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Non-perturbative Renormalization of Bilinear Operators with Improved Staggered Quarks

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We present renormalization factors for the bilinear operators obtained using the non-perturbative renormalization method (NPR) in the RI-MOM scheme with improved staggered fermions on the MILC asqtad lattices ($N_f = 2+1$). We use the MILC coarse ensembles with $20^3 \times 64$ geometry and $am_\ell/am_s = 0.01/0.05$. We obtain the wave function renormalization factor Z_q from the conserved vector current and the mass renormalization factor Z_m from the scalar bilinear operator. We also present preliminary results of renormalization factors for other bilinear operators.

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1. Introduction

Recent calculation of B_K in Refs. [1, 2] shows that the dominant systematic error ($\approx 4.4\%$) comes from the matching factor obtained using the one-loop perturbation theory. Hence, it becomes essential to reduce this error as much as possible. One possibility is to calculate the matching factor using the two-loop perturbation theory, which will reduce the systematic error down to the $\approx 0.9\%$ level. Another possibility is to obtain the matching factor using the non-perturbative renormalization method (NPR) with the RI-MOM [3] and RI-SMOM scheme [4], which is very likely to reduce the systematic error down to the $\approx 2\%$ level. Here, we present preliminary results of renormalization factors of bilinear operators calculated using NPR in the RI-MOM scheme with improved staggered fermions.

2. Bilinear Operator Renormalization

A bilinear operator of staggered fermions is defined as

$$O_i^{f_1 f_2}(y) = \sum_{AB} \sum_{c_1 c_2} \overline{\chi}_{i;c_1}^{f_1}(y_A) \overline{(\gamma_S \otimes \xi_F)}_{AB} [U_{i;AB}]_{c_1 c_2}(y) \chi_{i;c_2}^{f_2}(y_B)\,, \tag{2.1}$$

where i is a gauge configuration index. c_i are color indices and f_i flavor indices. The y represents a coordinate of the hypercube with its lattice spacing $2a$. The indices A, B are hypercubic vectors such as $A = (0,1,1,0)$ for example. Here, we use the notation of $y_A = 2y + A$. $[U_{i;AB}]_{c_1 c_2}(y)$ is a gauge link, an average of the shortest paths which connect y_A and y_B as products of HYP-smeared fat links. γ_S represents the spin and ξ_F the taste. Here, $\chi(y_B)$ represents the staggered fermion field.

We define the Green's function as

$$G_{i;c_1 c_2}^{f_1 f_2}(x_1, x_2, y) = \langle \chi_{i;c_1}^{f_1}(x_1) O_i^{f_1 f_2}(y) \overline{\chi}_{i;c_2}^{f_2}(x_2) \rangle\,, \tag{2.2}$$

where x_1 and x_2 represents coordinates on the original lattice with its lattice spacing a.

$$x_1, x_2 \in \mathbb{Z}^4\,, \qquad y \in \mathbb{W}^4 \tag{2.3}$$

$\mathbb{Z}^4$ denotes coordinate space with its lattice spacing a, and $\mathbb{W}^4$ denotes hypercube coordinate space with its lattice spacing $2a$.

Now we define $\widetilde{p}$ and $\widetilde{q}$ as momenta defined in the reduced Brillouin zone. Then $p = \widetilde{p} + \pi_A$ and $q = \widetilde{q} + \pi_B$, where $\pi_A = \dfrac{\pi}{a} A$. Here, the domain of various momenta is defined as

$$p, q \in (-\frac{\pi}{a}, \frac{\pi}{a}]^4\,, \qquad \widetilde{p}, \widetilde{q} \in (-\frac{\pi}{2a}, \frac{\pi}{2a}]^4\,. \tag{2.4}$$

First, we apply the Fourier transformation to the Green's function G as follows.

$$F_{i;c_1 c_2}^{f_1 f_2}(\widetilde{p} + \pi_A, \widetilde{q} + \pi_B, y) \equiv a^8 \sum_{x_1, x_2 \in \mathbb{Z}^4} e^{i(\widetilde{p} + \pi_A) x_1} e^{-i(\widetilde{q} + \pi_B) x_2} G_{i;c_1 c_2}^{f_1 f_2}(x_1, x_2, y)\,. \tag{2.5}$$

