

Perturbative analysis of twisted volume reduced theories

Margarita García Pérez*

Instituto de Física Teórica UAM/CSIC, Universidad Autónoma de Madrid, E-28048-Madrid, Spain

E-mail: margarita.garcia@uam.es

Antonio González-Arroyo

Instituto de Física Teórica UAM/CSIC and Departamento de Física Teórica, C-15, Universidad Autónoma de Madrid, E-28049-Madrid, Spain

E-mail: antonio.gonzalez-arroyo@uam.es

Masanori Okawa

Graduate School of Science, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan

E-mail: okawa@sci.hiroshima-u.ac.jp

We discuss the perturbative expansion of $SU(N)$ Yang-Mills theories defined on a d -dimensional torus of linear size l with twisted boundary conditions, generalizing previous results in the literature. For a specific class of twist tensors depending on a single integer flux value k , we show that perturbative results to all orders depend on the combination $lN^{2/d}$ and a flux-dependent angle $\tilde{\theta}$. This implies a new kind of volume independence that holds at finite N and for fixed values of $\tilde{\theta}$. Our results also provide interesting information about the possible occurrence of tachyonic instabilities at one-loop order. We support the prescription that instabilities are avoided, if the large N limit is taken keeping $\tilde{\theta} > \tilde{\theta}_c$, and appropriately scaling the magnetic flux k with N . Numerical results in 2+1 dimensions provide a test of how these ideas extend into the non-perturbative regime.

31st International Symposium on Lattice Field Theory LATTICE 2013

July 29 - August 3, 2013

Mainz, Germany

*Speaker.

1. Introduction

In a recent paper [1], focused on the study of 2+1 dimensional $SU(N)$ Yang-Mills theory defined on a 2-d torus, we have analyzed the interplay between the rank of the group N and the finite volume effects in the presence of a non-trivial magnetic flux. We have presented perturbative and numerical evidence of a kind of volume independence in the theory, reflected in the combined Nl dependence of physical quantities, l being the size of the 2-dimensional spatial manifold. Here, we will generalize the perturbative results to the case of a four dimensional set-up with the kind of twisted boundary conditions relevant for Twisted Eguchi-Kawai (TEK) reduction, as put forward by two of the present authors [2, 3]. We will start by presenting the perturbative set-up for a Yang-Mills theory with twisted boundary conditions, and the interplay between finite volume and finite N effects in this context. We will continue by discussing possible caveats to the volume independence conjecture, including the presence of tachyonic instabilities [4] and the appearance of symmetry breaking in the TEK model for large values of N [5]. We will argue that the prescription to scale the magnetic twist with N , put forward in Refs. [2, 3] to prevent the latter, also succeeds in avoiding the tachyonic behaviour. We will end by presenting some numerical results in 2+1 dimensions that provide a test on the realization of volume independence at a non-perturbative level.

2. Perturbative set-up

To set the stage, let us start with a brief and general introduction on the formulation of the $SU(N)$ Yang-Mills theory with twisted boundary conditions. The notation and the discussion will follow the review [6]. We start with a manifold composed of a d -dimensional torus, of lengths l_μ , times some non-compact extended directions. The latter will not be relevant for our purposes and will be neglected in the discussion below. Gauge fields on this base space satisfy periodicity conditions in the compact directions given by:

$$A_\mu(x + l_\nu \hat{e}_\nu) = \Omega_\nu(x) A_\mu(x) \Omega_\nu^\dagger(x) + i \Omega_\nu(x) \partial_\mu \Omega_\nu^\dagger(x), \quad (2.1)$$

where the $SU(N)$ twist matrices $\Omega_\mu(x)$ are subject to the consistency conditions:

$$\Omega_\mu(x + l_\nu \hat{e}_\nu) \Omega_\nu(x) = Z_{\mu\nu} \Omega_\nu(x + l_\mu \hat{e}_\mu) \Omega_\mu(x), \quad (2.2)$$

with $Z_{\mu\nu} = \exp\{2\pi i n_{\mu\nu}/N\}$, and $n_{\mu\nu} \in \mathbb{Z}_N$. In what follows, we will focus on the case of constant twist matrices $\Omega_\mu(x) = \Gamma_\mu$. They are known under the name of twist-eaters. For the so-called irreducible twists, it can be shown that they are uniquely defined modulo global gauge transformations (similarity transformations) and multiplication by an element of \mathbb{Z}_N [6]. We will be considering here the case of even number of twisted compactified directions d . If we choose $N = L^{d/2}$, with $L \in \mathbb{Z}$, the twist $n_{\mu\nu} = \varepsilon_{\mu\nu} k N / L$, with $\varepsilon_{\mu\nu} = \Theta(\nu - \mu) - \Theta(\mu - \nu)$ (where Θ is the step function), is irreducible if k and L are co-prime. In what follows we will focus on this specific twist choice.

