Comparing Tensor Renormalization Group and Monte Carlo calculations for spin and gauge models

Yannick Meurice[*a][†]**, Alan Denbleyker** [a]**, Yuzhi Liu** [a,c,d]**, Tao Xiang**[b]**, Zhiyuan Xie**[b]**, Ji-Feng Yu**[b]**, Judah Unmuth-Yockey**[a]**, Haiyuan Zou** [a]

[a] *Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52240, USA*

[b] *Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing 100190, China*

[c] *Fermi National Accelerator Laboratory, Batavia, IL 60510, USA*

[d] *Department of Physics, University of Colorado, Boulder, CO 80309, USA*

We show that the Tensor Renormalization Group (TRG) method can be applied to O(N) spin models, principal chiral models and pure gauge theories (Z2, U(1) and SU(2)) on (hyper) cubic lattices. We explain that contrarily to some common belief, it is very difficult to write compact formulas expressing the blockspinning of lattice models. We show that in contrast to other approaches, the TRG formulation allows us to write exact blocking formulas with numerically controllable truncations. The basic reason is that the TRG blocking separates neatly the degrees of freedom inside the block and which are integrated over, from those kept to communicate with the neighboring blocks. We argue that the TRG is a method that can handle large volumes, which is crucial to approach quasi-conformal systems. The method can also get rid of some sign problems. We discuss recent results regarding the critical properties of the 2D O(2) nonlinear sigma model with complex β and chemical potential. As some of these results appeared in a recently published paper (PRD 88, 056005) and two recent preprints (arXiv:1309.4963 and arXiv:1309.6623), these proceedings rather emphasize the conceptual aspects of our ongoing effort.

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[*]Speaker.

[†]Current email address: yannick-meurice@uiowa.edu

1. Introduction: blocking in configuration space is hard!

The block spin idea played a central role in the development of the RG method by the late Ken Wilson. However, successful applications of the method were possible without requiring a numerical implementation of the original idea. Blocking in configuration space is hard! Anyone who believes that blocking is a straightforward procedure should attempt to write a simple algorithm for the two-dimensional Ising model on a square lattice. For this model, blocking means replacing four spins in a 2x2 square block by a single variable and write an effective energy function (or at least some effective measure) for the new block variables.

A possible strategy is to consider 2 by 2 blocks in a $A - B$ checkerboard. First, we consider the B blocks as fixed backgrounds. We can then block spin in the A blocks. For instance, $Prob(\phi_A = 4|\phi_i^{background}) \propto \exp(\beta(4 + \sum_{i=1}^{8} \phi_i^{background}))$ and more complicated expressions for the intermediate values of ϕ_A. So far, we have blocked 1/2 of the total number of spins. The next step is to try to block spin in the B blocks. Using the results of step 1 we can block spin in a given B block surrounded by 4 A blocks but this set of 5 blocks is surrounded by 20 background spins. We have blocked 5/8 of the total number of spins but the procedure becomes more intricate at each step and finding the effective energy function is nontrivial. This is explained in more detail and illustrated in [1]. The goal of this discussion is only to illustrate the generic difficulty of blocking in configuration space.

It is possible to invent approximations where no new interactions are generated by the blocking process. Well-known examples are the Migdal-Kadanoff approximation [2, 3], the approximate recursion formula [4] or other hierarchical approximations [5, 6]. However, in these examples, the lack of reference to an exact procedure makes the systematic improvement of these approximations difficult. In contrast, the TRG formulation allows us to write exact blocking formulas with numerically controllable truncations. The basic reason is that the TRG blocking separates neatly the degrees of freedom inside the block (which are integrated over), from those kept to communicate with the neighboring blocks. The TRG approach of classical lattice models was introduced in Refs. [7, 8, 9] motivated by tensor states developed in RG studies of quantum models [10]. Improved methods to take into account the environment were proposed in Refs. [11, 12, 13]. In Ref. [1], we showed that TRG methods can be applied to models studied by lattice gauge theorists, namely spin models ($O(N)$ and principal chiral models) and pure gauge models (Ising, $U(1)$ and $SU(2)$). The case of actions quadratic in Grassmann variables is briefly discussed in [14].

