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Improved Lattice Radial Quantization

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Lattice radial quantization was proposed in a recent paper by Brower, Fleming and Neuberger [1] as a nonperturbative method especially suited to numerically solve Euclidean conformal field theories. The lessons learned from the lattice radial quantization of the 3D Ising model on a longitudinal cylinder with 2D Icosahedral cross-section suggested the need for an improved discretization. We consider here the use of the Finite Element Methods(FEM) to descretize the universally-equivalent ϕ^4 Lagrangian on $\mathbb{R}\times\mathbb{S}^2$. It is argued that this lattice regularization will approach the exact conformal theory at the Wilson-Fisher fixed point in the continuum. Numerical tests are underway to support this conjecture.

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1. Introduction

Conformal or near conformal behavior in field theory lies at the heart of many challenging theoretical and phenomenological problems. Models for possible strong dynamics for electro-weak symmetry breaking as a replacement of the elementary Higgs of the Standard Model are often built on near-conformal theories. A large variety of extra-dimensional models use the AdS/CFT correspondence to introduce large scale separations. However conventional lattice methods near an IR conformal fixed point are difficult, precisely because of the growing separation of the length scales between the UV and IR. A recent paper by Brower, Fleming and Neuberger(BFN) explored replacing the traditional Euclidean lattice in favor of one suited to *Radial Quantization* [1]. In radial quantization the dilatation operator plays the role of the Hamiltonian. The potential advantage for lattice simulations is now the dilatation operator generates translations in $\log r$ so a finite radial lattice separates scales *exponentially* in the number of lattice sites.

For an exactly conformal field theory, the idea is straightforward. The flat metric for any Euclidean field theory on $\mathbb{R}^D$ can obviously be expressed in radial coordinates,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = r_0^2 e^{2t}(dt^2 + d\Omega_{D-1}^2)\ , \tag{1.1}$$

where $t = \log(r/r_0)$, introducing an arbitrary reference scale r_0, and where $d\Omega^2_{D-1}$ is the metric on the $\mathbb{S}^{D-1}$ sphere of unit radius. However for an exactly conformal field theory, the Weyl transformation can also be used to remove the conformal factor, e^{2t}. As a result, a CFT can be legitimately mapped from the D-dimensional Euclidean space $\mathbb{R}^D$ to perform radial quantization on a cylindrical manifold, $\mathbb{R}\times\mathbb{S}^{D-1}$.

On $\mathbb{R}\times\mathbb{S}^{D-1}$, dilatations, inversion at the origin, and rotations on $\mathbb{S}^{D-1}$ are manifest. The extension to the full conformal group O(D+1,1), including Poincaré invariance, is now a consequence of the dynamics. Ref. [1] explores the circumstances when, and when not, full conformal invariance is achieved. As a first example, the large N solution to the 2D O(N) sigma model, which (like massless QCD) has a classically conformal Lagrangian with a quantum conformal anomaly, is considered. Comparing the traditional Euclidean $\mathbb{R}^2$ quantization to radial quantization on $\mathbb{R}\times\mathbb{S}^1$, it was recognized that the quantum anomaly forces the traditional $\mathbb{R}^2$ quantization to choose the Lorentz symmetry (O(2) rotations), whereas radial $\mathbb{R}\times\mathbb{S}^1$ quantization chooses dilatation invariance. Neither quantization scheme had the full conformal symmetry. Repeating this exercise for the 3D O(N) model, having no anomaly, apparently gives the full conformal symmetry group, O(4,1), for both. Calculations on the large N expansion in 3D are underway to prove this.

