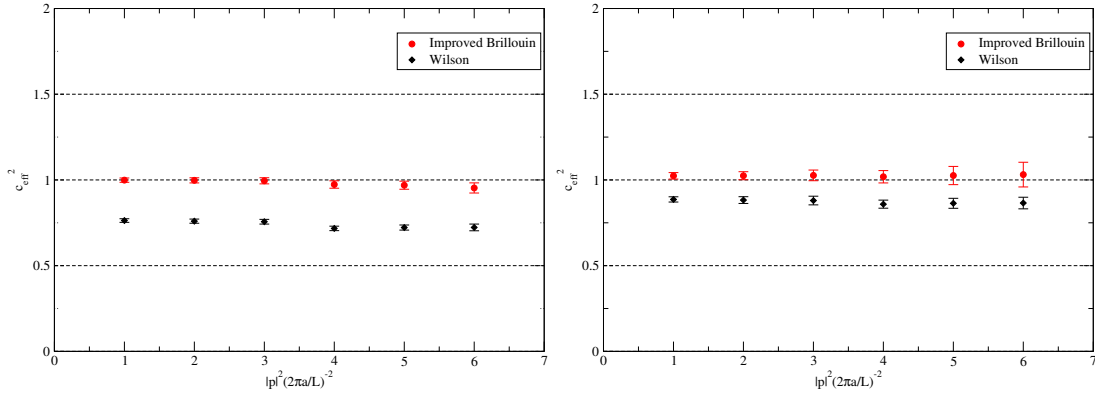


Figure 3: Effective speed of light as a function of normalized momentum squared at $a^{-1} = 1.97$ GeV (left) and $a^{-1} = 2.81$ GeV (right). In each panel, data for improved Brillouin (filled circles) and Wilson (filled diamonds) fermions are plotted.

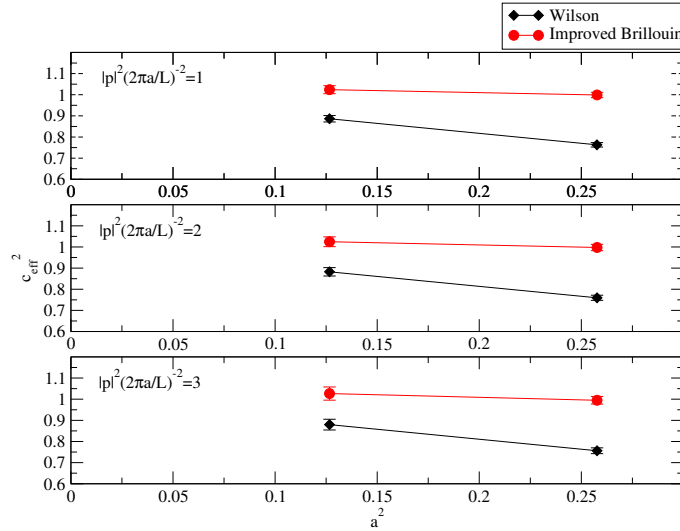


Figure 4: Scaling of c_{eff}^2 calculated at $|\vec{p}|^2 = 1$ (upper), 2 (middle) and 3 (lower) in units of $(2\pi/L)^2$. In each panel, data for improved Brillouin (filled circles) and Wilson (filled diamonds) are plotted as a function of a^2 [GeV^{-2}].

quenched configurations at $1/a = 1.97$ and 2.81 GeV. Three steps of stout smearing [8] are applied for the gauge links.

The results are shown in Figure 3. Here c_{eff}^2 is plotted against $|\vec{p}|^2$ for Wilson (black) and the improved Brillouin (red) fermions. Already at $1/a = 2.0$ GeV (left) the dispersion relation of the 3-GeV meson follows that of continuum theory, $c = 1$, very precisely (within the statistical error) when the improved Brillouin fermion is employed. With the Wilson fermion, the deviation is as large as 30%. Such large deviation is allowed in the effective theory approaches [7], where the rest mass m_1 and the kinetic mass m_2 are treated differently and only m_2 is taken as physical. With the improved Brillouin fermion, this is not necessary. Scaling towards the continuum limit is