Using the conjugate gradient algorithm, we calculate the Eq.(2.5) with momentum sources of $\widetilde{p}$ and $\widetilde{q}$ for each gauge configuration. Then, we apply the Fourier transformation to $F_{i;c_1 c_2}^{f_1 f_2}(\widetilde{p} + \pi_A, \widetilde{q} +$

$\pi_B, y)$ with respect to y. After that, we make an average over the gauge configurations such that there is no gluon field left uncontracted.

$$\begin{aligned} H^{f_1 f_2}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B, \widetilde{k}) &\equiv \frac{1}{N}\sum_{i=1}^{N}(2a)^4 \sum_{y\in\mathbb{W}^4} e^{-i\widetilde{k}y} F^{f_1 f_2}_{i;c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B, y) \\ &= \widetilde{\delta}^4(\widetilde{p}-\widetilde{q}-\widetilde{k})\widetilde{H}^{f_1 f_2}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B), \end{aligned} \tag{2.6}$$

where N is the number of the gauge configurations and $\widetilde{k}$ belongs to the reduced Brillouin zone.

We define

$$\widetilde{\delta}^4(\widetilde{p}) \equiv (2a)^4 \sum_{z\in\mathbb{W}^4} e^{i\widetilde{p}z}. \tag{2.7}$$

Since the momentum conservation law is well respected in the reduced Brillouin zone ($\widetilde{k} = \widetilde{p}-\widetilde{q}$), we can rewrite H as follows.

$$\begin{aligned} H^{f_1 f_2}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B, \widetilde{k}=\widetilde{p}-\widetilde{q}) &= \widetilde{\delta}^4(0)\widetilde{H}^{f_1 f_2}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B) \\ &= V\widetilde{H}^{f_1 f_2}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B), \end{aligned} \tag{2.8}$$

where $V = \widetilde{\delta}^4(0)$ is 4-dimensional volume factor. We call $\widetilde{H}$ the unamputated Green's function in this paper.

Using $\widetilde{H}$ and the inverse quark propagators, we can obtain the amputated Green's function as follows.

$$\begin{aligned} \widetilde{\Lambda}^{f_1 f_2}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B) = \sum_{\substack{C,D,\\E,F}} \sum_{c'_1 c'_2} [\widetilde{S}^{f_1}(\widetilde{p})]^{-1}_{AC;c_1 c'_1} \cdot \widetilde{H}^{f_1 f_2}_{c'_1 c'_2}(\widetilde{p}+\pi_C, \widetilde{q}+\pi_D) \\ \cdot \overline{\overline{(\gamma_5\otimes\xi_5)}}_{DF} [[\widetilde{S}^{f_2}(\widetilde{q})]^{-1}]^{\dagger}_{FE;c'_2 c_2} \overline{\overline{(\gamma_5\otimes\xi_5)}}_{EB}, \end{aligned} \tag{2.9}$$

where $\widetilde{S}(\widetilde{p})$ is the quark propagator in the momentum space [5]. Let us define the projection operator $\mathbb{P}$ as follows.

$$\mathbb{P}^{\beta}_{BA;c_2 c_1} \equiv \frac{1}{48}\overline{\overline{(\gamma_{S'}\otimes\xi_{F'})}}^{\dagger}_{BA}\delta_{c_2 c_1} \tag{2.10}$$

$$\Gamma^{\alpha\beta}(\widetilde{p},\widetilde{q}) \equiv \sum_{A,B}\sum_{c_1 c_2}[\widetilde{\Lambda}^{\alpha}_{c_1 c_2}(\widetilde{p}+\pi_A, \widetilde{q}+\pi_B)\mathbb{P}^{\beta}_{BA;c_2 c_1}], \tag{2.11}$$

where α and β represent bilinear operators with various spins and tastes. Here, we call Γ the projected amputated Green's function.

The renormalized Green's function is related to the bare one as follows.

$$\Gamma^{\alpha\sigma}_R(\widetilde{p},\widetilde{q}) = \sum_{\beta} Z_q^{-1} Z_O^{\alpha\beta}\Gamma^{\beta\sigma}_B(\widetilde{p},\widetilde{q}). \tag{2.12}$$

Here, the subscript R (B) denotes a renormalized (bare) quantity. Z_q is the wave function renormalization factor for the quark fields, and $Z_O^{\alpha\beta}$ is the renormalization factor matrix element which represents the mixing between the α and β operators.

