PROCEEDINGS OF SCIENCE

Poincaré symmetries and the Yang-Mills gradient flow

A. Patella*
School of Computing and Mathematics & Centre for Mathematical Science, Plymouth University, Plymouth PL4 8AA, UK
PH-TH, CERN, CH-1211 Geneva 23, Switzerland
E-mail: agostino.patella@cern.ch

L. Del Debbio
Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK
E-mail: luigi.del.debbio@ed.ac.uk

A. Rago
School of Computing and Mathematics & Centre for Mathematical Science, Plymouth University, Plymouth PL4 8AA, UK
E-mail: antonio.rago@plymouth.ac.uk

The latest developments have shown how to use the gradient flow (or Wilson flow, on the lattice) for the exploration of symmetries, and the definition of the corresponding renormalized Noether currents. In particular infinitesimal translations can be introduced along the gradient flow for gauge theories, and the corresponding Ward identities can be derived. When applied to lattice gauge theories, this approach leads to a possible strategy to renormalize the energy-momentum tensor nonperturbatively, and to study dilatations and scale invariance.

31st International Symposium on Lattice Field Theory - LATTICE 2013
July 29 - August 3, 2013
Mainz, Germany

*Speaker.

PoS(LATTICE 2013)324

1. Introduction

The lattice regulator provides a unique framework to investigate non-perturbative properties of non-Abelian gauge theories. However this formulation explicitly breaks Poincaré symmetries at finite lattice spacing, and their restoration can be recovered only in the continuum limit.

As a consequence the naive discretization of the energy-momentum tensor on the lattice requires renormalization [1]. The correct (or renormalized) energy-momentum tensor on the lattice is obtained as a linear combination of all operators with dimension less or equal than four, whose mixing with the naively-discretized operator is allowed by lattice symmetries:

$$(\hat{T}_{\mu\rho})_R = \sum_i c_i \left\{ \hat{T}^{(i)}_{\mu\rho} - \langle \hat{T}^{(i)}_{\mu\rho} \rangle \right\} \ , \tag{1.1}$$

In this proceedings we will be interested only in the pure gauge theory. The list of the $\hat{T}^{(i)}$ operators is reported in appendix A. The assumption that Poincaré symmetries are restored in the continuum limit implies that it is possible to tune the coefficients c_i in such a way that the Poincaré Ward identities are satisfied up to terms that vanish in the continuum limit. In particular the translation Ward identity (TWI) has to be recovered:

$$\langle \phi_1(x_1)_R \cdots \phi_k(x_k)_R \int_{\partial V} dS_\mu(x) (\hat{T}_{\mu\rho})_R(x) \rangle = - \sum_{x_i \in V} \frac{\partial}{\partial x_{i\rho}} \langle \phi_1(x_1)_R \cdots \phi_k(x_k)_R \rangle + O(a^2) \ , \tag{1.2}$$

where $(\phi_i)_R$ are renormalized (gauge-invariant) local operators, the integral $\int_{\partial V} dS_\mu(x)$ is intended to be replaced by some lattice discretization, and all operators appear in points separated from each other by a non-zero distance in physical units. Also the boundary of the volume V needs to be kept away from all points x_i. If any two operators coalesce, extra divergences appear in the Ward identity due to contact terms that have to be subtracted in order to obtain a finite continuum limit. This is of course a technical complication that one wants to avoid as much as possible.

The goal of this proceedings is to discuss the possibility of using the Yang-Mills gradient flow (or Wilson flow, on the lattice) [2 – 4] in order to avoid contact terms in the TWI, allowing to write a local (i.e. not integrated) TWI which might be useful to determine the coefficients c_i numerically. We will also briefly discuss the dilatation Ward identity (DWI) and its relationship with the gradient flow. For the derivation of the equations presented in this proceeding we refer the reader to [5].

Alternative strategies have been recently proposed to renormalize the energy-momentum tensor on the lattice [6 – 10], based on the idea that a discrete subgroup of translations is preserved on the lattice, and acts on the partition function by shifting the boundary conditions for all fields.