The periodicity constraint:

$$A_\mu(x + l_\nu \hat{v}) = \Gamma_\nu A_\mu(x) \Gamma_\nu^\dagger, \quad (2.3)$$

can be resolved by introducing a basis of the space of $N \times N$ matrices satisfying:

$$\Gamma_\mu \hat{\Gamma}(p^{(c)}) \Gamma_\mu^\dagger = e^{i l_\mu p_\mu^{(c)}} \hat{\Gamma}(p^{(c)}) \quad . \quad (2.4)$$

The basis can be constructed in terms of the twist matrices as:

$$\hat{\Gamma}(p^{(c)}) = \frac{1}{\sqrt{2N}} e^{i\alpha(p^{(c)})} \Gamma_0^{s_0} \dots \Gamma_{d-1}^{s_{d-1}} \quad (2.5)$$

with $\alpha(p^{(c)})$ an arbitrary phase factor, and:

$$p_\mu^{(c)} = \frac{2\pi}{Ll_\mu} \varepsilon_{\mu\nu} k s_\nu, \quad (2.6)$$

with s_ν integers defined modulo L . In the case of irreducible twists, one can show that there are N^2 linearly independent such matrices which, excluding the identity, provide a basis of the $SU(N)$ Lie algebra. The phase factors, $\alpha(p^{(c)})$, can be chosen to satisfy the following commutation relations:

$$[\hat{\Gamma}(p), \hat{\Gamma}(q)] = iF(p, q, -p - q) \hat{\Gamma}(p + q), \quad (2.7)$$

with

$$F(p, q, -p - q) = -\sqrt{\frac{2}{N}} \sin\left(\frac{\theta_{\mu\nu}}{2} p_\mu q_\nu\right), \quad (2.8)$$

playing the role of the $SU(N)$ structure constants in this particular basis. Here,

$$\theta_{\mu\nu} = \frac{L^2 l_\mu l_\nu}{4\pi^2} \times \tilde{\varepsilon}_{\mu\nu} \tilde{\theta}, \quad (2.9)$$

with $\tilde{\varepsilon}_{\mu\nu} \varepsilon_{\nu\sigma} = \delta_{\mu\sigma}$, and the angle $\tilde{\theta} = 2\pi\bar{k}/L$, where \bar{k} is an integer satisfying $k\bar{k} = 1 \pmod{L}$.

We can now expand the gauge fields in this basis:

$$A_\nu(x) = \frac{1}{\sqrt{V}} \sum_p e^{ip \cdot x} \hat{A}_\nu(p) \hat{\Gamma}(p), \quad (2.10)$$

with momenta decomposed as $p_\mu = p_\mu^{(s)} + p_\mu^{(c)}$, the sum of a colour-momentum part $p_\mu^{(c)}$, and a spatial-momentum part quantized in the usual way: $p_\mu^{(s)} = 2\pi m_\mu / l_\mu$, with $m_\mu \in \mathbb{Z}$. The prime implies that the zero colour-momentum component is excluded from the sum. Neglecting this issue, the momentum is quantized as if the theory lived in an effective torus with sizes $l_\mu^{\text{eff}} = Ll_\mu$. Note also that the Fourier coefficients $\hat{A}_\nu(p)$ are just complex numbers, so that all the effect of the colour is translated into the momentum dependence of the group structure constants.

One can now easily generalize the Feynman rules derived for $d = 2$ in Ref. [1], to arrive to the conclusion that all vertices in perturbation theory are proportional to the factor:

$$\frac{g}{\sqrt{V}} F(p, q, -p - q) = -\sqrt{\frac{2\lambda}{V_{\text{eff}}}} \sin\left(\frac{\theta_{\mu\nu}}{2} p_\mu q_\nu\right), \quad (2.11)$$

where $V_{\text{eff}} \equiv \prod_\mu l_\mu^{\text{eff}}$. This peculiar momentum dependent Feynman rules relate the twisted theory with a non-commutative Yang-Mills theory with non-commutativity parameter $\theta_{\mu\nu}$ [7]-[9].