A second example considered in ref. [1] was the lattice implementation of radial quantization for the 3D Ising model at the Wilson-Fisher fixed point. A discrete lattice was used, on a cylinder with longitudinal coordinate in $t = \log r$ and uniform equilateral triangular refinement of the icosahedral approximation to $\mathbb{S}^2$. Here the numerical methods proved generally quite accurate, but as noted in the conclusion, there were small departures away from the full restoration of the conformal group in the continuum. In particular, the scaling dimensions for the two irreducible icosahedral representations for the 3rd descendant of the Z_2 odd primary did not converge to a single $l = 3$ representation of O(3) in the continuum. Apparently this lattice implementation in the continuum limit yields a radial quantization of a critical Ising model on a transverse icosahedron, which quite naturally failed to realize the full conformal symmetry. In this presentation, we seek to remove

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this obstruction by the use of a Finite Element Method (FEM) for the universally-equivalent ϕ^4 Lagrangian in order to allow convergence to O(3) symmetry on $\mathbb{S}^2$. This improved radial quantization, we conjecture, will approach the exact conformal field theory at the Wilson-Fisher fixed point with full O(4,1) conformal symmetry in the continuum limit without fine-tuning of irrelevant operators on the lattice.

2. Finite Element Lagrangian

The problem we face is how to put a quantum field theory on a tessellation of a curved manifold. The finite element method (FEM) is an established mathematical method to discretize differential equations on an irregular mesh [2]. It is not clear that a variation of FEM can be formulated for the Ising model,because there is no notion of a locally smooth field configuration. Therefore, we first replace our Ising model by the universally-equivalent ϕ^4 Lagrangian. The Lagrangian on a smooth manifold is given by

$$S = \int d^D x \sqrt{-g}[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda(\phi^2 - \frac{\mu^2}{2\lambda})^2] \; . \tag{2.1}$$

On a flat $\mathbb{R}^3$ manifold, a suitable discretization by a regular cubic lattice is achieved by replacing the kinetic term by a nearest neighbor finite difference approximation. The justification for this can be understood by performing a local Taylor series expansion at each site x for the kinetic term in the classical equation of motion,

$$a^{-2} \sum_{\pm\hat{\mu}} [\phi(x) - \phi(x + a\hat{\mu})] \simeq -\nabla^2 \phi(x) + O(a^2) \; , \tag{2.2}$$

and observing that rotational symmetry is restored up to $O(a^2)$ corrections and that at the quantum level this introduces symmetry breaking only by irrelevant operators. The reason for this success is traced to the fact that the hypercubic group preserves an infinitely growing subgroup of the Poincaré group (cubic rotations plus discrete translations). Poincaré invariance is restored as an accidental symmetry.

On a sphere there is no analogous infinite subgroup of the rotation group, so a new method is needed. Instead we consider a finite element approach to the lattice discretization of the $\mathbb{R} \times S^2$ manifold to address this problem. The simplest option is to use first order piecewise linear finite elements on each triangle projected locally onto its tangent plane as illustrated in Fig. 1. Introducing local Cartesian coordinates, ξ_a, on each tangent plane, we approximate the field $\phi(x)$ as a sum over the three non-zero linear elements $\phi(x) = \sum_i \phi_i \, L_i(\xi_1, \xi_2)$ on each triangle, where the projection is defined by the function $x^\mu(\xi_a)$. For example if we label the vertices by $(1,2,3)$ we may orient the coordinate system with vertices at $(\xi_1, \xi_2) = (0,0), (l_{12}, 0)$ and (a_1, a_2) respectively by an appropriate 2D Euclidean transformation (rotation plus translation). Now the elements take the explicit form,

$$L_1(\xi_a) = [l_{12} - \xi_1 - (l_{12} - \xi_1)(\xi_2/a_2)]/l_{12} \; , \quad L_2(\xi_a) = [\xi_1 - a_1(y/a_2)]/l_{12} \; , \quad L_3(\xi_a) = \xi_2/a_2$$

such that $L_i(x,y)$ is 1 on vertex i and zero on the other two. A straightforward calculation gives the contribution of this triangle to the kinetic term,

$$\int_A d^2\xi \; \partial_a \phi(\xi) \partial_a \phi(\xi) = \frac{1}{8A_{123}} [(l_{23}^2 + l_{31}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}] \; , \tag{2.3}$$