2. Gradient flow

The Yang-Mills gradient flow is defined by the set of equations:

$$\begin{cases} \partial_t B_{t,\mu}(x) = D_{t,\nu} G_{t,\nu\mu}(x) \\ B_{0,\mu}(x) = A_\mu(x) \end{cases} \ , \tag{2.1}$$

where $G_{t,\nu\mu}$ and $D_{t,\nu}$ are respectively field strength and covariant derivative constructed with the field $B_{t,\mu}$, and t is an additional parameter with the dimensions of a $[length]^2$ which we will refer to as flow time.

A lattice-discretized version of the flow eq. (2.1) can be written as well [2]. Given a gauge configuration $U(x,\mu)$ as initial condition, the discretized flow equation produces a new gauge configuration $V_t(x,\mu)$ for each value of the flow time t. The new configuration $V_t(x,\mu)$ can be used to construct a new class of gauge-invariant operators. The remarkable feature of the gradient flow is that gauge-invariant operators that are local functions of the field $V_t(x,\mu)$ at some fixed and positive flow time t are finite in the continuum limit, i.e. do not require renormalization. It is worth to stress that the parameter t has to be kept fixed in physical units as the continuum limit is approached.

3. Infinitesimal translations of probes at positive flow time

Local translations are generated by the differential operator $\delta_{x,\rho}$ acting on the fundamental fields like:

$$\delta_{x,\rho} A_\mu(y) = F_{\rho\mu}(x)\delta(x-y) \ . \tag{3.1}$$

Strictly speaking this operator is the sum of a canonical infinitesimal translation and a field-dependent infinitesimal gauge transformation, and therefore it reduces to a canonical infinitesimal translation when applied to any gauge-invariant operator.

In terms of the differential operator $\delta_{x,\rho}$, the TWI can be written in the following schematic form (in a regularization that preserves Poincaré symmetry, e.g. in dimensional regularization):

$$\langle \partial_\mu T_{\mu\rho}(x) \ [\ \cdots\ ] \rangle = -\langle \delta_{x,\rho} [\ \cdots\ ] \rangle \ . \tag{3.2}$$

A possible naive discretization of the differential operator $\delta_{x,\rho}$ is given by:

$$\hat{\delta}_{x,\rho} U_\mu(y) = a^{-3} \hat{F}_{\rho\mu}(x) U_\mu(x) \delta_{x,y} \ , \tag{3.3}$$

where $\hat{F}_{\mu\nu}$ is some discretization of the field strength (for example one can use the properly-normalized clover plaquette). When applied to any renormalized local operator $\phi(x)_R$ with dimension d_ϕ, the differential operator $\hat{\delta}_{x,\rho}$ generates extra contact terms due to mixing of $\hat{\delta}_{x,\rho}\phi(x)_R$ with operators with dimension less or equal than $d_\phi + 5$, which need to be properly renormalized before taking the continuum limit.

The analysis of the divergences simplifies drastically if we consider probe operators defined at positive flow time. It can be shown [5] that, if $\phi_t(x)$ is a local function of the gauge field at positive flow time t, then the operator $\hat{\delta}_{x,\rho}\phi_t(y)$ renormalizes multiplicatively:

$$[\hat{\delta}_{x,\rho}\phi_t(y)]_R = Z_\delta \hat{\delta}_{x,\rho}\phi_t(y) \ , \tag{3.4}$$

and the (finite) renormalization constant Z_δ does not depend on the probe $\phi_t(y)$. This observation allows us to write a discretized version of the local TWI (in the case of one probe):

$$\langle \partial_\mu [\hat{T}_{\mu\rho}(x)]_R \ \phi_t(0) \rangle = -Z_\delta \langle \hat{\delta}_{x,\rho}\phi_t(0) \rangle + O(a^2) \ , \tag{3.5}$$

that is finite in the continuum limit even for coinciding points $x=0$. Notice that $\phi_t(0)$ is not a local operator in terms of the fields at $t=0$, therefore $[\hat{\delta}_{x,\rho}\phi_t(0)]_R$ is not zero when $x \neq 0$. As the gradient flow is essentially a smoothing procedure with a Gaussian decaying kernel, $[\hat{\delta}_{x,\rho}\phi_t(0)]_R$ vanishes for large x as $e^{-x^2/(4t)}$. Clearly one can use equation (3.5) for different probes, different space-time points x, and different flow times t to determine the coefficients c_i appearing in the definition of the renormalized energy-momentum tensor up to an overall normalization.