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$n(x,y,z,t)$	$a\lvert\tilde{p}\rvert$	GeV	$n(x,y,z,t)$	$a\lvert\tilde{p}\rvert$	GeV	$n(x,y,z,t)$	$a\lvert\tilde{p}\rvert$	GeV
$(1,0,1,3)$	0.5330	0.8835	$(1,1,1,6)$	0.8019	1.3291	$(2,2,2,8)$	1.3421	2.2243
$(1,1,1,2)$	0.5785	0.9588	$(1,2,1,5)$	0.9128	1.5128	$(2,2,2,9)$	1.4018	2.3233
$(1,1,1,3)$	0.6187	1.0254	$(1,2,2,4)$	1.0210	1.6922	$(2,3,2,7)$	1.4663	2.4302
$(1,1,1,4)$	0.6710	1.1122	$(2,1,2,6)$	1.1114	1.8420	$(3,3,3,9)$	1.8562	3.0764
$(1,1,1,5)$	0.7328	1.2146	$(2,2,2,7)$	1.2871	2.1332			

Table 1: The list of momenta used for our analysis. The first column is the four vectors in the units of $(\frac{2\pi}{L_s},\frac{2\pi}{L_s},\frac{2\pi}{L_s},\frac{2\pi}{L_t})$, where L_s (L_t) is the number of sites in the spatial (temporal) direction.

The RI-MOM scheme prescription is

$$\Gamma_R^{\alpha\sigma}(\tilde{p},\tilde{p}) = \Gamma_{tree}^{\alpha\sigma}(\tilde{p},\tilde{p}) = \delta^{\alpha\sigma}\,, \tag{2.13}$$

where $\Gamma_{tree}^{\alpha\sigma}(\tilde{p},\tilde{p})$ is the projected amputated Green's function at the tree level. Therefore, the renormalization factor is obtained from the following equation.

$$Z_O^{\alpha\beta} = Z_q \cdot [\Gamma_B^{-1}(\tilde{p},\tilde{p})]^{\alpha\beta}\,. \tag{2.14}$$

3. Numerical Results

We use the $20^3 \times 64$ MILC asqtad lattice ($a \approx 0.12$ fm, $am_\ell/am_s = 0.01/0.05$). And we use HYP-smeared staggered fermions as valence quarks. We perform the measurement for 5 valence quark masses (0.01, 0.02, 0.03, 0.04, 0.05), and 14 external momenta which are given in Table 1. The number of gauge configurations is 30. We do the uncorrelated fitting and use the jackknife resampling method to estimate statistical errors.

3.1 Wave Function Renormalization Factor Z_q

Let us consider the conserved vector current ($V_\mu \otimes S$). The renormalization factor of the conserved currents is unity. Therefore, we can obtain the wave function renormalization factor Z_q of the staggered quark fields from the Eq.(2.14).

$$Z_q^{\text{RI-MOM}} = \Gamma_B^{\alpha\beta}(\tilde{p},\tilde{p})\,, \tag{3.1}$$

where $\alpha = \beta = (V_\mu \otimes S)$. Here, the superscript RI-MOM denotes that the wave function renormalization factor Z_q is defined in the RI-MOM scheme.

We convert the raw data in the RI-MOM scheme into the scale-invariant (SI) data by removing the scale-dependent part of the RG evolution matrix as follows.

$$Z_q^{\text{SI}} = \frac{c(\alpha_s(\mu_0))}{c(\alpha_s(\mu))} Z_q^{\text{RI-MOM}}(\mu), \qquad (\mu_0 = 2\text{GeV}, \quad \mu^2 = \tilde{p}^2) \tag{3.2}$$

This Wilson coefficient $c(x)$ is calculated using four-loop anomalous dimension given in Refs. [3, 6]. In this paper, we choose $\mu_0 = 2$GeV to compare results with those of other groups.