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where l_{ij} are the lengths of the edges and A_{ijk} is the area of the $(1,2,3)$ spherical triangle projected onto the tangent plane. Note that the resulting FEM integral is indeed invariant under Euclidean group (rotation and translations) on the 2D tangent plane, as assumed in our choice of tangent plane coordinates. To complete the FEM expression, the lattice for the radial coordinate was chosen as

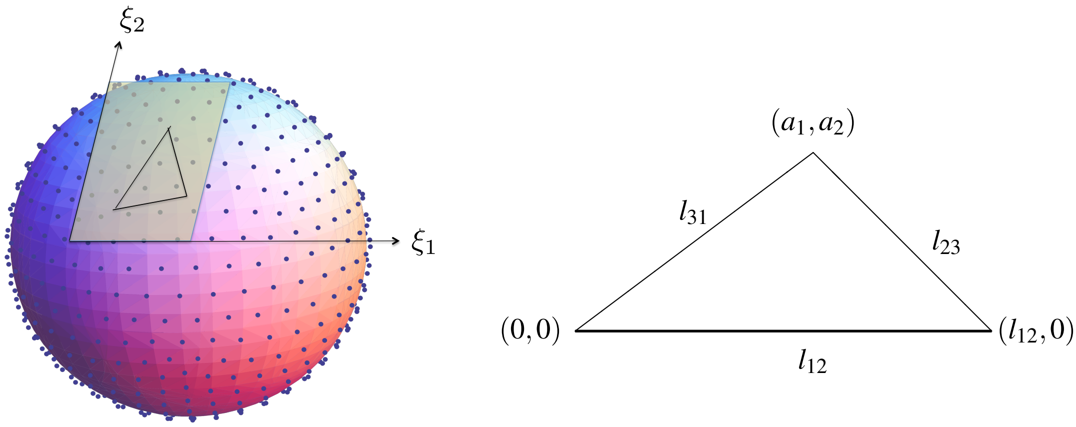

Figure 1: . The coordinates on the sphere x^μ for each spherical triangle are mapped one to one by radial projection on to the tangent plane Cartesian coordinates $\xi_a(x)$.

before with a uniform spacing and the kinetic term, $(\partial_t\phi)^2$, represented by a finite difference. To account for the density $\sqrt{-g}$ in the continuum both the radial kinetic term and the potential terms are weighted by the local volume element, ω_x, computed as $1/3$ the sum of the areas of triangles adjacent to the site x. The resulting discrete FEM action is given by

$$S = \frac{1}{2}\sum_{t,\langle x\,y\rangle} K_{x,y}(\phi_{t,x}-\phi_{t,y})^2 + \frac{1}{2}\sum_{t,x}\omega_x(\phi_{t+1,x}-\phi_{t,x})^2 + \lambda\sum_{t,x}\omega_x(\phi^2_{t,x}-\mu^2/2\lambda)^2\,, \tag{2.4}$$

where (t,x) denotes lattice sites: $t = 0,\cdots,N_t-1$ enumerates the spheres and $x = 1,\cdots,2+10s^2$, the sites on each sphere with refinement level s, starting with $s = 1$ for the unrefined icosahedron. Consequently $1/s$ plays the role of the lattice spacing: $a \sim 1/s$. $K_{x,y}$ are the FEM weights for each link $\langle x\,y\rangle$ on the triangulated $\mathbb{S}^2$ computed from the adjacent triangles using Eq. 2.3. One can prove that the Legendre functions, $Y(\theta_x,\phi_x) \equiv Y_{lm}(\hat{x})$, evaluated on the vertices on the sphere, converge to the continuum orthonormal basis,

$$\lim_{s\to\infty}\sum_{x=1}^{2+10s^2}\omega_x Y^*(\hat{x})_{l'm'}Y_{lm}(\hat{x}) = 4\pi\delta_{l'l}\delta_{m'm} \tag{2.5}$$

in the zero lattice spacing limit.