In these proceedings, we review the general procedure with the simple example of the 2D Ising model (Sec. 2). More complicated examples (2D $O(2)$ model and 3D Abelian gauge theories) are discussed in Sec. 3. The TRG method has the potential to handle situations where there is a sign problem. We briefly discuss recent results for spin models at complex β and with chemical potential in Sec. 4. More details can be found in recent preprints[16, 15].

2. TRG blocking: it's simple and exact!

The first step in the TRG formulation of spin models is to use character expansions and assign degrees of freedom to the links. This first step is similar to what is done when we construct dual formulations, however, we do not introduce dual variables here. In the case of the Ising model, we

write for each link:

$$\exp(\beta\sigma_1\sigma_2) = \cosh(\beta)(1 + \sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2) =$$
$$cosh(\beta)\sum_{n_{12}=0,1}(\sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2)^{n_{12}}. \tag{2.1}$$

We then regroup the four terms involving a given spin σ_i and sum over its two values ±1. The results can be expressed in terms of a tensor: $T^{(i)}_{xx'yy'}$ which can be visualized as a cross attached to the site i with the four legs covering half of the four links attached to i. The entire lattice can be tiled with such crosses. The horizontal indices x, x' and vertical indices y, y' take the values 0 and 1 as the index n_{12}.

$$T^{(i)}_{xx'yy'} = f_x f_{x'} f_y f_{y'}\delta\left(\mathrm{mod}[x+x'+y+y',2]\right) , \tag{2.2}$$

where $f_0 = 1$ and $f_1 = \sqrt{\tanh(\beta)}$. The delta symbol is 1 if $x+x'+y+y'$ is zero modulo 2 and zero otherwise. The partition function can be written exactly as

$$Z = (\cosh(\beta))^{2V} Tr\prod_i T^{(i)}_{xx'yy'} , \tag{2.3}$$

where Tr means contractions (sums over 0 and 1) over the links. This representation reproduces the closed paths of the high temperature expansion. TRG blocking separates the degrees of freedom inside the block which are integrated over, from those kept to communicate with the neighboring blocks. This is represented graphically in Fig. 1.
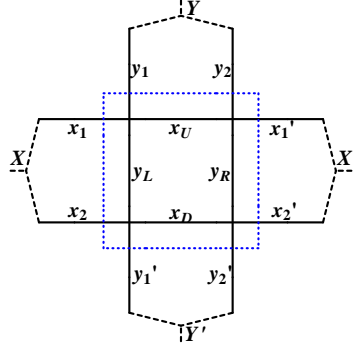
Figure 1: Graphical representation of $T'_{XX'YY'}$.

Blocking defines a new rank-4 tensor $T'_{XX'YY'}$ where each index now takes four values. Its explicit form is:

$$T'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)} = \tag{2.4}$$
$$\sum_{x_U,x_D,x_R,x_L} T_{x_1 x_U y_1 y_L} T_{x_U x'_1 y_2 y_R} T_{x_D x'_2 y_R y'_2} T_{x_2 x_D y_L y'_1} ,$$

where $X(x_2,x_2)$ is a notation for the product states e. g. , $X(0,0) = 1$, $X(1,1) = 2$, $X(1,0) = 3$, $X(0,1) = 4$. The partition function can now be written exactly as

$$Z = (\cosh(\beta))^{2V} Tr\prod_{2i} T'^{(2i)}_{XX'YY'} , \tag{2.5}$$

where $2i$ denotes the sites of the coarser lattice with twice the lattice spacing of the original lattice. The new expression has exactly the same form as the previous one except for the fact that the indices run over more values.

