Subtleties of Non-Abelian Gauge Theories in Cold-Atomic Lattices

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I point out two of the subtleties referred to in the title. The first is that gauge-invariant magnetic systems may realized under general circumstances, as suggested by a simple theorem. The second subtlety is that care is needed to identify the field theory simulated by a cold-atomic lattice gauge system. Though the simplest such model confines in $2+1$ dimensions, it has non-relativistic "gluon" excitations. Time-reversal invariance is spontaneously broken in this system. The confinement mechanism is related to an extra U(1) gauge invariance. There is a model, suggested long ago by D. Rohlich and me, which is known to have relativistic spin waves. One of the outstanding theoretical problems is a better determination of the energy-momentum relation of spin waves in different magnetic gauge systems.

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1. Introduction

The rapid advance of ultracold-atom technology [1] has made atomic lattice systems with dynamical gauge fields a serious prospect [2], [3], [4]. Such systems, first proposed by Horn [5] and examined by Banks and Zaks [6] are Heisenberg-type magnets. They were generalized and given the names "gauge magnets" [7] and "quantum-link models" [8], [9]. I use the first moniker here, but they both mean the same thing. Several groups of people "discovered" gauge magnets (including D. Rohrlich and me), without realizing they were not the first. Only a finite number of representations of the gauge group exist. I think it is fair to include the remarkable proposed realization of the Kogut-Susskind Hamiltonian lattice gauge theory [3] among these systems, because (for practical reasons) the latter has a finite number of states per link.

I want to direct your attention to two subtle aspects of the subject. First, gauge magnets may describe general models of hopping, tightly-bound particles. There is an old example [10], but this is a special case of a more general theorem.

The second aspect is that non-Abelian gauge magnets are not necessarily Yang-Mills theories. The simplest example of an $SU(2) \times U(1)$ gauge magnet has non-relativistic "gluons", whose Lagrangian can be found [7]. With a link term, there are two phases [6]; simple arguments indicate that both phases confine (there may be a deconfined phase in $3+1$ dimensions). Confinement, however, can be understood in terms of the $U(1)$ part of the gauge group - the quarks in this theory must also have Abelian charge. There is a simple model with relativistic gluons; though its Yang-Mills action may contain topological terms. We need a better understanding of how the energy of a gauge magnet spin wave depends on its momentum. The coherent-state methods devised thus far [7] cry out for improvement.

Nothing in the previous paragraph is controversial. Finding the field theory describing a quantum spin system is an art, not a science. The accepted wisdom was that all translation-invariant XXX spin-s chains are gapless. That wisdom was wrong. Haldane [11] argued that the integer-spin case is the O(3) sigma model with $\theta = 0$, which is gapped (but *some* integrable cases of integer spin chains *are* gapless [12]).

Spin-wave methods can't yield the phase diagram of a quantum system, but are useful for understanding ultraviolet behavior. They help identifying the field theory, though not the vacuum.

2. Non-Abelian Gauge Magnets

The simplest SU(2) gauge magnets have four states on each link of a spacial latttice (x, j). Each state corresponds to a component of a Dirac spinor. Operators $\gamma^\mu(x)$ satisfy the anticommutation relations $[\gamma^\mu, \gamma^\nu]_+ = 2\delta^{\mu\nu}$, where $\mu, \nu = 0, 1, 2, 3$. Products of these operators provide the closed Lie algebra SU(4), which can also act on the one-link space of states. The other generators of this SU(4) are the 4×4 operators $[\gamma^\mu, \gamma^\nu]_+ = 2\delta^{\mu\nu}$, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\rho^\mu = -i\gamma^5\gamma^\mu$ and $\sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]_-$, $\Sigma^{b\pm} = \frac{1}{4}\varepsilon^{bcf}\sigma^{cf} \pm \sigma^{0b}$, where $b, c, f = 1, 2, 3$.

