Vector correlator and scale determination in lattice QCD

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We implement a proposal made in [9] to determine the lattice spacing by matching the lattice vector correlator at a reference distance scale with the same correlator obtained by a dispersion relation based on the *R*-ratio determined experimentally. We work with the isovector current, requiring a separation of the isovector hadronic final states on the phenomenological side. We also discuss the finite-size effect on the correlator, which must be controlled in order for the method to be applicable.

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1. Introduction

Determining the lattice spacing in physical units is a central task in almost any lattice QCD calculation. Pragmatically, the question is, which dimensionful but renormalized quantity can be determined with the highest overall accuracy on the lattice, statistical and systematic errors both being taken into account. *Relative scale setting* does not require the quantity to be known experimentally, and is sufficient to compare all other dimensionful quantities across different lattice spacings. Quite a few choices of this kind are available: the length scales r_0 [1], r_1 [2] are based on the static potential. The renormalized eigenvalue density of the Dirac operator ν_R/V [3] is also an attractive quantity for this purpose, and so are t_0 [4] and w_0 [5], which are derived from the Wilson flow.

For the purpose of *absolute scale setting* however, the quantity must be known experimentally to a high degree of precision, both in terms of statistical and systematic precision. Commonly used quantities are spectroscopic quantities such as the Omega mass [6], and the decay constants of the light pseudoscalar mesons, F_π (see [7] for a recent calculation of this quantity) and F_K [8]. Note also that at a time when most lattice calculations were done in the quenched approximation, r_0 was used for absolute scale setting. What quantities are considered to be known in a model-independent way on the phenomenological side thus depends to some extent on the targeted accuracy.

Here we explore a proposal to set the absolute scale using the vector correlator [9]. The point of view taken here is that for the time being, phenomenology is more accurate than lattice QCD in calculating this correlator in the Euclidean domain, and that it is therefore reasonable to set the scale in this way. One immediate advantage of this method is that a whole *function* can be compared between the lattice calculation and the phenomenological extraction from the reaction $e^+e^- \to$ hadrons and τ decays via a dispersion relation. The details of the scale-setting condition can thus be optimized for the needs of lattice calculations.

2. Scale setting via the isovector vector correlator

Let $j_\mu^{\mathrm{em}}(x)$ be the electromagnetic current of hadrons. Its Euclidean correlator,

$$G^{\mathrm{em}}(t) \equiv \int \mathrm{d}^3\boldsymbol{x} \, \langle j_z^{\mathrm{em}}(t,\boldsymbol{x}) j_z^{\mathrm{em}\,\dagger}(0) \rangle,$$

admits the spectral representation:

$$G^{\mathrm{em}}(t) = \int_0^\infty \mathrm{d}\omega \, \omega^2 \rho(\omega^2) e^{-\omega|t|},$$

$$\rho(s) = \frac{R(s)}{12\pi^2}, \qquad R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha(s)^2/(3s)}.$$

These equations provide the connection between lattice QCD observables and phenomenology. If all exclusive channels are measured on the experimental side, the isoscalar/isovector flavor separation can be done in a model independent way (assuming isospin symmetry). We therefore introduce $R_1(s)$, defined as $R(s)$, but where the final hadronic state is required to be isovector.

On the lattice, we need only compute the Wick-connected diagram for the isovector correlator $G(t)$, and will therefore focus on this case. Since we will use $N_{\mathrm{f}} = 2$ ensembles, the isovector

β	label	L/a	m_π[MeV]
5.2	A_4	32	380
	A_5	32	330
5.3	F_6	48	310
	F_7	48	270
	G_8	64	190
5.5	N_5	48	440
	N_6	48	340
	O_7	64	270

Table 1: List of the CLS ensembles used, with the linear spatial extent L and the pion masses. All ensembles have a time extent $T = 2L$ and are such that $m_\pi L > 4$.

vector current cannot create a hadronic state with hidden strangeness (e.g. $K\bar{K}$ pairs), and therefore final states containing kaons were removed from the spectral density on the phenomenological side. Needless to say, the absence of strange quarks in our simulations leads to a systematic uncertainty in the comparison of lattice and phenomenological results.