This multiplicative normalization can be fixed by requiring that the operator $Z_\delta \hat{\delta}_{x,\rho}$ generates an infinitesimal translation with the correct normalization:

$$Z_\delta \int d^4x \; \hat{\delta}_{x,\rho}\phi_t(y) = \partial_\rho \phi_t(y) + O(a^2) \; . \tag{3.6}$$

This strategy would require the calculation of 3-point functions (two probes and an energy-momentum tensor). Perhaps a better way to approach this problem is to look at the DWI.

4. Infinitesimal dilatations of probes at positive flow time

Some trivial algebraic manipulation of the local TWI (3.5) yields the local DWI:

$$\langle \partial_\mu [x_\rho \hat{T}_{\mu\rho}(x)]_R \; \phi_t(0)\rangle_c = -Z_\delta x_\rho \langle \hat{\delta}_{x,\rho}\phi_t(0)\rangle + \langle \phi_t(0) \; [\hat{T}_{\mu\mu}(x)]_R\rangle_c + O(a^2) \; . \tag{4.1}$$

As the pure gauge theory has a mass gap, all terms in the previous equation are integrable over space-time:

$$Z_\delta \langle \int d^4x \; x_\rho \hat{\delta}_{x,\rho}\phi_t(0)\rangle = \langle \phi_t(0) \int d^4x \; [\hat{T}_{\mu\mu}(x)]_R\rangle_c + O(a^2) \; , \tag{4.2}$$

where no contact terms arise in the r.h.s. The operator $Z_\delta \int d^4x \; x_\rho \hat{\delta}_{x,\rho}$ generates an infinitesimal dilatation when acting on gauge-invariant operators. The flow equation is invariant under dilatations, provided that the fields and the flow time are rescaled with their respective engineering dimensions along with the space-time coordinates, which implies (for details, see [5]):

$$Z_\delta \int d^4x \; x_\rho \hat{\delta}_{x,\rho}\phi_t(y) = \left\{ 2t\frac{d}{dt} + y_\rho \frac{\partial}{\partial y_\rho} + d_\phi \right\} \phi_t(y) + O(a^2) \; . \tag{4.3}$$

Putting everything together one obtains the anomalous DWI for a single probe at positive flow time:

$$\left(2t\frac{d}{dt} + d_\phi\right) \langle \phi_t(0)\rangle = Z_\delta \int d^4x \; x_\rho \langle \hat{\delta}_{x,\rho}\phi_t(0)\rangle = \langle \phi_t(0) \int d^Dx \; [\hat{T}_{\mu\mu}(x)]_R\rangle_c + O(a^2) \; . \tag{4.4}$$

This equation can be used to determine Z_δ or, equivalently, the overall normalization of the coefficients c_i. It is very interesting to notice that the derivative with respect to the time flow is essentially equivalent to the insertion of the trace of the renormalized energy-momentum tensor.

5. Conclusions

In this proceedings we have outlined a possible strategy to renormalize the energy-momentum tensor on the lattice, based on the use of Ward identities for probe observables defined along the gradient flow. Contact terms that plague local Ward identities for translations and dilatations disappear when probes at positive flow time are considered. Beside the standard renormalization of the energy-momentum tensor, only an extra finite multiplicative renormalization needs to be determined.

