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The hadronic vacuum polarization with twisted boundary conditions

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The leading-order hadronic contribution to the anomalous magnetic moment of the muon is given by a weighted integral over the subtracted hadronic vacuum polarization. This integral is dominated by euclidean momenta of order the muon mass which are not available on current lattice volumes with periodic boundary conditions. Twisted boundary conditions can in principle help access momenta of any size even in a finite volume. We investigate the implementation of twisted boundary conditions both numerically (using all-mode averaging for improved statistics) and analytically, and present our initial results.

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1. Introduction

The leading-order hadronic (HLO) contribution to the anomalous magnetic moment of the muon $a_\mu = (g-2)/2$ of the muon is given by the integral [1, 2][1]

$$\begin{aligned} a_\mu^{\rm HLO} &= 4\alpha^2 \int_0^\infty dp^2\, f(p^2)\left(\Pi^{\rm em}(0) - \Pi^{\rm em}(p^2)\right) , \\ f(p^2) &= m_\mu^2 p^2 Z^3(p^2)\, \frac{1 - p^2 Z(p^2)}{1 + m_\mu^2 p^2 Z^2(p^2)} , \\ Z(p^2) &= \frac{\sqrt{(p^2)^2 + 4m_\mu^2 p^2} - p^2}{2m_\mu^2 p^2} , \end{aligned} \tag{1.1}$$

where m_μ is the muon mass, and for non-zero momenta $\Pi^{\rm em}(p^2)$ is defined from the hadronic contribution to the electromagnetic vacuum polarization $\Pi^{\rm em}_{\mu\nu}(p)$:

$$\Pi^{\rm em}_{\mu\nu}(p) = \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\Pi^{\rm em}(p^2) \tag{1.2}$$

in momentum space. Here p is the euclidean momentum flowing through the vacuum polarization.

The integrand in Eq. (1.1) is dominated by momenta of order the muon mass; it typically looks as shown in Fig. 1, with the peak located at $p^2 \approx (m_\mu/2)^2$. The figure includes lattice data for the vacuum polarization on the $64^3 \times 144$, $1/a = 3.35$ GeV MILC Asqtad ensemble and the curve is a [1,1] Padé fit to this data [4]. The resulting value for a_μ is extremely sensitive to the fitting choice,

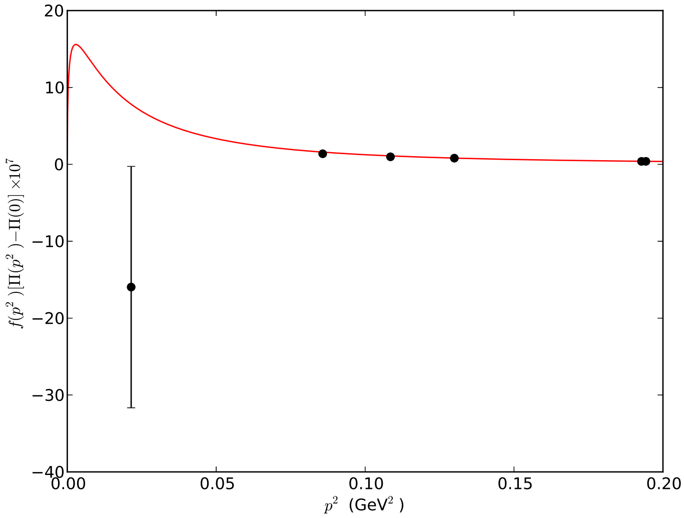


Figure 1: The integrand in Eq. (1.1) for the $64^3 \times 144$, $1/a = 3.35$ GeV MILC Asqtad ensemble. The data shown are lattice results for the vacuum polarization and the curve is a [1,1] Padé fit to this data.

[1]For an overview of lattice computations of the muon anomalous magnetic moment, see Ref. [3] and references therein.

and while a fit may give a "good result" (*i.e.,* may have a reasonable χ^2 per degree of freedom), it has been recently shown in Ref. [5] that such fits may not reproduce the correct result.

For a precision computation of this integral using lattice QCD, one would therefore like to access the region of this peak. In a finite volume with periodic boundary conditions, for the smallest available non-vanishing momentum to be at this peak, we require $L \approx 25$ fm, which is out of reach of present lattice computations, if the lattice spacing a is chosen to be such that one is reasonably close to the continuum limit. Clearly, a different method for reaching such small momenta is needed, and here we discuss the use of twisted boundary conditions in order to vary momenta arbitrarily in a finite volume. More details on this work can be found in Ref. [6].