In practice, this exact procedure will require too much memory after a few iterations and one needs to introduce truncations. For the Ising model on square and cubic lattices, a truncation method based on Higher Order SVD (HOTRG) gives very accurate results [13] . It was also noticed that the Higher Order SVD [14] sharply singles out a surprisingly small subspace of dimension two. In the two states limit, the transformation can be handled analytically yielding a value 0.964 for the critical exponent ν much closer to the exact value 1 than 1.338 obtained in Migdal-Kadanoff approximations. Alternative blocking procedures that preserve the isotropy can improve the accuracy to $\nu = 0.987$ and 0.993.

3. More examples

The previous results can be generalized for the $O(2)$ model. The partition function reads

$$Z = \int \prod_i \frac{d\theta_i}{2\pi} e^{\beta \sum_{<ij>} \cos(\theta_i - \theta_j)} . \tag{3.1}$$

For each link, we can use

$$e^{\beta \cos(\theta_i - \theta_j)} = \sum_{n_{ij}=-\infty}^{+\infty} e^{in_{ij}(\theta_i - \theta_j)} I_{n_{ij}}(\beta) , \tag{3.2}$$

where the I_n are the modified Bessel functions of the first kind. In two dimensions, we obtain the factorizable expression for the initial tensor:

$$T^i_{n_{ix},n_{ix'},n_{iy},n_{iy'}} = \sqrt{I_{n_{ix}}(\beta)}\sqrt{I_{n_{iy}}(\beta)}\sqrt{I_{n_{ix'}}(\beta)}\sqrt{I_{n_{iy'}}(\beta)}\delta_{n_{ix}+n_{iy},n_{ix'}+n_{iy'}} . \tag{3.3}$$

The partition function and the blocking of the tensor are similar to the Ising model. The only difference is that the sums run over the integers. As the $I_n(\beta)$ decay rapidly for large n and fixed β (namely like $1/n!$) there is no convergence issue. The generalization to higher dimensions is straightforward. In Ref. [1], exact blocking formulas are also provided for the $O(3)$ model and the $SU(2)$ principal chiral model.

The TRG method can also be used to handle gauge theories. We briefly describe the so-called symmetric formulation for Abelian groups of Ref. [1] where the $SU(2)$ case is also discussed. A blocking procedure can be constructed by sequentially combining two cubes into one in each of the directions. On the links in the blocked direction of the new lattice formed by two cubes, two parallel A tensors form the new A' tensor with product states (capital letters). Each tensor element can be written as

$$A'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)} = A_{x_1 x'_1 y_1 y'_1} \times A_{x_2 x'_2 y_2 y'_2}. \tag{3.4}$$

On the new faces, two $\tilde{B}$ tensors and one A tensor form a new $\tilde{B}'$ tensor,

$$\tilde{B}'_{xx'Y(y_1,y_2)Y'(y'_1,y'_2)Z(z_1,z_2,z_3)Z'(z'_1,z'_2,z'_3)} = \sum_{x_3,x'_3} \tilde{B}_{xx_3 y_1 y'_1 z_1 z'_1} A_{x_3 x'_3 z_3 z'_3} \tilde{B}_{x'_3 x' y_2 y'_2 z_2 z'_2}.$$

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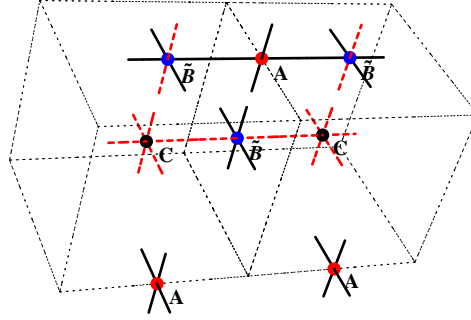
Figure 2: Illustration of the blocking procedure

