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Schrödinger functional boundary conditions and improvement of the $\mathrm{SU}(N)$ pure gauge action for $N > 3$

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The leading method to study the running coupling constant of non-abelian gauge theories is based on the Schrödinger functional scheme. However, the boundary conditions and $\mathscr{O}(a)$ improvement have not been systematically generalized for theories with more than three colors. These theories have applications in BSM model building as well as in the large N limit. We have studied the boundary conditions and improvement for the pure Yang-Mills theory within the Schrödinger functional scheme. We have determined for all values of N the boundary fields which provide high signal/noise ratio. Additionally, we have calculated the improvement coefficient c_t for the pure gauge to one loop order for $\mathrm{SU}(N)$ gauge theories with $N = 2,\ldots,8$ from which $N \geq 4$ are previously unknown.

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1. Introduction

Recently there has been an interest in the scaling properties of the gauge coupling in SU(N) theories with more than three colors [1, 2, 3]. For a review of large N gauge theories see [4]. The studies are motivated by their applications in Beyond Standard Model (BSM) physics and by the need to understand the scaling of the coupling constant in the large N limit.

The main tool for measuring the evolution of the coupling constant as a function of the scale on the lattice is the Schrödinger functional method. As is well-known this method suffers from $\mathcal{O}(a)$ lattice artifacts that originate from the boundary terms. They can be removed by adding improvement terms to the action and tuning appropriately the improvement coefficients [5]. Until now the boundary coefficient necessary to improve the pure gauge theory was only known for $N = 2$ and 3 [5, 6].

In these proceedings, we present the preliminary results of calculating the improvement coefficient c_t to one loop order in perturbation theory for the pure gauge theory with $N = 2, \ldots, 8$. After implementing $\mathcal{O}(a)$ improvement we see that the discretization effects are reduced for all considered values of N. In addition, we have done a study with a general N to find Schrödinger functional boundary fields with high signal/noise ratio that could be used in lattice simulations.

2. Theory

We use the standard $\mathcal{O}(a)$ improved SU(N) Wilson gauge action in the Schrödinger functional scheme

$$S = S_G + \delta S_{G,b}, \tag{2.1}$$

where

$$S_G = \frac{1}{g_0^2} \sum_p \mathrm{Tr}[1 - U(p)], \tag{2.2}$$

$$\delta S_{G,b} = \frac{1}{g_0^2}(c_t - 1) \sum_{p_t} \mathrm{Tr}[1 - U(p_t)]. \tag{2.3}$$

$U(p_t)$ refer to the timelike plaquettes on the $T = 0$ and $T = L$ boundaries. Additionally the perturbative expansion of the improvement coefficient c_t is

$$c_t = 1 + c_t^{(1,0)} g_0^2 + \mathcal{O}(g_0^4). \tag{2.4}$$

The gauge fixing procedure leads to the addition of gauge fixing S_{gf} and Faddeev-Popov ghost S_{FP} terms to the action.

In the Schrödinger functional (SF) scheme the boundary conditions in the temporal boundaries are taken to be abelian and spatially constant [5], given by

$$U_k(t = 0, \vec{x}) = \exp[aC_k], \quad U_k(t = L, \vec{x}) = \exp[aC_k'],$$

with

$$C_k = \frac{\mathrm{i}}{L}\mathrm{diag}(\phi_1(\eta), \ldots, \phi_n(\eta)), \quad C_k' = \frac{\mathrm{i}}{L}\mathrm{diag}(\phi_1'(\eta), \ldots, \phi_n'(\eta)).$$

The phases ϕ_i and ϕ_i' depend on an internal parameter η. This choice of boundary conditions induces a background field which is a unique minimum of the action provided that the phases ϕ and ϕ' lay within the fundamental domain [5]. See Section 3 for more details. Boundary conditions in the spatial directions are taken to be periodic.