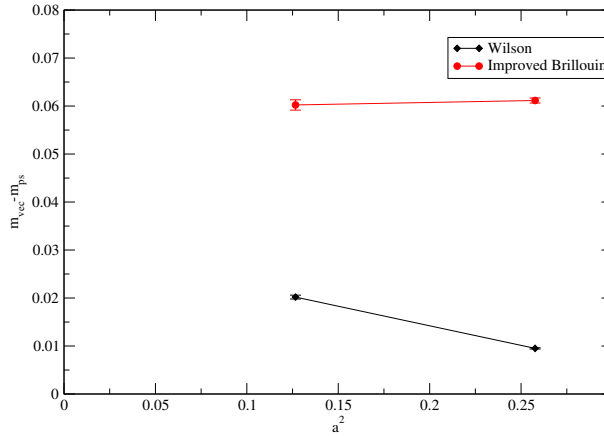


Figure 5: Continuum scaling of the hyperfine splitting $m_{vec} - m_{ps}$ [GeV]. Results with the Wilson (black) and improved Brillouin (red) fermions are plotted as a function of a^2 [GeV $^{-2}$].

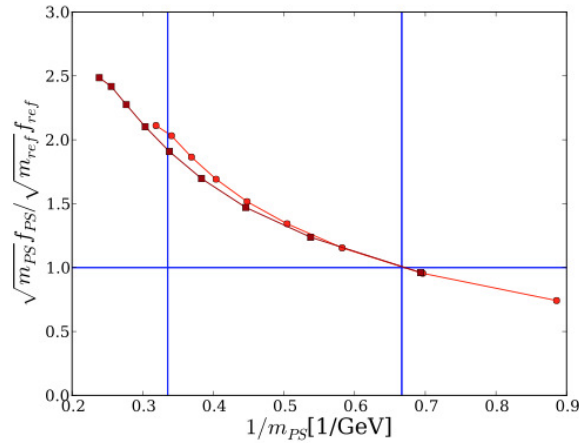


Figure 6: Heavy-heavy pseudo-scalar meson decay constants calculated at two lattice spacings, 2.0 GeV (red) and 2.8 GeV (black), with the domain-wall fermion. The results for $f_{PS}\sqrt{m_{PS}}$ are normalized by the value at $m_{PS} = 1.5$ GeV and plotted as a function of $1/m_{PS}$.

demonstrated in Figure 4. In three panels, the results at normalized momentum squared are shown as a function of a^2 . With the Brillouin fermion, we do not see any deviations from the continuum at the level of 1%, which is the size of the statistical error.

In Figure 5, we show a similar scaling test of the two formulations for the hyperfine splitting $m_{vec} - m_{ps}$ of the 3-GeV heavy-heavy meson. Also for this quantity, the scaling towards the continuum limit is much better with the improved Brillouin fermion.

Finally, we briefly describe a calculation of the heavy-heavy decay constant using the domain-wall fermion. It is well known that the domain-wall fermion mechanism breaks down at large am ($\simeq 0.5$) [9, 10, 11], but the real question is where it shows up in numerical calculations. In Figure 6 we plot the decay constant $f_{PS}\sqrt{m_{PS}}$ as a function of $1/m_{PS}$. The data are normalized by the value at $m_{PS} = 1.5$ GeV, so that the renormalization constant cancels out. We observe a

good indication of scaling of f_{ps} with the pseudo-scalar mass $m_{ps} \sim 3\text{GeV}$, which suggests that the lattices of $1/a = 2 - 4\text{GeV}$ could be used for the direct extraction of the properties of D mesons. A complete continuum limit study of this and alternative heavy quark discretizations is needed to come to a final conclusion on this matter.

4. Summary and plans

Relativistic formulation for heavy quark has an advantage that no tuning of parameters depending on the heavy quark mass is necessary. Therefore, as fine-lattice dynamical QCD simulations has become realistic, such a brute-force approach could be a powerful alternative to the effective theory approaches, provided that the possible $(am)^n$ corrections are under control.

We are performing various scaling tests of relativistic formulations on quenched lattices, and so far have obtained promising results. In the future we plan to extend the study by adding finer lattices and more choices of fermion formulations.

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