At the center, two C tensors and one $\tilde{B}$ tensor form a new C' tensor,

$$C'_{xx'Y(y_1,y_2,y_3)Y'(y'_1,y'_2,y'_3)Z(z_1,z_2,z_3)Z'(z'_1,z'_2,z'_3)} = \sum_{x_2,x'_2} C_{xx_2y_1y'_1z_1z'_1}\tilde{B}_{x_2x'_2y_2y'_2z_2z'_2}C_{x'_2x'y_3y'_3z_3z'_3}.$$

4. Work in progress and conclusions

Successful numerical applications were mentioned at the conference and are now reported in recent preprints [15, 16]. For the 2D Ising at complex β (sign problem) and finite volume, the partition function obtained with the TRG has been compared to the exact Onsager-Kaufman solution. This is illustrated in Fig. 3. In general, the TRG can reach much larger values of Imβ than MC. The only regime where caution needs to be used is in the vicinity of Fisher's zeros where all the contributions tend to cancel and the results are more sensitive to truncation effects. This is explained at length in Ref. [16]. For the 2D $O(2)$ at finite volume, the Fisher's zero have been obtained and compared with those obtained by MC and reweighting. The good agreement is illustrated in Fig. 4. The critical properties of the 2D $O(2)$ at large volume have been compared with MC [15]. More recently, calculations for the 2D $O(2)$ with chemical potential (another sign problem) have been compared to results obtained with the worm algorithm [17] and an excellent agreement was found [18]. We plan to study modulated phases in clock models with imaginary chemical potential and compare with results presented at the conference [19].

In summary, the TRG method allows us to achieve the initial Wilsonian program (block spinning) in an exact way. It applies to most classical lattice models. The HOTRG procedure allows controllable truncations. Successful numerical implementations at large volumes and for models with sign problem are very promising. In general the implementation of the HOTRG involves computational demands that are very different from those of conventional MC methods. Namely, memory considerations overwhelm CPU performance. On the analytical side, the somehow irregular behavior at intermediate truncations remains [20] to be understood. Many new applications
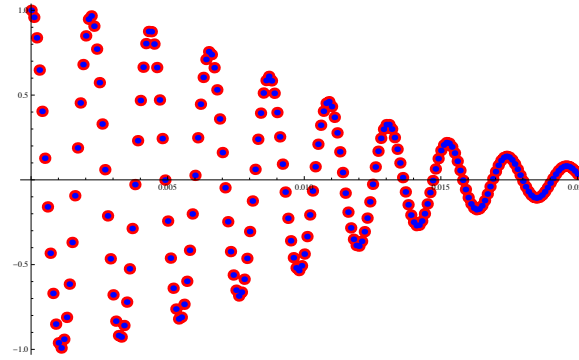
Figure 3: The real part of the partition function for $\beta = 0.3 + ix$ vs. x; red: result from HOTRG with 20 states; blue: exact solution (Onsager-Kaufmann).
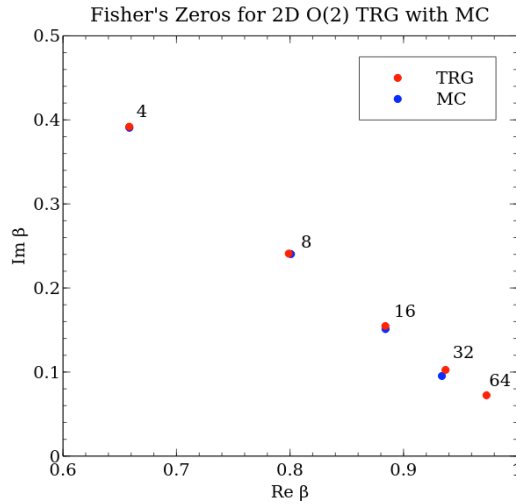
Figure 4: Fisher's zeros of XY model with $L = 4, 8, 16, 32, 64$ for 30 states compared to MC

are possible and we expect that the TRG will become a useful complement of the conventional MC methods.

Acknowledgements

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