The effective action has a perturbative expansion of the form[1]

$$\Gamma = -\ln\left\{\int D[\psi]D[\bar{\psi}]D[U]D[c]D[\bar{c}]e^{-S}\right\} = g_0^{-2}\Gamma_0 + \Gamma_1 + \mathcal{O}(g_0^2). \tag{2.5}$$

A renormalized coupling can be defined in the Schrödinger functional scheme as a derivative of the effective action (2.5) respect to the parameter η,

$$g^2 = \frac{\partial\Gamma_0}{\partial\eta}\Big/\frac{\partial\Gamma}{\partial\eta} = g_0^2 - g_0^4\frac{\partial\Gamma_1}{\partial\eta}\Big/\frac{\partial\Gamma_0}{\partial\eta} + \mathcal{O}(g_0^6). \tag{2.6}$$

In lattice studies it is common to use the lattice step scaling function $\Sigma(u,s,L/a)$, which describes the evolution of the renormalized coupling constant under a change of scale by a factor s:

$$\begin{aligned}\Sigma(u,s,L/a) &= g^2(g_0,sL/a)|_{g^2(g_0,L/a)=u}\\ &= u + \Sigma_{1,0}(s,L/a)u^2 + \mathcal{O}(u^3).\end{aligned} \tag{2.7}$$

We also use the function

$$\delta_0(L/a) = \frac{\Sigma_{1,0}(2,L/a)}{\sigma_{1,0}(2)}, \tag{2.8}$$

to measure the convergence of the lattice step scaling function $\Sigma_{1,0}(2,L/a)$ to its continuum limit $\sigma_{1,0}(2) = 2b_{0,0}\ln 2$. In the previous equation $b_{0,0} = 11N_c/(48\pi^2)$ is the one loop coefficient of the pure gauge β-function.

3. Boundary fields

Boundary fields ϕ and ϕ' are within the fundamental domain if they satisfy the equations [5]

$$\phi_1 < \phi_2 < \ldots < \phi_n, \quad |\phi_i - \phi_j| < 2\pi, \quad \sum_{i=1}^{N}\phi_i = 0. \tag{3.1}$$

Such vectors ϕ form a $N-1$ simplex with vertices

$$\begin{aligned}\mathbf{X}_1 &= \tfrac{2\pi}{N}(-N+1,1,1,\ldots,1)\\ \mathbf{X}_2 &= \tfrac{2\pi}{N}(-N+2,-N+2,2,\ldots,2)\\ \mathbf{X}_3 &= \tfrac{2\pi}{N}(-N+3,-N+3,-N+3,3,\ldots,3,)\\ &\vdots\\ \mathbf{X}_{N-1} &= \tfrac{2\pi}{N}(-1,-1,\ldots,-1,N-1)\\ \mathbf{X}_N &= (0,0,\ldots,0).\end{aligned} \tag{3.2}$$

For SU(4) the fundamental domain is shown in figure 1.

[1] We refer to the original literature [5] for details on the perturbative expansion of the effective action.

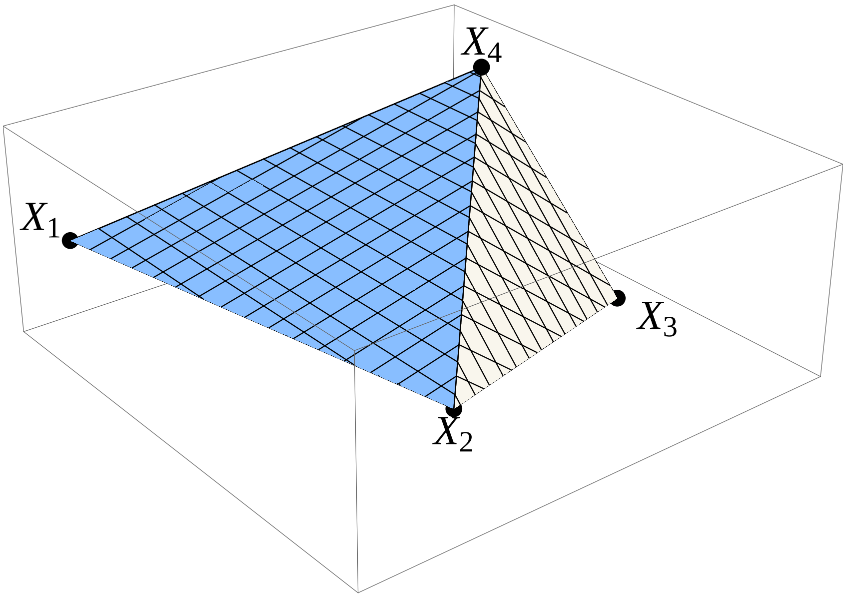

Figure 1: Fundamental domain of SU(4)