2. Twisted Boundary Conditions

Given the electromagnetic current,

$$J_\mu^{\rm em}(x) = \sum_i Q_i \overline{q}_i(x)\gamma_\mu q_i(x) \ , \tag{2.1}$$

in which i runs over quark flavors, and quark q_i has charge $Q_i e$, we wish to calculate the connected part of the two-point function in a finite volume, but with an arbitrary choice of momentum.[2] In order to do this, we will employ quarks which satisfy twisted boundary conditions [7, 8, 9],

$$q_t(x) = e^{-i\theta_\mu}\, q_t(x+\hat{\mu}L_\mu) \ , \qquad \overline{q}_t(x) = \overline{q}_t(x+\hat{\mu}L_\mu)\, e^{i\theta_\mu} \ , \tag{2.2}$$

where the subscript t indicates that the quark field q_t obeys twisted boundary conditions, L_μ is the linear size of the volume in the μ direction ($\hat{\mu}$ denotes the unit vector in the μ direction), and $\theta_\mu \in [0, 2\pi)$ is the twist angle in that direction. For a plane wave $u(p)e^{ipx}$, the boundary conditions (2.2) lead to the allowed values for the momenta (we set $a = 1$)

$$p_\mu = \frac{2\pi n_\mu + \theta_\mu}{L_\mu} \ , \qquad n_\mu \in \{0, 1, \ldots, L_\mu - 1\} \ . \tag{2.3}$$

The twist angle can be chosen differently for the two quark lines in the connected part of the vacuum polarization, resulting in a continuously variable momentum flowing through the diagram. (Clearly, this trick does not work for the disconnected part.) If this momentum is chosen to be of the form (2.3), then allowing θ_μ to vary over the range between 0 and 2π allows p_μ to vary continuously between $2\pi n_\mu/L_\mu$ and $2\pi(n_\mu+1)/L_\mu$. This momentum is realized if, for example, we choose the anti-quark line in the vacuum polarization to satisfy periodic boundary conditions (*i.e.*, Eq. (2.2) with $\theta_\mu = 0$ for all μ), and the quark line twisted boundary conditions with twist angles θ_μ.

Thus, we define two currents[3]

$$j_\mu^+(x) = \frac{1}{2}\left[\overline{q}(x)\gamma_\mu U_\mu(x) q_t(x+\hat{\mu}) + \overline{q}(x+\hat{\mu})\gamma_\mu U_\mu^\dagger(x) q_t(x)\right] \ , \tag{2.4}$$

$$j_\mu^-(x) = \frac{1}{2}\left[\overline{q}_t(x)\gamma_\mu U_\mu(x) q(x+\hat{\mu}) + \overline{q}_t(x+\hat{\mu})\gamma_\mu U_\mu^\dagger(x) q(x)\right] \ . \tag{2.5}$$

[2] This method is only useful for the connected part of the two-point function, although for a full calculation of the photon vacuum polarization one must also look at the disconnected part.

[3] Note this is shown here for naïve quarks, but the arguments that follow would hold for any other discretization in which a conserved vector current can be defined. For example, for staggered quarks we merely make the replacement $\gamma_\mu \to \eta_\mu(x)$ and carry through the argument.

In the case where we remove the twist ($\theta_\mu = 0$), these become equal to each other and the standard conserved vector current used for lattice calculations.

We thus consider a mixed-action theory, where we have periodic sea quarks and (quenched) twisted valence quarks. Formally this amounts to N_s quarks with periodic boundary conditions, N_v quarks with twisted boundary conditions, and N_v ghost quarks with the same twisted boundary conditions. The ghost quarks thus cancel the fermionic determinant of the twisted quarks. Then, under the field transformations,

$$\begin{aligned} \delta q(x) &= i\alpha^+(x)e^{-i\theta x/L}q_t(x)\,, \qquad \delta\overline{q}(x) = -i\alpha^-(x)e^{i\theta x/L}\overline{q}_t(x)\,, \\ \delta q_t(x) &= i\alpha^-(x)e^{i\theta x/L}q(x)\,, \qquad \delta\overline{q}_t(x) = -i\alpha^+(x)e^{-i\theta x/L}\overline{q}(x)\,, \end{aligned} \tag{2.6}$$

in which we abbreviate $\theta x/L = \sum_\mu \theta_\mu x_\mu / L_\mu$. We obtain, following the standard procedure, the Ward-Takahashi Identity (WTI)