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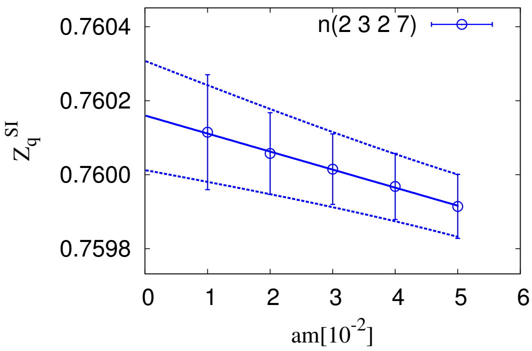
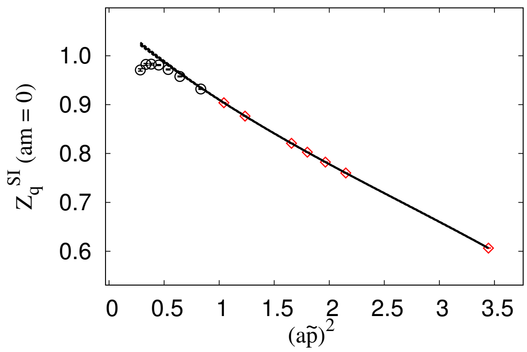
In general, the data of Z_q depends on the quark mass and the momentum. First, we fit the data with respect to quark mass for a fixed momentum to the linear function f_{Z_q} as follows.

$$f_{Z_q}(m, a, \tilde{p}) = a_1 + a_2 \cdot am\,, \tag{3.3}$$

where a_i is a function of $\tilde{p}$. We call this m-fit. We present the m-fit results in Fig. 1(a), and the uncorrelated fitting has $\chi^2/\mathrm{dof} = 0.0024(62)$.



(a) m-fit



(b) p-fit

Figure 1: Z_q: The red points are within the fitting range which satisfies $(a\tilde{p})^2 > 1$.

After the m-fit, we take the chiral limit values which corresponds to $a_1(a, \tilde{p})$ and fit them to the following functional form.

$$f_{Z_q}(am = 0, a\tilde{p}) = b_1 + b_2(a\tilde{p})^2 + b_3((a\tilde{p})^2)^2 + b_4((a\tilde{p})^2)^3 \tag{3.4}$$

We call this procedure p-fit. To avoid the non-perturbative effects at small momentum region, we choose the momentum window as $(a\tilde{p})^2 > 1$. We present the p-fit results in Fig. 1(b), and the uncorrelated fitting has $\chi^2/\mathrm{dof} = 0.06(16)$.

In Eq. 3.4, we assume that those terms of $\mathscr{O}((a\tilde{p})^2)$ and higher order are pure lattice artifacts. Hence, we take the b_1 as the wave function renormalization factor Z_q at $\mu = 2$ GeV in the RI-MOM scheme. We find out that $Z_q = b_1 = 1.0764(44)$, where the error is purely statistical.

3.2 Mass Renormalization Factor Z_m

By the Ward identity, the mass renormalization factor is

$$Z_m = \frac{1}{Z_{S\otimes S}}\,, \tag{3.5}$$

where $Z_{S\otimes S}$ is a renormalization factor of scalar bilinear operator with scalar taste. From the Eq.(2.14),

$$(Z_q \cdot Z_m)^{\text{RI-MOM}} = \left(\frac{Z_q}{Z_{S\otimes S}}\right)^{\text{RI-MOM}} = \Gamma_{S\otimes S}(\tilde{p}, \tilde{p})\,, \tag{3.6}$$

where $Z_{S\otimes S} \equiv Z_O^{\alpha\beta}$ with $\alpha = \beta = (S\otimes S)$, and $\Gamma_{S\otimes S} = \Gamma_B^{\alpha\beta}$ with $\alpha = \beta = (S\otimes S)$. To obtain the scale-invariant(SI) quantity, we divide $(Z_q \cdot Z_m)^{\text{RI-MOM}}$ by the RG running factors.

$$(Z_q \cdot Z_m)^{\text{SI}} = \frac{c(\alpha_s(\mu_0))}{c(\alpha_s(\mu))} \cdot \frac{d(\alpha_s(\mu_0))}{d(\alpha_s(\mu))} (Z_q \cdot Z_m)^{\text{RI-MOM}}(\mu), \quad (\mu_0 = 2\text{GeV}, \quad \mu^2 = \tilde{p}^2) \tag{3.7}$$

where $d(x)$ is the Wilson coefficient calculated using the quark mass anomalous dimension at the four-loop level [3, 6].