We start from the conjecture that the signal to noise ratio is maximized if ϕ and ϕ' are chosen, s.t. they are as far from the edges of the fundamental domain and each other as possible [1, 5, 6]. To find such points we first want to define a mapping which mirrors the point in the fundamental domain. We start by defining a mapping $R_{i,j}(\phi)$ that reflects the points in the fundamental domain with respect to a $(N-2)$ dimensional hyperplane. The hyperplane $R_{i,j}(\phi)$ goes through vertices $\mathbf{X}_k$, $k \neq i, j$ and intersects the line connecting $\mathbf{X}_i$ and $\mathbf{X}_j$ at the middle. In figure 2 we show all the possible mappings $R_{i,j}(\phi)$ on SU(4).

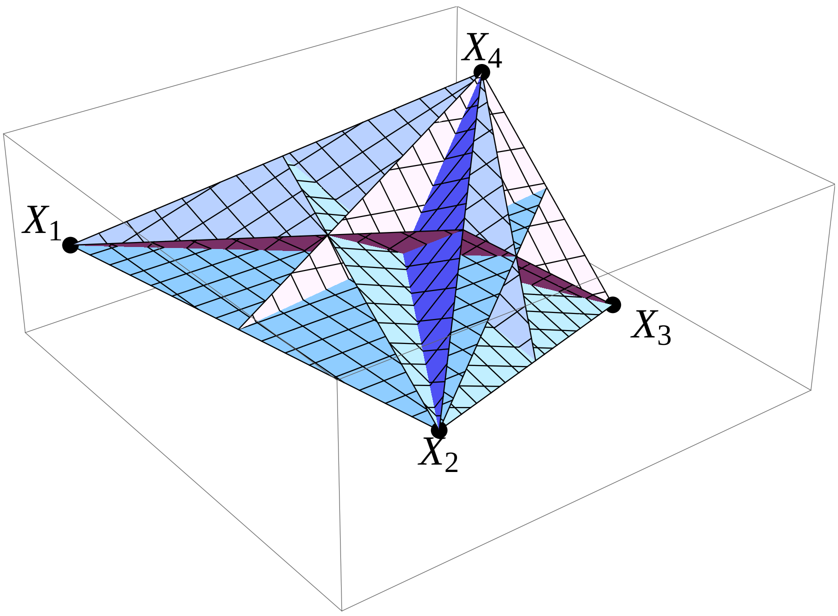

Figure 2: All possible $R_{i,j}(\phi)$ hyperplanes on the fundamental domain of SU(4)

The function $R_{i,j}(\phi)$ is not a mapping from the fundamental domain to itself, but we can define a composite mapping[2]

$$M(\phi) = \left(R_{1,N-1} \circ R_{2,N-2} \circ \ldots \circ R_{[N/2],N-[N/2]}\right)(\phi), \tag{3.3}$$

that has this property. The value of the field ϕ' is then derived from ϕ using $M(\phi)$ which has a simple form $\phi'_i = \phi_{N-i+1}$.