$$\begin{aligned} \sum_\mu \partial_\mu^- \left\langle j_\mu^+(x) j_\nu^-(y)\right\rangle &+ \frac{1}{2}\delta(x-y)\left\langle \overline{q}_t(y+\hat\nu)\gamma_\nu U_\nu^\dagger(y)q_t(y) - \overline{q}(y)\gamma_\nu U_\nu(y)q(y+\hat\nu)\right\rangle \\ &- \frac{1}{2}\delta(x-\hat\nu-y)\left\langle \overline{q}(y+\hat\nu)\gamma_\nu U_\nu^\dagger(y)q(y) - \overline{q}_t(y)\gamma_\nu U_\nu(y)q_t(y+\hat\nu)\right\rangle = 0\,, \end{aligned} \tag{2.7}$$

where ∂_μ^- is the backward lattice derivative, which for this paper always acts on x: $\partial_\mu^- f(x) = f(x) - f(x-\hat\mu)$.

From this WTI, we define the vacuum polarization function as

$$\begin{aligned} \Pi_{\mu\nu}^{+-}(x-y) = \left\langle j_\mu^+(x) j_\nu^-(y)\right\rangle - \frac{1}{4}\delta_{\mu\nu}\delta(x-y)\Big(&\left\langle \overline{q}(y)\gamma_\nu U_\nu(y)q(y+\hat\nu) - \overline{q}(y+\hat\nu)\gamma_\nu U_\nu^\dagger(y)q(y)\right\rangle \\ &+ \left\langle \overline{q}_t(y)\gamma_\nu U_\nu(y)q_t(y+\hat\nu) - \overline{q}_t(y+\hat\nu)\gamma_\nu U_\nu^\dagger(y)q_t(y)\right\rangle\Big)\,. \end{aligned} \tag{2.8}$$

In the case where we set the twist to zero in all directions, $\theta_\mu = 0$, this definition reduces to the standard result for the vacuum polarization and is transverse. However in the twisted case, $\Pi_{\mu\nu}^{+-}(x-y)$ is not transverse, but instead obeys the identity

$$\sum_\mu \partial_\mu^- \Pi_{\mu\nu}^{+-}(x-y) + \frac{1}{4}\left(\delta(x-y) + \delta(x-\hat\nu-y)\right)\left\langle j_\nu^t(y) - j_\nu(y)\right\rangle = 0\,, \tag{2.9}$$

in which $j_\nu(x)$ and $j_\nu^t(x)$ are currents defined by

$$\begin{aligned} j_\mu(x) &= \frac{1}{2}\left(\overline{q}(x)\gamma_\mu U_\mu(x)q(x+\hat\mu) + \overline{q}(x+\hat\mu)\gamma_\mu U_\mu^\dagger(x)q(x)\right)\,, \\ j_\mu^t(x) &= \frac{1}{2}\left(\overline{q}_t(x)\gamma_\mu U_\mu(x)q_t(x+\hat\mu) + \overline{q}_t(x+\hat\mu)\gamma_\mu U_\mu^\dagger(x)q_t(x)\right)\,. \end{aligned} \tag{2.10}$$

It is important to note that other choices for $\Pi_{\mu\nu}^{+-}(x-y)$ are possible, but there will always be a non-vanishing contact term in the WTI. The reason is that the contact term in Eq. (2.9) (or, equivalently, in Eq. (2.7)) cannot be written as a total derivative, because the fact that q and q_t fields satisfy different boundary conditions breaks explicitly the isospin-like symmetry that otherwise would exist. (For $\alpha^\pm$ constant and $\theta = 0$, Eq. (2.6) is an isospin-like symmetry of the action. As a check, we see that for $q_t = q$, *i.e.*, for $\theta = 0$, the contact term in Eq. (2.9) vanishes.) The resulting non-transverse part of $\Pi_{\mu\nu}^{+-}$ therefore will need to be subtracted.