3. Spectral Properties of the Laplacian on FEM Sphere

Several comments are in order. Contrary to the finite difference approximation on a hypercubic lattice in $\mathbb{R}^D$ in Eq. 2.2 above, the Taylor series expansion of the kinetic term on FEM sphere

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does *not* yield a local rotationally invariant Laplace operator $\nabla^2\phi + O(a^2)$. It is straightforward to perform an explicit Taylor series expansion,

$$a^{-2} \sum_{y\in\langle xy\rangle} K(x,y)[\phi(x) - \phi(y)] \simeq -c_{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + O(a^2) \tag{3.1}$$

to evaluate the symmetric matrix, $c_{\mu\nu}$, as a function of the length of sides of the adjacent triangles. Rotational invariance is generally violated at leading order ($c_{11} \neq c_{22}, c_{12} \neq 0$), except for a few vertices lying on the symmetry axes of the original icosahedron. However, local rotational invariance of this Taylor expansion is not essential to a faithful continuum limit for lattice field theory. For a sequence of more and more refined lattice operators, it is sufficient to require that the spectrum of eigenvalues, well below the cut-off, converge to the exact continuum values. Of course on the regular hypercubic lattice, this spectral condition is also trivially satisfied as can be see by diagonalizing Eq. 2.2 in Fourier space: $\sum_\mu 4a^{-2}\sin^2(ak_\mu/2) \simeq k^2 + O(a^2)$. What we need is the analogous spectral property for our FEM discrete Laplacian on the sphere.

Indeed in FEM literature, there are many theorems on spectral convergence. The general take-away lesson is that if the sequence of grid refinements have simplices that uniformly shrink to zero (*e.g.* a 2D area bounded by $O(a^2)$) and obey an appropriate "shape-regular" condition (*e.g.* in 2D bounds on ratios of angles) then the spectra of the FEM Laplace operator will converge as a^{2p} for p-order finite elements and their eigenvectors will converge to their continuum form as a^p. To test this FEM lore for our extension to the sphere, we are studying the spectral properties numerically. The generalized eigenvalue condition on the FEM Laplacian for a sphere is: $K_{xy}\phi_n(y) = \lambda_n\omega_x\phi_n(x)$. In Fig.2, we compare the lowest 64 eigenvalues for the unweighted ($K_{xy} = 1$) finite difference operator, used in Ref. [1], with that of the improved FEM operator. The $l = 0$ (Singlet) , $l = 1$ (Triplet) and $l = 2$ (Quintet) are in irreducible representations of the icosahedral group and are in excellent agreement for both. However for the $l \geq 3$ eigenvalues, the $2l+1$ degeneracy is clearly much improved by the finite element method even with a modest refinement level of $s = 8$.

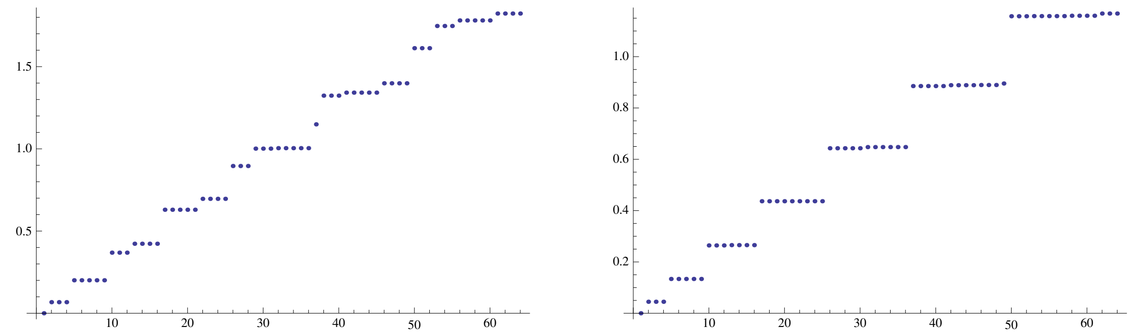

Figure 2: . For $s = 8$, the comparison of the first 64 eigenvalues for the unimproved spectrum discrete icosahedral Laplace operator on the left vs the FEM operator on the sphere on the right.