We choose ϕ to be in the middle of a line connecting $\mathbf{X}_1$ and the centeroid of the fundamental domain and associate a flow[3]

$$\begin{aligned} t(\eta) &= \frac{\eta N}{2\pi(N-2)}(\mathbf{X}_1 - \mathbf{X}_{N-1}) \\ &= \left(-\eta, \frac{2\eta}{N-2}, \ldots, \frac{2\eta}{N-2}, -\eta\right), \end{aligned} \tag{3.4}$$

to the direction which gets mirrored by $R_{1,N-1}(\phi)$ transformation and points outside from the fundamental domain. As an example boundary fields of $\mathrm{SU}(4)$ are

$$\phi = \begin{cases} -\eta - 9\pi/8, \\ \eta + \pi/8, \\ \eta + 3\pi/8, \\ -\eta + 5\pi/8, \end{cases} \quad \phi' = \begin{cases} \eta - 5\pi/8, \\ -\eta - 3\pi/8, \\ -\eta - \pi/8, \\ \eta + 9\pi/8. \end{cases} \tag{3.5}$$

Note that these boundary fields are different than those used in [1]. The possible improvement in the signal/noise ratio should be determined with lattice simulations.

4. Boundary improvement

Improvement coefficient c_t is previously know for $N = 2,3$ to one loop order in perturbation theory. These values have been calculated by Lüscher et. al. in [5] for $\mathrm{SU}(2)$ and in [6] for $\mathrm{SU}(3)$ resulting in $c_t^{(1,0)}(\mathrm{SU}(2)) = -0.0543(5)$ and $c_t^{(1,0)}(\mathrm{SU}(3)) = -0.08900(5)$. The method used in [5] and [6] is also applicable to $N > 3$ with some modifications.

The details of the calculation of $c_t^{(1,0)}$ will be given in [10]. The calculation goes along the lines of [5]. The idea of the process is to calculate $p_{1,0}(L/a) = \frac{\partial \Gamma_1(L/a)}{\partial \eta} / \frac{\partial \Gamma_0(L/a)}{\partial \eta}$ from (2.6) as a function of the lattice size L/a. This is done by solving a second order difference relation to several different operators. In this way we are able to solve $p_{1,0}(L/a)$ and consequently the running coupling g^2 to one loop order in perturbation theory for a range in $L/a \in \{6, 8, 10, \ldots, 64\}$.

The variable $p_{1,0}(L/a)$ has an asymptotic expansion in L/a [5]

$$p_{1,0}(L/a) \sim \sum_{n=0}^{\infty} (r_n + s_n \ln(L/a)) \left(\frac{a}{L}\right)^n, \tag{4.1}$$

where $s_0 = 2b_{0,0}$ and $s_1 = 0$. The coefficient $c_t^{(1,0)}$ is determined by demanding linear cutoff effects to be absent in (4.1), i.e. $r_1 = 2c_t^{(1,0)}$. The problem is then to extract the coefficient r_1 as accurately as possible from the $p_{1,0}(L/a)$ data. To do this we used the "Blocking" method described in [7]. Our preliminary results are shown in table 1.

[2] $R_{i,i}(\phi)$ is the identity mapping and $[x]$ means the integer part of x

[3] The normalization $\frac{\eta N}{2\pi(N-2)}$ is chosen such that the coefficients of η the standard case of $\mathrm{SU}(3)$.

N	$c_t^{(1,0)}$	$\delta c_t^{(1,0)}$
2	-0.0543	0.0002
3	-0.088	0.005
4	-0.1220	0.0002
5	-0.154	0.004
6	-0.1859	0.0008
7	-0.218	0.004
8	-0.249	0.004

Table 1: The values of $c_t^{(1,0)}$ and estimated errors $\delta c_t^{(1,0)}$ for $N = 2, \ldots, 8$.

We expect $c_t^{(1,0)} = AC_2(R) + BC_2(G) = \tilde{A}N + \tilde{B}/N$. This is motivated by the fact that the Feynman diagrams involved are proportional to these Casimir invariants. Also it has been shown in [8] that the fermionic part of $c_t^{(1)}$ is proportional to the Casimir invariant $T(R)$. See also [9]. A plot of the values of $c_t^{(1,0)}$ as a function of N and our fit to the data is shown in 3.