3. Volume independence

Let us now analyze the dependence of the results on the rank of the gauge group N and the sizes of the torus l_μ . Recalling the definition of $\theta_{\mu\nu}$ in Eq. (2.9) and the momentum quantization rule, it is clear that all the N and l_μ dependence enters only through the combination $l_\mu^{\text{eff}} = Ll_\mu$, and the angle $\tilde{\theta} = 2\pi\bar{k}/L$. This implies that the perturbative expansion, at fixed $\tilde{\theta}$ and l^{eff} , depends in an indistinguishable way on N and the torus size. We dub this phenomenon *volume reduction* or *volume independence* at finite N . A limiting case would be TEK reduction which applies to a discretized version of the Yang-Mills theory in which $l_\mu = a$ (the lattice spacing). As a matter of fact, TEK models have been used as a regularized version of non-commutative gauge theories with non-commutativity parameter:

$$\theta_{\mu\nu}^{\text{TEK}} = \frac{L^2 a^2}{4\pi^2} \times \tilde{\epsilon}_{\mu\nu} \tilde{\theta}, \quad (3.1)$$

Beyond perturbation theory, there are, however, possible caveats to the volume independence conjecture. As mentioned previously, several authors realized the presence of tachyonic instabilities in certain non-commutative theories [4]. These extend to ordinary theories with twisted boundary conditions and present a menace to the volume independence mechanism. The problem occurs at one loop in perturbation theory. In Ref. [1] we saw how the problem does not arise in 2+1 dimensions if one adopts the large N prescription given in Ref. [3]. The argument extends to 4 dimensions as well, and goes as follows. The transverse part of the 2-point vertex function has a non-zero value at leading order, since twisted boundary conditions eliminate zero-momentum gluons. This contribution is proportional to $|p|^2 \sim 4\pi^2/l_{\text{eff}}^2$. At one loop, the self-energy contribution is negative and proportional to

$$\lambda \frac{\tilde{p}_\mu \tilde{p}_\nu}{|\tilde{p}|^4} \quad (3.2)$$

where $\tilde{p}_\mu = \theta_{\mu\nu} p_\nu$. Instability arises if the second term is larger than the first. This occurs for large enough λ where the calculation is unreliable. However, since the first term goes to zero as l_{eff} goes to infinity, one might wonder if instability could arise in that limit. The prescription given in Ref. [3] amounts to taking the $l_{\text{eff}} \rightarrow \infty$ limit with $\theta_{\mu\nu}$ given by Eq. (2.9) and keeping $\tilde{\theta}$ fixed. Plugging this expression in Eq. (3.2), one sees that the negative self-energy also goes to zero as l_{eff} goes to infinity and the critical λ remains finite, and of order $\tilde{\theta}^2$. Thus, as supported by our numerical results, no instability should arise for $\tilde{\theta} > \tilde{\theta}_c$. The previous evidence for instability occurred when taking a different limit, in which $\tilde{\theta}$ was decreasing to zero as $1/L$.

Additional complications could arise from the fact that it is impossible to strictly keep $\tilde{\theta}$ fixed as N changes. This is so because $\tilde{\theta}$ is a rational number with coprime rational factors \bar{k} and L . Smoothness of physical quantities on $\tilde{\theta}$ is thus required for volume independence to hold. Our results for the electric flux energies and the perturbative glueball spectrum in 2+1 dimensions exhibit such a smooth dependence, but the issue is difficult to settle in general terms and it has indeed been discussed profusely in the context of non-commutative gauge theories without conclusive results (see e.g. [10], [11]). Finally, effects arising from non-perturbative physics might not respect the l^{eff} dependence. There are indications that this is so for certain twist choices in TEK models, where reduction fails due to spontaneous symmetry breaking at large N [5]. It has been recently shown, however, that symmetry breaking can be avoided if (in addition to keeping $\tilde{\theta} > \tilde{\theta}_c$) the magnetic flux is scaled with L , as L goes to infinity [2, 3].

Most of these questions can only be addressed non-perturbatively. In what follows we will summarize the results of a numerical analysis for the case of 2+1 dimensions [1]. The analysis for $d = 4$ is ongoing and will be presented elsewhere.