The lattice gauge fields are quantum connection or parallel-transport operators, defined as

$$V_j(x) = U_j(x) + \alpha_j(x)U_j^5(x) = [\gamma_j^0(x) \otimes \mathbb{1} - i\vec{\gamma}_j(x) \cdot \otimes \vec{\tau}] + \alpha_j(x)[\rho_j^0(x) \otimes \mathbb{1} - i\vec{\rho}_j(x) \cdot \otimes \vec{\tau}],$$

where $\alpha_j(x)$ is an arbitrary complex number. The Hamiltonian describes an $SU(2) \times U(1)$ gauge theory. It has the form

$$H = J \sum_{x,\, j \neq k} \text{Tr } V_j(x) V_k(x + \hat{j}a) V_j(x + \hat{k}a)^\dagger V_k(x)^\dagger + K \sum_{x,j} \gamma_j^5(x). \tag{2.1}$$

The triplet of SU(2) Gauss' law operators and the single U(1) Gauss' law operators are

$$G^b(x) = \sum_{j=1}^d [\Sigma_j^{b\,+}(x) - \Sigma_j^{b\,-}(x - \hat{j}a)], \;\; G^5(x) = \sum_{j=1}^d [\gamma_j^5(x) - \gamma_j^5(x - \hat{j}a)], \tag{2.2}$$

respectively. The Gauss' law operators (2.2) commute with the Hamiltonian. Therefore the state of the entire system $\Psi(t)$ satisfies both $G^b(x)\Psi(t) = S^b(x)\Psi(t)$ and $G^5\Psi(t) = S^5(x)\Psi(t)$, where the charges $S^b(x)$ and $S^5(x)$ are determined by the initial state $\Psi(0)$. Notice that (2.1) is nontrivial when $K = 0$.

   In Reference [10], a particular representation of the operators on links used was $\gamma^0 = (T^+ + T^-) \otimes \mathbb{1}$, $\vec{\gamma} = i(T^+ - T^-) \otimes 2\vec{S}$, where $\vec{S}$ is the spin of a particle which can fill one of two vacancies on a link, and $T^\pm$ moves the particle between the two vacancies. The particles are SU(2) rishons, described for SU(N) gauge magnets in [9]. The motivation of introducing these particles in Reference [10] was to show how gauge magnets could arise in the low-energy limit of hopping particles on a lattice. A pictorial representation is:
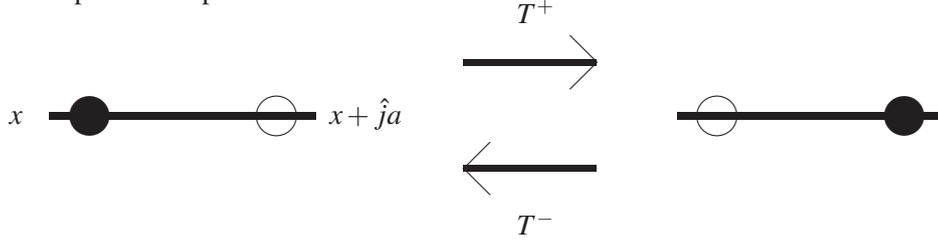
where the solid circle is the spinning particle and the empty circle is a vacancy. This pictorial representation shows that the SU(2) and U(1) Gauss' law are restrictions on the total spin and the particle number, respectively, adjacent to a lattice vertex [10], [9]. The U(1) Gauss' law condition, for the case of no sources, means that the arrangement of particles around a site satisfies the "ice" or "six-vertex" condition in the physical states spanning the Hilbert space. This means that adjacent to one site x, only the following configurations (labeled 1 to 6) appear:
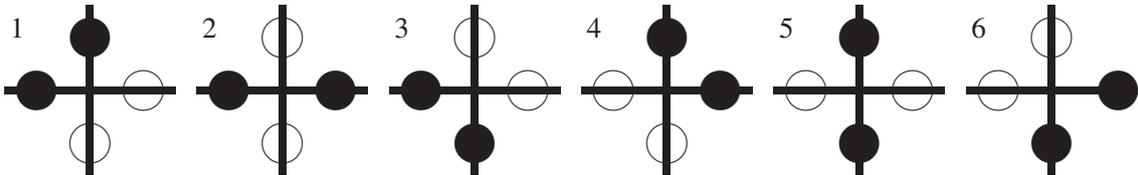

If a color source is present at x, then the number of adjacent particles is one or three and the spin state of the source and the adjacent particles is a singlet.