In Fig. 3, we plot the diagonal matrix elements, $c_{lm} = \sum_{\langle x\,y\rangle} Y^*_{lm}(\hat{x})K(x,y)Y_{lm}(\hat{y})$, as a function of l. On the left, for $s = 8$, all $2 + 10s^2 = 642$ diagonal elements are included and on the right for $s = 128$ the diagonal elements for $l = 0, \cdots, 32$ averaged over m match the continuum form, $l(l+1)$, up to $O(10^{-4})$ corrections. A more thorough analysis of the rate of convergence will be reported in the future, where we will show that this lattice discretization of the Lagrangian on

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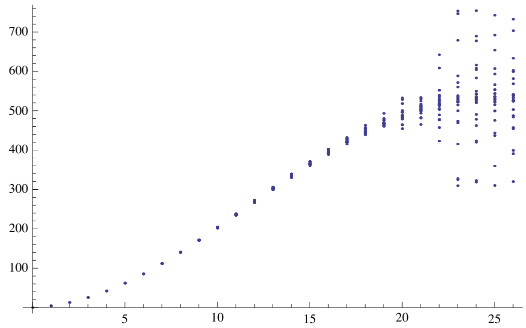

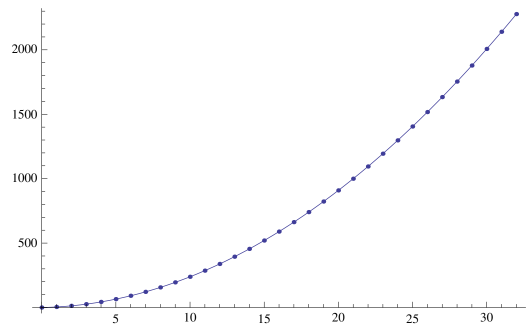

Figure 3: . On the left all the diagonal elements, $c_{lm} = \sum_{\langle x\,y\rangle} Y^*_{lm}(\hat{x})K(x,y)Y_{lm}(\hat{y})$, are plotted against l for $s = 8$ and on the right k_{lm} averaged over m are fitted to $l + 1.00012\, l^2 - 1.34281 \times 10^{-7}\, l^3 - 0.57244 \times 10^{-7}\, l^4$ for $s = 128$ and $l \leq 32$.

$\mathbb{R} \times \mathbb{S}^2$ converges to the continuum for all modes a finite fraction below the cut-off. We conjecture that this is sufficient to guarantee the correct continuum universality at the Wilson-Fisher fixed point.

In a series of pioneering papers, Christ, Friedberg and Lee [3, 4] developed a similar approach to place quantum field theory on random simplicial lattices in flat space. We note however that their application to random lattices violates the "shape-regular" constraint needed here to ensure good spectral properties.

4. Discussion

It is plausible that FEM lattice radial quantization of ϕ^4 considered here will recover the exact CFT with full O(4,1) invariance at the Wilson-Fisher fixed point in the continuum limit. Still both analytical and numerical methods should be pursued to test this conjecture. We note that even in the continuum, the map from $\mathbb{R}^3$ to $\mathbb{R} \times \mathbb{S}^2$ raises questions for ϕ^4 theory, since both μ and λ have dimension of mass. The classical Lagrangian on $\mathbb{R}^3$ is not conformally invariant so after performing the Weyl transformation, we have simply dropped the radial dependence in $r\mu = r_0 e^t \mu$ and $r\lambda = r_0 e^t \lambda$, replacing them by t-independent dimensionless parameters. The implicit assumption we have made is that both traditional $\mathbb{R}^3$ and radial $\mathbb{R} \times \mathbb{S}^2$ quantum theories so defined have identical CFT's at their Wilson- Fisher fixed point in spite of the fact that each have inequivalent deformations for μ and λ away from the fixed point. To test this idea, we are now performing analytical calculations for the 3D non-compact O(N) model for both the traditional and radial quantization in the large N expansion. While this will not prove our conjecture for $N = 1$, it is a useful first step.