4. Non-perturbative results in 2+1 dimensional SU(N) Yang-Mills theory

Our analysis will be focused on the study of the electric flux energies, \mathcal{E} , extracted from Polyakov loop correlators. It is easy to show that these operators carry electric flux $e_i = -\varepsilon_{ijn}\bar{k}$, determined by the gluon colour momentum $\vec{p} = 2\pi\vec{n}/(Nl)$. In perturbation theory at one-loop, a compact formula, exhibiting a smooth $\tilde{\theta}$ dependence, has been derived in Ref. [1]:

$$\frac{\mathcal{E}^2}{\lambda^2} = \frac{|\vec{n}|^2}{4x^2} - \frac{1}{x}G\left(\frac{\vec{e}}{N}\right), \quad (4.1)$$

where

$$G(z) = \frac{1}{16\pi^2} \int_0^\infty \frac{dt}{\sqrt{t}} \left(\theta_3^2(0,t) - \prod_{i=1}^2 \theta_3(z, it) - \frac{1}{t} \right), \quad (4.2)$$

in terms of the Jacobi theta function: $\theta_3(z, it) = \sum_{n \in \mathbf{Z}} \exp\{-t\pi n^2 + 2\pi inz\}$. We have introduced the dimensionless parameter $x = Nl\lambda/4\pi$. Note that in 3 dimensions the coupling constant is dimensionful. Thus, all energy scales can be expressed in units of λ and the resulting dimensionless quantities should appear in perturbation theory as a power series in λl . Combining this information with volume independence, we conclude that the relevant scale parameter in perturbation theory is precisely x (for a similar statement involving $Nl\Lambda_{QCD}$ in 4-d see [12] and [10]).

In Ref. [1] we have presented evidence that x -scaling holds beyond perturbation theory for choices of the twist that do not exhibit tachyonic instability. As an illustration, we present in the left plot of Fig. 1 an analysis of electric-flux states with minimal momentum $\vec{p} = (2\pi/Nl, 0)$. We display the x dependence of the combination $x^2\mathcal{E}^2/\lambda^2$, for $(N, \bar{k}) = (7, 2)$ and $(17, 5)$, corresponding to very close values of $\tilde{\theta}$. We stress the striking similarity of the results despite the very different values of N . Other results, for varying N and \bar{k} , confirm this conclusion, giving support to the conjecture of a universal x -dependence for fixed $\tilde{\theta}$. The data show that the small- x behaviour follows the perturbative formula, starting at the tree-level result $x^2\mathcal{E}^2/\lambda^2 = 0.25$. At large torus sizes, we expect a linear growth of the energy that can be cast in a form that also exhibits x -scaling:

$$\frac{\mathcal{E}(\vec{e}/N)}{\lambda} = 4\pi x \frac{\sigma'}{\lambda^2} \phi\left(\frac{\vec{e}}{N}\right), \quad (4.3)$$

where the string tension, for electric flux \vec{e} , has been parametrized as: $\sigma_{\vec{e}} = N\sigma'\phi(\vec{e}/N)$. Higher order string corrections to this formula can also be taken into account, including the contribution of Kalb-Ramond B -fields which play an important role in the twisted set-up [10]. Indeed, the observation that the B -field contribution amounts precisely to the perturbative tree-level term in Eq. (4.1) has guided us in the search for an x -dependent parameterization that fits very well the data. The reader is referred to Ref. [1] for further details and a full account of the results. Incidentally, let us mention that we have analyzed the dependence of the string tension on the electric flux, finding a clear preference for Sine scaling with $\phi(z) = \sin(\pi z)/\pi$, over Casimir scaling with $\phi(z) = z(1-z)$.

Our results allow to analyze in detail the issue of tachyonic instabilities. The general perturbative argument presented in Sec. 3 can be refined using Eq. (4.1). As already discussed, the one-loop