3. The hopping-parameter expansion

The Hubbard model has a lattice Hamiltonian with nearest-neighbor hopping spin-1/2 particles, with a local Coulomb interaction. For half-filling, with a repulsive Coulomb term, perturbation theory in the hopping term yields an effective Heisenberg antiferromagnet.

There is a similar mechanism to produce a gauge magnet from a lattice model of hopping particles, in the low-energy limit [10]. Consider a (not necessarily regular) lattice of sites, at which a particle may sit. Suppose the particle is has a vector index, making a vector N-plet. The particle creation and annihilation operators at a site j can be written as $c_{j,\alpha}^{\dagger}$ and $c_{j,\alpha}$, where $\alpha = 1, \ldots, N$. The spin or isospin at a site j is $S_j^a = \sum_{\alpha\beta} c_{j,\alpha}^{\dagger}(S^a)_{\alpha\beta} c_{j,\beta}$, where the $N \times N$ matrices $(S^a)_{\alpha\beta}$ are generators of the symmetry group. The lattice is subdivided into cells or "bags" on the lattice, labeled by F. These cells are connected sets of sites. The sets are disjoint, covering the entire lattice. Each cell is surrounded by a red boundary in the figure:
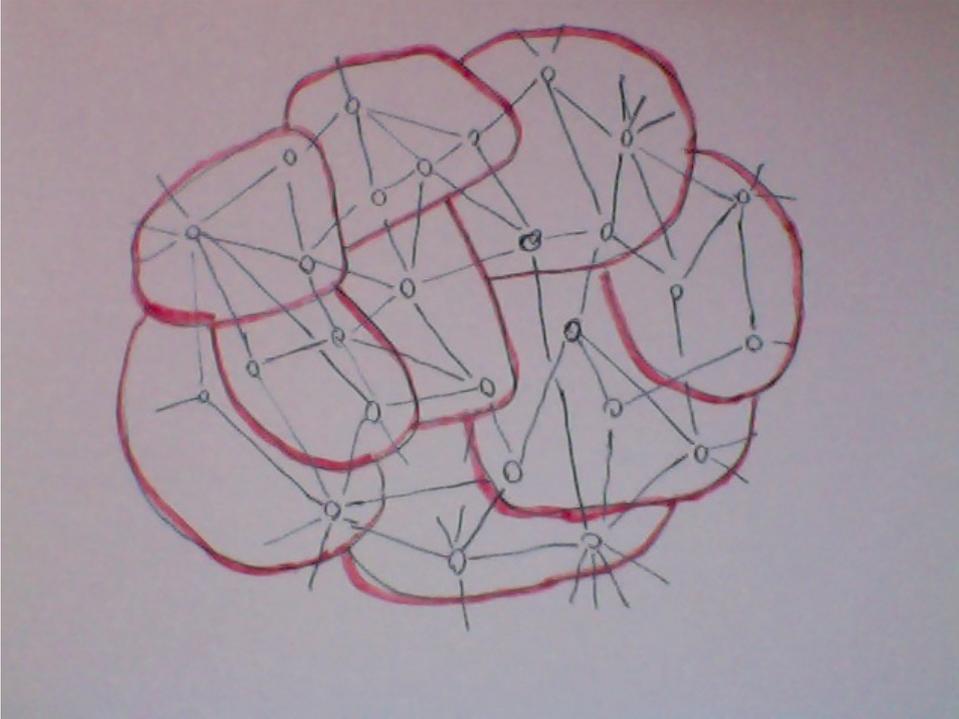
The Hamiltonian has a nearest-neighbor hopping term, with hopping parameter t and a term acting on each cell F of the lattice:

$$H = -t \sum_{\langle i,j \rangle} \sum_{\alpha} c_{i,\alpha}^{\dagger} c_{j,\alpha} + U \sum_{F} V_F, \tag{3.1}$$

where V_F must satisfy certain properties, namely: 1. the eigenvector of V_F with the smallest eigenvalue (or one-cell ground state) is a singlet of the symmetry and 2. the eigenvectors of V_F without the lowest eigenvalue are multiplets of the symmetry, with eigenvalues at least of order 1. Let's label the different multiplets by the letter $m = 1, 2, \ldots$. There are a finite number of these, which depends on the number of sites in one cell F. Since the multiplets are degenerate, the Hamiltonian commutes with a projection operator P_F^m onto the multiplet labeled by m.