Among many possible choices [9], one convenient definition to set the scale based on the vector correlator is

$$f(\tau_1) \equiv 3.25, \qquad f(t) \equiv t \cdot m_{\text{eff}}(t) \equiv -\frac{t}{G(t)}\frac{dG}{dt}. \tag{2.1}$$

Any multiplicative renormalization factor on the vector current obviously cancels out in this quantity. Summing up the contribution of the exclusive $I = 1$ hadronic states in e^+e^- data up to about 2 GeV yields[1]

$$\tau_1 = 0.727(9)\,\text{fm}. \tag{2.2}$$

The region above 2 GeV makes an almost negligible contribution at Euclidean time τ_1 and can safely be treated using perturbation theory. The specific choice of 3.25 is motivated by an approximate minimization of the relative error on τ_1. The propagation of the statistical error on the effective mass onto τ_1 is given by

$$\frac{\delta\tau_1}{\tau_1} = \left(\frac{\tau_1}{f}\frac{df}{d\tau_1}\right)^{-1}\frac{\delta m_{\text{eff}}(\tau_1)}{m_{\text{eff}}(\tau_1)} \overset{f(\tau_1)=3.25}{\approx} 2.2\,\frac{\delta m_{\text{eff}}(\tau_1)}{m_{\text{eff}}(\tau_1)}, \tag{2.3}$$

where the estimate of the slope is taken from the phenomenological curve. Thus for physical pion masses, a one percent error on the effective mass translates into a 2.2 percent error on the scale determination. This loss of precision is due to the relatively flat behavior of the function $f(t)$. Clearly, $f(t)$ would be a constant in a scale-invariant theory, in particular it would be 3 for non-interacting massless quarks. This approximately t independent behavior 'accidentally' extends far beyond the region of validity of perturbation theory due to a shallow minimum around $t = 0.5$ fm. A preliminary study shows that this undesirable feature is largely absent in the $N_{\text{f}} = 2 + 1$ flavor theory. For large (but not asymptotic) t, $\frac{t}{f(t)}\frac{df}{dt} \approx 1$ since $f(t) \approx m_\rho t$, but then the relative error on the effective mass tends to be significantly larger.

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[1]A detailed description of this analysis will be published in a forthcoming paper.
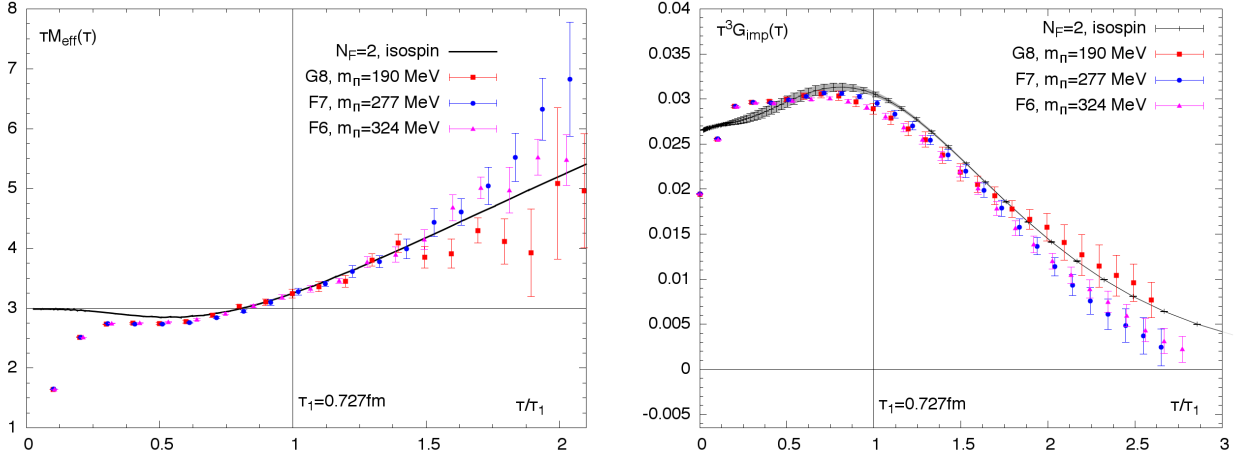
Figure 1: Left panel: scale determination for three ensembles at $\beta = 5.3$ with pion masses. Right panel: comparison of the correlator $t^3 G(t)$ computed on the lattice, and its phenomenological determination from $e^+ e^-$ data. The normalization is such that $G(t) = (4\pi^2 t^3)^{-1}$ for non-interacting quarks.
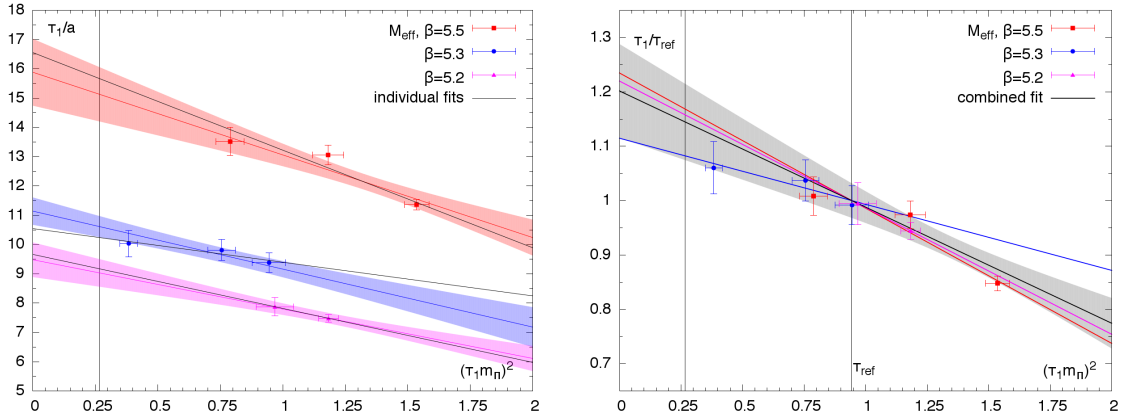