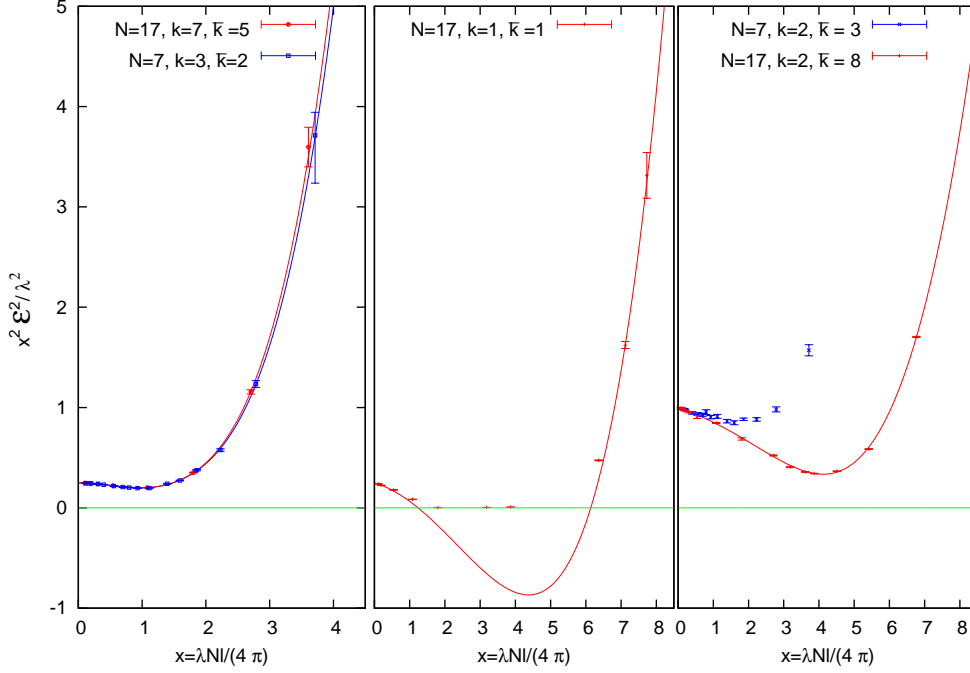


Figure 1: We display $x^2 \mathcal{E}^2 / \lambda^2$, as a function of x , for electric flux states with momentum: $p = (2\pi/Nl, 0)$ (Left, Center), and $p = (4\pi/Nl, 0)$ (Right).

correction, being negative, could give rise to a tachyonic excitation. This occurs above a critical coupling: $x_c(\vec{e}) = |\vec{n}|^2 / (4G(\vec{e}/N))$. The quantity $G(\vec{z})$ diverges as $1/|\vec{z}|$ for small $|\vec{z}|$. This would seem to unavoidably drive x_c to zero as N goes to infinity. However, given the relation $n_i = -\varepsilon_{ijk} e_j$, it suffices to scale $k \sim \sqrt{N}$ to push x_c into the non-perturbative domain. On the opposite side, if we look at minimal momentum $|\vec{n}| = 1$, the electric flux is given by \bar{k} and the critical coupling occurs at $x_c(\vec{e}) = 4\pi^2 \bar{k} / N \equiv 2\pi \tilde{\theta}$. Thus, keeping $\tilde{\theta} > \tilde{\theta}_c$ the perturbative instability is avoided.

One can still worry about the possible appearance of non-perturbative instability. An argument based on the effective string description has been used in Ref. [1] to indicate that this is avoided if: $|\vec{n}| |\vec{e}| (N - |\vec{e}|) / N^2 > 1/12$. Still, a full proof should rely on numerical results. In our previous work [1], we have performed simulations at several values of $\tilde{\theta}$ and k/N . All the results with $\tilde{\theta} > 2\pi/7$ and $k/N > 2/17$ showed no indication of instability. A representative sample is presented in Fig. 1. In addition to the stable case already discussed, we display two cases in which k/N becomes small. They correspond to electric flux $|\vec{e}| = 1$, with $k = \bar{k} = 1$ (central plot) and $k = 2$, $\bar{k} = (N - 1)/2$ (right plot). In the first case (with $N = 17$), as the energy squared decreases from the perturbative tree-level value it touches, at some point, zero. This signals the appearance of an instability. Of course, the energy does never become tachyonic and what we observe instead is a region where the electric flux has a small mass, possibly reflecting a non zero vacuum expectation value of the Polyakov loop. In the large x regime though, the linearly rising potential overcomes this behaviour and restores the standard x -dependence. In the second case, we display two values of $k/N = 2/7$ and $2/17$. They are still large enough to prevent the appearance of instability. However, we observe a strong decrease in the energy of electric flux with decreasing k/N . Assuming this trend continues, one expects an instability to set-in below a given value of this ratio.

5. Conclusions

We have given a unified description of the perturbative expansion of $SU(N)$ Yang-Mills theory on an even dimensional torus traversed by $Z(N)$ magnetic flux through each plane. We stress the emergence of an effective size parameter combining spatial and group degrees of freedom. Our results, valid at finite N , provide important information on large N reduced models and the twisted volume reduction mechanism. In particular, they support the prescription given in Ref. [3] on how the flux has to scale with the rank N . A few numerical results in 2+1 dimensions for the energies of electric flux sectors support the applicability of these ideas at the non-perturbative level. A good description is given of the evolution of these energies at all scales.

Acknowledgments

M.G.P. and A.G-A acknowledge support from the grants FPA2012-31686 and FPA2012-31880, the MINECO Centro de Excelencia Severo Ochoa Program SEV-2012-0249, the Comunidad Autónoma de Madrid HEPHACOS S2009/ESP-1473, and the EU PITN-GA-2009-238353 (STRONGnet). They participate in the Consolider-Ingenio 2010 CPAN (CSD2007-00042). M. O. is supported by the Japanese MEXT grant No 23540310. We acknowledge the use of the IFT clusters.

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