Theorem: The effective Hamiltonian obtained in degenerate perturbation theory in t, is a gauge magnet, where the Gauss law operator is

$$G_F^a = \sum_{j \in F} S_j^a - \sum_m P_F^m \left(\sum_{j \in F} S_j^a \right) P_F^m \sim (\vec{D} \cdot \vec{E})^a - \rho^a. \tag{3.2}$$

Sketch of proof: This operator annihilates the lowest energy states of the effective theory, provided $t \ll U$. The degenerate multiplets are color sources. Gauss' law $G_F^a \Psi = 0$ is tautological (it is true by definition). The commutation relations of the operators (3.2) are precisely what generators of gauge transformations must satisfy. Finally, taking the low-energy limit means that (3.2) will commute with the effective Hamiltonian. This effective Hamiltonian will have terms of order t^m/U^{m-1} on polygons (plaquettes) with m sides. A gauge-invariant matter coupling may appear at order t^2/U. *Note:* for the simplest gauge magnets, the role of the cell F is played by the set of particle vacancies adjacent to a lattice site.

The main implication of the theorem is that gauge invariance is nothing special. The theorem does not guarantee that gauge invariance is truly dynamical, *i.e.*, the spin waves of the effective theory have the quantum numbers of gluons[1]. Nonetheless, it does suggest that dynamical gauge invariance, *e.g.*, that of Reference [10], can occur in a variety of contexts.

I should mention that there is an alternative proposal for producing gauge magnets from Hubbard-type cold-atomic systems [13].

4. Confinement and U(1) gauge symmetry

Suppose a single static quark is at x. Then Gauss' law implies that the number of particles on the link vacancies adjacent to x is one or three (so that total spin a singlet).

Let's consider the limit of (2.1) as $K \to -\infty$. In this limit, only vertex 4 has finite energy. Notice that keeping only one such vertex breaks parity. This means that a quark must produce a line of vertices going to the boundary (or an antiquark, if present) each of which has energy of order K. Hence quarks are confined. In the illustration below, a quark-antiquark pair forces a line of vertices other than vertex 4 to connect the sources. Thus confinement occurs with a string tension of order $|K|/a$ [4].
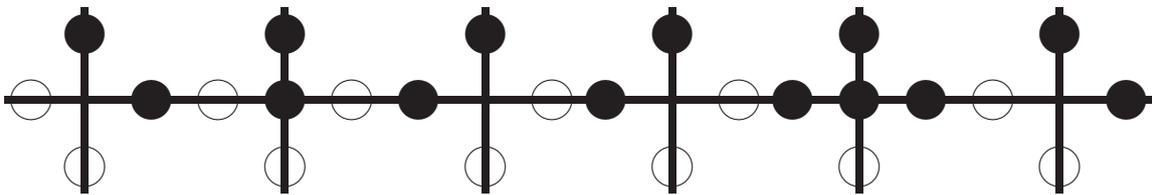
Similarly, in the limit that $K \to +\infty$, vertex 3 is dominant and a similar mechanism produces confinement.

This confinement mechanism is inherently Abelian. The argument works in U(1) theories with electric charges but no quarks. As $|K/J|$ decreases, there can be a phase transition to a phase in which one type of vertex is no longer longer frozen into the system. It may be that this is the

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[1]Indeed, Alessio Celi, Luca Tagliacozzo and I found an example where gauge invariance is not dynamical (unpublished).

transition has been already found [6]. Is the second phase deconfined phase? I think it is clear that the answer is no. The reason is that discrete rotation invariance is broken, in the same way that this happens in quantum-dimer models [14], which also have a U(1) gauge invariance. There can be special choices of K/J for which the correlation length becomes infinity where deconfinement occurs, but these are not generic.