To further test our lattice FEM construction for radial quantization, we are also pursuing Monte Carlos simulations using the mixed cluster/Metropolis algorithm of Brower and Tamayo [5]. To date we have located the critical surface in the bare (μ^2, λ) plane as illustrated in Fig. 4 on the right. On the left we compare this with the phase plane of ϕ^4 in the continuum on $\mathbb{R}^3$ determined with the ε expansion. The similarity between the two critical surfaces is reassuring but we are just beginning to do high precision simulations to strength this comparison. To locate the fixed point in the radial quantized critical surface, we are employing the methods of Hasenbusch [6, 7]. By

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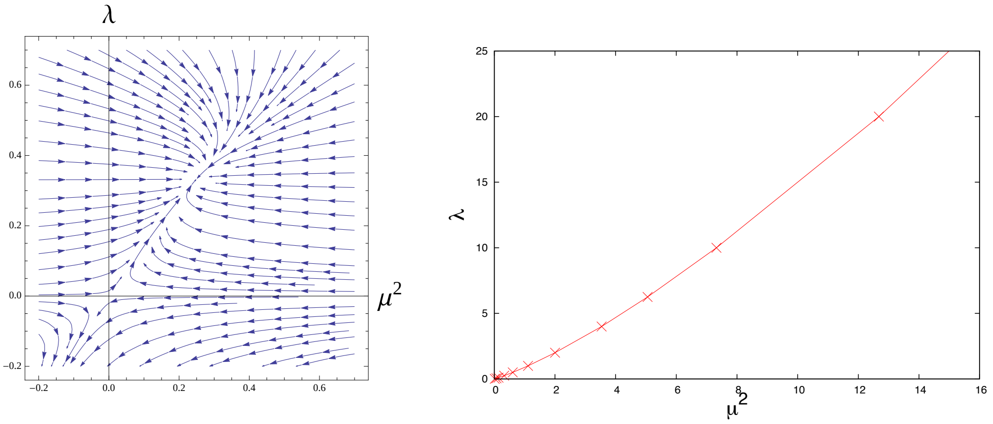

Figure 4: On the left is the RG flow in the (λ, μ^2) phase plan in the continuum epsilon expansion on $\mathbb{R}^3$ compared on the right with the critical surface for the lattice radial theory on $\mathbb{R} \times \mathbb{S}^2$.

setting λ to its fix point value, we seek an improved action to help to compute more accurately the anomalous dimensions (or critical exponents) in both the even and odd Z_2 sectors. Our first goal is to determine if the defect in the 3rd descendant found on the Ising icosahedron is removed by the FEM improvement action when extrapolated to the continuum.

We are also generalizing the FEM method to place both gauge fields and fermions on $\mathbb{S}^{D-1}$ in order to be able to apply radial quantization to a wide class of gauge theories with fermionic matter. For the scalar field the generalization of FEM to 4D field theories on $\mathbb{R} \times \mathbb{S}^3$ using tetrahedral simplices is not difficult. However the generalization of FEM methods to higher spin fermionic and gauge field is more involved. It requires a more detailed use of geometric objects, most importantly the verbein $e^a_\mu(x) = \partial_\mu \xi^a(x)$, relating the curved manifold to the local tangent plane, familiar to the application of continuum field theory in general relativity. Nonetheless we are optimistic that the FEM can be applied very generally to lattice radial quantization and that we can in due time explore numerically a wide range of consequences for conformal and IR conformal theories.

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