Figure 2: Left: chiral extrapolation of τ_1/a at three different values of β. Right: test of the cutoff effects on the quark mass dependence of τ_1.

We are applying the scale setting method for the first time. We use $N_{\rm f} = 2$ ensembles of $O(a)$ improved Wilson fermions generated as part of the CLS effort[2] (see for instance [10] for a more detailed description and references). We choose the local vector current at the source and the conserved vector current at the sink, as in [10].

Figure 1 shows the lattice data for the function $f(t)$, with t rescaled in units of τ_1. Results from three ensembles at the same lattice spacing and for different pion masses are shown. We observe a significant interval around $t = \tau_1$ where the phenomenological curve and the lattice data are statistically consistent, with little dependence on the pion mass observed. This is an encouraging feature of the method. It implies that the result for τ_1/a is insensitive to the precise choice (2.1) within the statistical uncertainty.

For the chiral extrapolation of τ_1/a viewed as a function of $x \equiv (\tau_1 m_\pi)^2$, we adopt the point

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[2]https://twiki.cern.ch/twiki/bin/view/CLS/WebIntro

β	a/fm from τ_1	a/fm from m_Ω [11]	a/fm from F_K [8]
5.5	0.048(3)	0.050(2)(2)	0.0486(4)(5)
5.3	0.0685(23)	0.063(2)(2)	0.0658(7)(7)
5.2	0.081(4)	0.079(3)(2)	0.0755(9)(7)

Table 2: Comparison of lattice scale determination between our (preliminary) results and results based on the Ω baryon mass [11] and the kaon decay constant F_K [8].

of view that we Taylor expand this quantity around an intermediate value of x. The analysis then proceeds similarly as for r_0 [8], with the exception that we extrapolate τ_1/a to the physical value of x rather than $x = 0$. Given the number of data points we have and their accuracy, we restrict ourselves to linear order in the Taylor expansion. In the left panel of Fig. 2, we illustrate the extrapolation assuming that the pion mass dependence has no cutoff effects. This assumption is tested in the right panel, where the different τ_1 values are interpolated to a common, reference value of $\tau_1 m_\pi$, yielding $\tau_{1,\mathrm{ref}}$. The dependence of $\tau_1/\tau_{1,\mathrm{ref}}$ on $\tau_1 m_\pi$ should then be universal. Within the rather large uncertainties, this is what is observed. We note that there is some evidence for a flattening of the m_π dependence at $\beta = 5.3$, where we have access to a pion mass below 200 MeV. Higher statistics are needed to establish this effect with confidence.

The results for the lattice spacings, where the absence of cutoff effects on the quark mass dependence of $a\tau_1$ has been assumed, are given in Table 2. Only a statistical error is quoted here for these preliminary results. Our results appearing in the first column are compared with those obtained previously via other reference quantities, the Omega baryon mass [11] and the kaon decay constant [8]. We note an overall agreement with the latter results. There is a slight tension at $\beta = 5.3$ between our value of the lattice spacing and the value obtained via the Omega mass. However, unlike the other scale determinations of Table 2, the present one included the G8 ensemble with a pion mass below 200 MeV. As can be seen from Fig. 2, without this ensemble, the lattice spacing would come out slightly smaller.

We can turn to the comparison of the correlation function $G(t)$ itself to its phenomenological determination from $e^+e^- \to$ hadrons data (right panel of Fig. 1). Apart from the expected large cutoff effects at short distance, we observe that the fall-off of the lattice correlators is faster for the larger pion masses, as one would expect. On the G8 ensemble at $m_\pi = 190$ MeV, the fall-off appears to be slower, but the statistical uncertainty becomes large beyond 1.2 fm.