5. Spin-wave frequency and wavelength

The simplest model of the type (2.1) is with $\alpha_j(x) = 0$ everywhere:

$$H = J \sum_{x,\, j\neq k} \mathrm{Tr}\, U_j(x)U_k(x+\hat{j}a)U_j(x+\hat{k}a)^\dagger U_k(x)^\dagger + K\sum_{x,j}\gamma_j^5(x). \tag{5.1}$$

A way to study spin waves is to find the Heisenberg equation of motion $i\partial_t B = [H, B]$ of a local operator B, defined on a single link. If γ^μ and ρ^μ are replaced, in the definition of U_j and $U_j^5(x)$, by the classical variables m^μ and n^μ respectively, with $n \cdot n = m \cdot m = 1$, $m \cdot n = 0$, these equations for m^μ and n^μ follow from the classical action [7], [15]:

$$S = \sum_{x,j} s \int dt \int_0^\infty du\; \varepsilon_{\alpha\beta\mu\nu}\, n_j^\alpha m_j^\beta \left(\frac{\partial n_j^\mu}{\partial t}\frac{\partial n_j^\nu}{\partial u} + \frac{\partial m_j^\mu}{\partial t}\frac{\partial m_j^\nu}{\partial u}\right)$$
$$- J \sum_{x,\, j\neq k} \mathrm{Tr}\, U_j(x,t)U_k(x+\hat{j}a,t)U_j(x+\hat{k}a,t)^\dagger U_k(x,t)^\dagger, \tag{5.2}$$

with spin $s = 1/2$. The first term in (5.2) is the Wess-Zumino action for relativistic spin [15]. If the equations of motion are linearized, the spin wave frequency E in terms of its wave number p is $|E| = 4Jp^2$. There is spontaneous symmetry breaking of of time-reversal symmetry, just as for a ferromagnet. The action (5.2) is first-order in time derivatives. Thus, at least in the semiclassical approximation, the spin waves are not Yang-Mills gluons, but nonrelativistic gauge Bosons. If the term $K\sum \gamma^5$ is included, this action is no longer sufficient to describe spin waves. The Heisenberg equations of motion still indicate a nonrelativistic relation between energy and momentum, however.

There is a gauge magnet with relativistic gluons [7]. In $2+1$ dimensions, it has the form

$$H = J_1 \sum_{x^1+x^2 \text{ even}} \mathrm{Tr}\, UUUU + J_2 \sum_{x^1+x^2 \text{ odd}} \mathrm{Tr}\, U^5 U^5 U^5 U^5, \tag{5.3}$$

where each term is on an elementary plaquette. This is a staggered model on a chessboard, with one type of term on red plaquettes, the other on black plaquettes. There is a similar model version in higher dimensions too. The spin waves are similar to those of a one-dimensional spin chain [16], and have speed of light $c = 8\sqrt{|J_1 J_2|}$ and mass gap $m = \frac{|J_1 - J_2|}{8|J_1 J_2|}$, respectively. This model has been speculated to be a Yang-Mills theory with a Chern-Simons term in $2+1$ dimensions [7].

6. Conclusions

I've tried to make two points. The first is that gauge magnets may be ubiquitous. It is fun to speculate that gauge invariance in particle physics arises this way, but I think not (a symmetry

principle, like supersymmetry, is needed to give all the gauge bosons the same speed of light). The second point is that better methods are needed for identifying the quantum field theory described by a gauge magnet.

Coherent state methods [7] need to be generalized. The formalism used thus far does not properly acommodate all the observables. Perhaps a Holstein-Primakoff method, analogous to that used for SU(2) ferromagnets and antiferromagnets exists. Such a method would go a long way towards yielding a convincing correspondence with a field theory. Such methods, of course, are only reliable only in the large-spin limit (in the models presented here, the spin is one-half). In the short term, perhaps this limitation is not so important. The ultimate development of such methods would be a correspondence between gauge magnets and gauge field theories similar to Haldane's correspondence between spin chains and the O(3) sigma model [11].

Acknowledgements: I would like to mention that Daniel Rohrlich first noticed there are also relativistic spin waves in (5.1), which are the photons of the U(1) gauge symmetry (though we did not think about this symmetry, at that time). I also am grateful for Alessio Celi, Luca Tagliacozzo and Maciej Lewenstein for teaching me about many of the advances which have occured recently in this subject. I thank Benni Reznik and Yanick Meurice for useful comments. Finally, I thank Axel Cortés Cubero for recent discussions about the relativistic model (5.3).

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