The main equations that we have analysed are:

$$\langle \partial_\mu [\hat{T}_{\mu\rho}(x)]_R \ \phi_t(0) \rangle = -Z_\delta \langle \hat{\delta}_{x,\rho} \phi_t(0) \rangle + O(a^2) \ , \tag{5.1}$$

$$\left(2t \frac{d}{dt} + d_\phi \right) \langle \phi_t(0) \rangle = Z_\delta \int d^4x \, x_\rho \langle \hat{\delta}_{x,\rho} \phi_t(0) \rangle = \langle \phi_t(0) \int d^D x \, [\hat{T}_{\mu\mu}(x)]_R \rangle_c + O(a^2) \ , \tag{5.2}$$

which in principle can be used to determine the three renormalization constants of the energy-momentum tensor, and the multiplicative renormalization constant Z_δ. Numerical simulations are needed to verify that this is a viable method in practice; they are deferred to future investigations.

We also point out that the extension of this work to theories with fermions does not present any additional challenge.

A. Mixing operators

In the case of QCD the operators $\hat{T}^{(i)}_{\mu\rho}$ have been classified and listed in [1]. For the pure gauge theory three operators are required:

$$\hat{T}^{(1)}_{\mu\rho} = -\frac{2}{g_0^2}\left\{ \mathrm{tr}\,\hat{F}_{\sigma\mu}\hat{F}_{\sigma\rho} - \frac{\delta_{\mu\rho}}{4}\sum_{\sigma\tau}\mathrm{tr}\,\hat{F}_{\sigma\tau}\hat{F}_{\sigma\tau} \right\} , \tag{A.1}$$

$$\hat{T}^{(2)}_{\mu\rho} = \delta_{\mu\rho}\sum_{\sigma\tau}\mathrm{tr}\,\hat{F}_{\sigma\tau}\hat{F}_{\sigma\tau} , \tag{A.2}$$

$$\hat{T}^{(3)}_{\mu\rho} = \delta_{\mu\rho}\sum_{\sigma}\mathrm{tr}\,\hat{F}_{\mu\sigma}\hat{F}_{\mu\sigma} , \tag{A.3}$$

where $\hat{F}_{\mu\nu}$ is some discretization of the field strength (for example one can use the properly-normalized clover plaquette).

The energy-momentum tensor on the lattice mixes with the identity as well. However the identity drops trivially out of the Poincaré Ward identities, reflecting the fact that the energy-momentum tensor is only defined up to a shift $(\hat{T}_{\mu\rho})_R \to (\hat{T}_{\mu\rho})_R + \alpha\delta_{\mu\rho}$. The subtraction in eq. (1.1) corresponds to the choice $\langle(\hat{T}_{\mu\rho})_R\rangle = 0$ (in decompactified flat space-time).

References

[1] S. Caracciolo, G. Curci, P. Menotti, and A. Pelissetto, *The energy momentum tensor for lattice gauge theories*, *Annals Phys.* **197** (1990) 119.

[2] M. Luscher, *Properties and uses of the Wilson flow in lattice QCD*, *JHEP* **1008** (2010) 071, [arXiv:1006.4518].

[3] M. Luscher and P. Weisz, *Perturbative analysis of the gradient flow in non-abelian gauge theories*, *JHEP* **1102** (2011) 051, [arXiv:1101.0963].

[4] M. Luscher, *Chiral symmetry and the Yang–Mills gradient flow*, *JHEP* **1304** (2013) 123, [arXiv:1302.5246].

[5] L. Del Debbio, A. Patella, and A. Rago, *Space-time symmetries and the Yang-Mills gradient flow*, arXiv:1306.1173.

[6] L. Giusti and H. B. Meyer, *Thermal momentum distribution from path integrals with shifted boundary conditions*, *Phys.Rev.Lett.* **106** (2011) 131601, [arXiv:1011.2727].

[7] L. Giusti and H. B. Meyer, *Implications of Poincare symmetry for thermal field theories in finite-volume*, *JHEP* **1301** (2013) 140, [arXiv:1211.6669].

[8] L. Giusti and M. Pepe, *Measuring the entropy from shifted boundary conditions*, arXiv:1311.1012.

[9] D. Robaina and H. B. Meyer, *Renormalization of the momentum density on the lattice using shifted boundary conditions*, arXiv:1310.6075.

[10] L. Giusti and H. B. Meyer, *Thermal field theories and shifted boundary conditions*, arXiv:1310.7818.