3. Systematic effects

Cutoff effects: At a pion mass of approximately 270 MeV, we present a comparison of two vector correlators, one computed using the lattice vector current $\bar\psi\gamma_\mu\psi$ at both source and sink and the other one using instead the conserved vector current at the sink (i.e. the discretization we used for the scale determination). Clearly, we expect any difference to be reduced when we go to a smaller lattice spacing. A comparison is shown in Fig. 3. The improvement terms in the vector current have not been included ($b_V = c_V = 0$ in the notation of [12]). However the correlators are tree-level improved, i.e. they have been divided by the free lattice correlator (see appendix B of [13]) and multiplied by the free continuum correlator (tree-level improvement was also applied in the
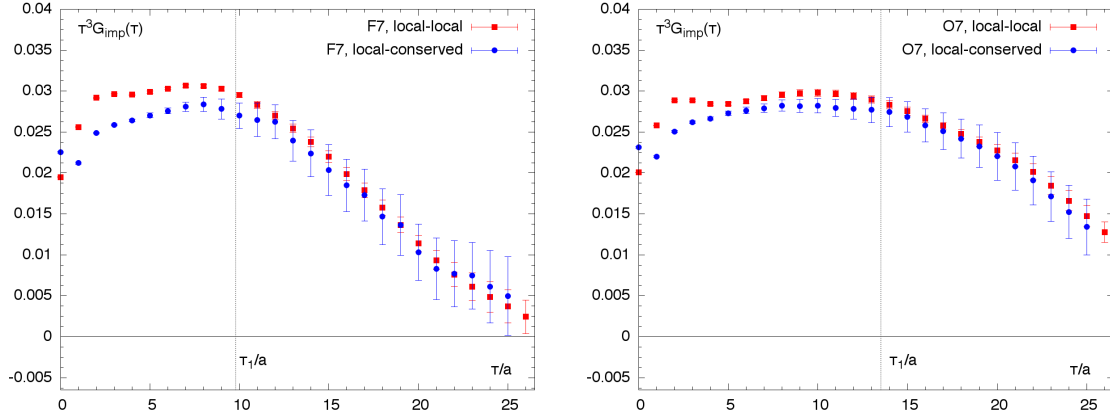
Figure 3: A comparison of two discretizations of the vector correlator at $m_\pi \approx 270\,\text{MeV}$. Left: $\beta = 5.3$, corresponding to $a \simeq 0.068\,\text{fm}$. Right: $\beta = 5.5$, correspondings to $a \simeq 0.048\,\text{fm}$.

right panel of Fig. 1). The cutoff effects are observed to be large for $t < 0.5\,\text{fm}$ but are reduced, as expected, when going from $\beta = 5.3$ to $\beta = 5.5$. It would be interesting to see whether the improvement term proportional to c_V would reduce the size of the cutoff effects.

Finite-size effects on the vector correlator are expected to be exponential at fixed time separation t [10], however the convergence to the infinite-volume limit is non-homogeneous in t. At large times, when the correlator is dominated by the few lowest-lying states, the dominant finite-size effects can be understood using Lüscher's finite-volume formalism [14, 15]. At the intermediate distance of about $0.7\,\text{fm}$, we expect to be in the regime where the finite-size effects are falling off exponentially in the box size. Obviously it would be desirable to check this explicitly.

4. Discussion

We have implemented a method to determine the energy scale in lattice QCD. The results are compatible with previous determinations on the same $N_\text{f} = 2$ ensembles. It is desirable at this point to improve the statistical accuracy, perhaps by using noise-reduction techniques. Also, O(a) improvement should be implemented at the level of the operator.

The method presented here straightforwardly generalizes to 2+1 and 2+1+1 flavors, where the main ambiguity of separating the different flavors disappears. Our scale determination method is relatively insensitive to the quark masses, see Fig. 2. The Omega mass, for instance, depends quite strongly on the strange quarks mass, therefore the uncertainty on the scale determined via this quantity must take into account the uncertainty in the strange quark mass setting.

For the isovector channel considered here, it is also attractive to use τ decay data on the phenomenology side [16]. We intend to investigate this possibility in a longer forthcoming publication.

While it is difficult, at present, for lattice calculations to compete with the high precision of $e^+ e^-$ data for the determination of $(g-2)_\mu$, flavor combinations in the vector and axial-vector channel that are not directly accessible experimentally should eventually be calculated on the lattice. In particular, this could have an impact on the precision determination of the running of the weak mixing angle.

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