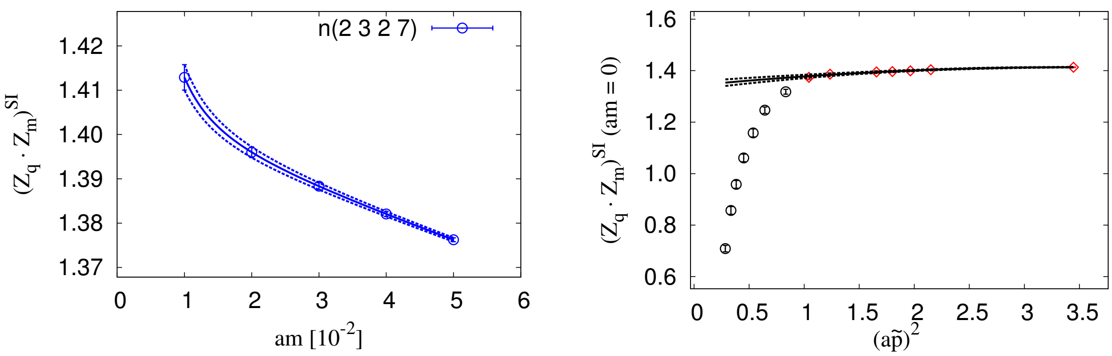
In the case of m-fit, we use the following fitting function:

$$f_{Z_q \cdot Z_m}(m, a, \tilde{p}) = c_1 + c_2(am) + c_3(am)^2 + c_4 \frac{1}{(am)^2} \tag{3.8}$$

where m is the valence quark mass. Here, note that the c_4 term comes from the chiral behavior of the chiral condensate which is proportional to $1/m^2$ due to zero modes [7]. The sea quark determinant contributes to the chiral condensate as follows,

$$\langle \bar{q} q \rangle \propto \frac{(am_\ell)^2 (am_s)^1}{(am_x)^2}, \tag{3.9}$$

where m_ℓ (m_s) is the light (strange) sea quark mass and m_x is the valence quark mass. Hence, as long as we take the chiral limit of m_ℓ and m_s at a fixed ratio of $m_\ell / m_x = 1$, then the c_4 term contribution vanishes safely. In Fig. 2(a), we show the m-fit results, and the uncorrelated fitting has $\chi^2/\text{dof} = 0.00008(51)$.



(a) m-fit (b) p-fit

Figure 2: $Z_q \cdot Z_m$

After the m-fit, we take the chiral limit values which correspond to c_1. We fit the data to the following fitting function with respect to $(a\tilde{p})^2$.

$$f_{Z_q \cdot Z_m}(am = 0, a\tilde{p}) = d_1 + d_2(a\tilde{p})^2 + d_3((a\tilde{p})^2)^2 \tag{3.10}$$

We call this procedure p-fit. In Fig. 2(b), we present the p-fit results and the fitting quality is $\chi^2/\text{dof} = 0.18(28)$. Our final result is $Z_m = 1.246(15)$, where the error is purely statistical.

α	$Z_O^{\alpha\alpha}$	(a)	(b)
$[S \times P]$	1.079(18)	0.00004(23)	0.19(48)
$[P \times A_\mu]$	0.8947(66)	0.00218(25)	0.032(74)
$[V_\mu \times V_\mu]$	0.982(11)	0.000003(17)	0.17(40)
$[A_\mu \times A_\nu]$	1.115(27)	0.0000006(33)	0.007(47)
$[T_{\mu\nu} \times T_{\rho\sigma}]$	1.293(16)	0.0000035(72)	0.008(42)

Table 2: $Z_O^{\alpha\alpha}$ for some bilinear operators. Here, $\mu \neq \nu \neq \rho \neq \sigma$. And (a) and (b) represent χ^2/dof for the m-fit and p-fit, respectively.

3.3 Renormalization Factors of Other Operators

We have done the first round data analysis for the complete set of bilinear operators. The renormalization factor of operators($Z_O^{\alpha\alpha}$) are calculated using Eq.(2.14) and we obtain Z_q using the conserved vector current. Part of the preliminary results are presented in Table 2.

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