3. Subtraction of contact term

In momentum space, we can decompose the vacuum polarization tensor as

$$\Pi^{+-}_{\mu\nu}(\hat{p}) = \left(\hat{p}^2\delta_{\mu\nu} - \hat{p}_\mu\hat{p}_\nu\right)\Pi^{+-}(\hat{p}^2) + \frac{\delta_{\mu\nu}}{a^2}X_\nu(\hat{p}) \ , \quad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right) \ , \tag{3.1}$$

and as such, we can determine X_ν by using the WTI in momentum space,

$$i\sum_\mu \hat{p}_\mu \Pi^{+-}_{\mu\nu}(\hat{p}) \ = \ -\cos(ap_\nu/2)\left\langle j^t_\nu(0)\right\rangle = i\frac{\hat{p}_\nu}{a^2}X_\nu(\hat{p}) \tag{3.2}$$

$$\Rightarrow X_\nu(\hat{p}) \ = \ \frac{i}{2}\cot(ap_\nu/2)a^3\left\langle j^t_\nu(0)\right\rangle \ . \tag{3.3}$$

There is a pole in X_ν only when $\pi n_\nu + \theta_\nu/2$ is equal to an integer multiple of $\pi L_\nu/a$, which is only possible if $\theta_\nu = 0$ for our allowed values of θ_ν, but then this term would vanish because for $\theta_\nu = 0$ the current from which $\Pi^{+-}_{\mu\nu}$ is constructed is conserved.

From dimensional analysis and axis-reversal symmetry, we see for small $\hat{\theta}_\mu \equiv \theta_\mu/L_\mu$:

$$\left\langle j^t_\nu(y)\right\rangle = -i\frac{c}{a^2}\hat{\theta}_\nu\left[1+\mathscr{O}(\hat{\theta}^2)\right] \ . \tag{3.4}$$

This must be odd under the interchange $\hat{\theta}_\nu \to -\hat{\theta}_\nu$, and we see that this vanishes when we take away the twisting (so that $\theta_\nu = 0$ for all ν). In that case, $\Pi^{+-}_{\mu\nu}$ is conserved.

We can determine the vacuum polarization at one-loop in perturbation theory to get an estimate for the size of this effect. In the twisted case, we have for N_c colors (again for $a = 1$),

$$\begin{aligned}\Pi^{+-}_{\mu\nu}(p) \ &= \ -\frac{N_c}{V}\sum_k \mathrm{tr}\left[\gamma_\mu \frac{\cos\left(k_\mu + p_\mu/2\right)}{i\sum_\kappa \gamma_\kappa \sin(k_\kappa + p_\kappa) + m}\gamma_\nu \frac{\cos\left(k_\nu + p_\nu/2\right)}{i\sum_\lambda \gamma_\lambda \sin k_\kappa + m}\right] \\ &\quad + \frac{i}{2}\delta_{\mu\nu}\frac{N_c}{V}\sum_k \mathrm{tr}\left[\gamma_\nu\left(\frac{\sin k_\nu}{i\sum_\kappa \gamma_\kappa \sin k_\kappa + m} + \frac{\sin\left(k_\nu + \hat{\theta}_\nu\right)}{i\sum_\kappa \gamma_\kappa \sin\left(k_\kappa + \hat{\theta}_\kappa\right) + m}\right)\right] \ ,\end{aligned} \tag{3.5}$$

and the WTI,

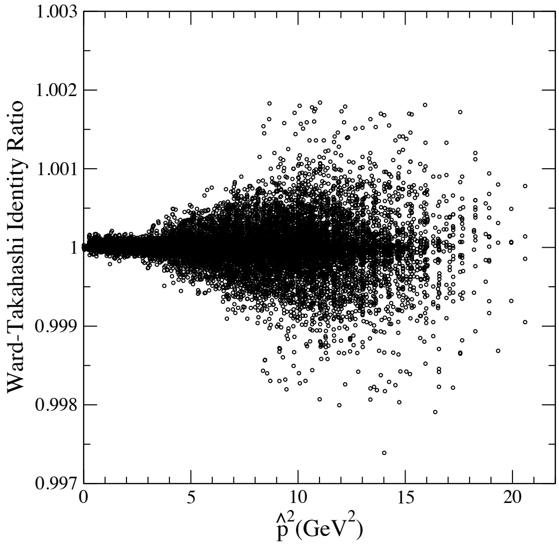
$$\begin{aligned}i\sum_\mu \hat{p}_\mu \Pi^{+-}_{\mu\nu}(p) \ &= \ -2i\cos\left(p_\nu/2\right)\frac{N_c}{V}\sum_k\left(\frac{\sin(2k_\nu)}{\sum_\kappa \sin^2 k_\kappa + m^2} - \frac{\sin(2(k_\nu + \hat{\theta}_\nu))}{\sum_\kappa \sin^2\left(k_\kappa + \hat{\theta}_k\right) + m^2}\right) \\ &= \ 2i\cos\left(p_\nu/2\right)\hat{\theta}\left[\frac{N_c}{V}\sum_k\left(\frac{2\cos(2k_\nu)}{\sum_k \sin^2 k_\kappa + m^2} - \frac{\sin^2(2k_\nu)}{(\sum_k \sin^2 k_\kappa + m^2)^2}\right)\right] + O(\hat{\theta}^3) \ .\end{aligned} \tag{3.6}$$