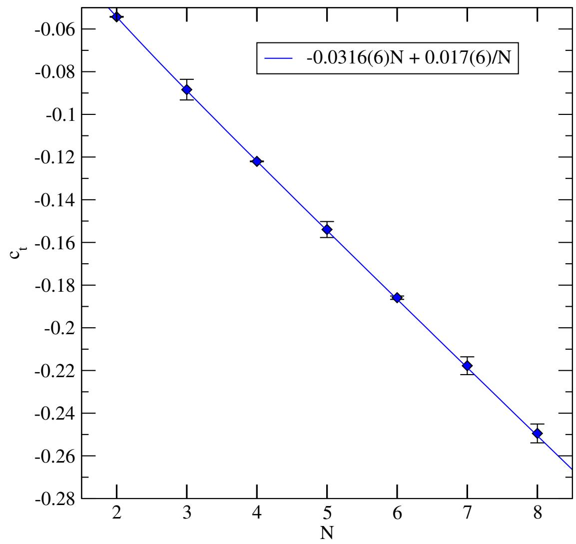

Figure 3: $c_t^{(1,0)}$ as a function of $\tilde{A}N + \tilde{B}/N$ fit to the data

We also want to be sure that setting c_t to the value that we obtained removes the $\mathcal{O}(a)$ terms from the lattice step scaling function. This can be seen from figure 4 where we have plotted δ_0 as a function of $(a/L)^2$. After the improvement δ_0 behaves linearly which is a clear indication that the leading terms are of the order $\mathcal{O}(a^2)$.

5. Summary and outlook

We have investigated the boundary conditions in general N and calculated the $\mathcal{O}(a)$ boundary improvement coefficients for $N = 2, \ldots, 8$ in the Schrödinger functional scheme. The precision in the determination of $c_t^{(1,0)}$ can be increased by using more than double precision floating point numbers in the numerical calculations. We are currently implementing these enhancements to improve our results.

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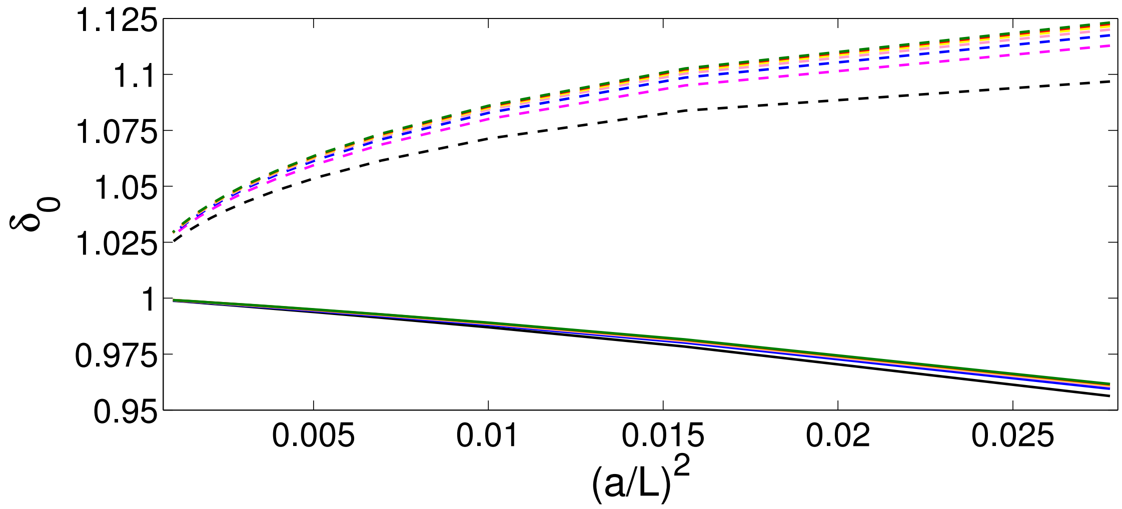

Figure 4: The unimproved (dashed) and improved (solid) one loop lattice step scaling function normalized to the continuum limit (δ_0) as a function of $(a/L)^2$ for $\mathrm{SU}(N)$ pure gauge with $N = 2$ (black), 3 (purple), 4 (blue), 5 (pink), 6 (yellow), 7 (red) and 8 (green).

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