For the MILC Asqtad ensemble with $V = 48^3 \times 144$ and a light quark mass of $am = 0.0036$, this gives

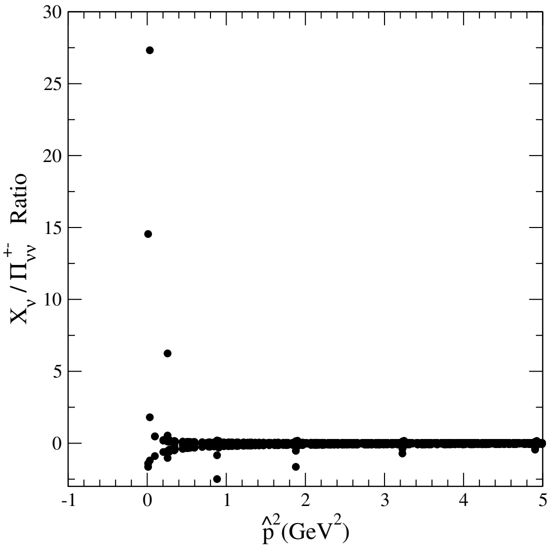
$$\left\langle j^t_\nu(0)\right\rangle = (7.30 \times 10^{-5})i$$

for a twist of $\theta_i = 0.28\pi$ in the spatial directions. Thus, generally this effect could be very small.

As the WTI holds on a configuration-by-configuration basis, it is straightforward to test Eq. (3.2) numerically. In Fig 2(a) we show the ratio of the right-hand side to the left-hand side of Eq. (3.1). This was performed on a typical configuration on an Asqtad MILC ensemble with $L^3 \times T =$



(a)



(b)

Figure 2: Numerical tests of the WTI on a single configuration. In (a) we show the ratio of the left-hand side and right-hand side of Eq. (3.2), while in (b) we show the ratio of the second term on the right-hand side and the left-hand side of Eq. (3.1). Both cases are on a single configuration on an Asqtad MILC ensemble with $L^3 \times T = 48^3 \times 144$, $1/a = 3.35$ GeV, $am = 0.0036$, $\theta_x = \theta_y = \theta_z = 0.28\pi, \theta_t = 0$.

$48^3 \times 144$, $1/a = 3.35$ GeV, $am = 0.0036$, $\theta_x = \theta_y = \theta_z = 0.28\pi, \theta_t = 0$. In this case, the stopping residual for the conjugate gradient was 10^{-8}. For small momenta this ratio is near one, and at most deviates from one by about 0.3% for larger momenta. The ratio is expected to numerically converge to one as the CG stopping criterion is reduced. As one is interested in using twisted boundary conditions for lower momenta this does not appear to introduce a significant systematic.

In Fig 2(b) we plot the quantity

$$\frac{X_\nu(\hat{p})}{a^2\Pi^{+-}_{\nu\nu}(\hat{p})} \tag{3.7}$$

on the same configuration as in Fig 2(a). For very small momenta, this counterterm can become quite significant, especially in the primary region of interest. While averaging over configurations seems to diminish this effect, this is still under investigation. Of course, even if averaging over an ensemble reduces the effect of the counterterm, one must worry about the systematic error introduced in such large cancellations during such averaging.

4. Conclusions

The use of twisted boundary conditions is promising in obtaining the connected portion of the leading hadronic contribution to the muon anomalous magnetic moment. While the introduction of twisted boundary conditions does not allow one to write a purely transverse vacuum polarization, it is straightforward to subtract the term which arises due to the partial twisting of the quarks.

Currently it appears as though averaging over an ensemble makes a large effect (on each configuration) negligible. The reason for this is under investigation, and there is no guarantee that it will be true for all ensembles. As such, when attempting to obtain a high-precision calculation of the muon $g-2$, it is imperative that one gets a measurement of the contact term that arises in the vacuum polarization and subtract it if it is not negligible, as small errors in the low momentum region can lead to large errors in the final determination of the muon $g-2$.

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