

Non- γ_5 hermiticity minimal doubling fermion

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We formulate new two dimensional fermions breaking γ_5 hermiticity, based on the minimal doubling fermion. We investigate their properties: (I) Symmetries, (II) eigenvalue distributions, and (III) the number of poles. As a simple application of the fermions, the Gross-Neveu model in two dimensions is studied using the fermion. We obtain the parity broken phase diagram called Aoki phase and the chiral broken phase diagram of the model with an imaginary chemical potential.

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1. Motivation

In lattice gauge theory, we often encounter situations that γ_5 hermiticity cannot be preserved, e.g., finite density theory, supersymmetric theory, and so on. Since γ_5 hermiticity is needed for reality of Dirac determinant, most of the observables are complex numbers in the case. In γ_5 hermiticity broken theory, such as finite density theory, the partition function does not converge, so that the analysis of high density region using lattice simulation breaks down even if the reweighting method are employed. The serious problem is called the sign problem. Additionally, in odd dimensions, γ_5 hermiticity cannot be defined because we cannot define an irreducible matrix which anti-commutes with all spatial γ -matrices. For the reasons mentioned above, it is important to know the effects of γ_5 hermiticity breaking. In this work, we consider some effects of γ_5 hermiticity breaking. To examine the effects, we firstly formulate two dimensional fermions without γ_5 hermiticity, based on the minimal doubling fermion [1, 2], and then investigate some properties of the fermions. We also apply the fermion to the Gross-Neveu model [3] and obtain parity broken and chiral broken phase diagrams.

2. Two dimensional non- γ_5 hermiticity fermion

Firstly, we define two dimensional fermions without γ_5 hermiticity (non- γ_5 hermiticity fermion), based on the minimal doubling fermion which preserves exact chiral symmetry [1, 2],

$$D_1^{(2)}(p) = \sum_{\mu=1,4} i \sin p_\mu \cdot \gamma_\mu + \kappa(1 - \cos p_1) \cdot \gamma_1, \quad (2.1)$$

$$D_2^{(2)}(p) = \sum_{\mu=1,4} i \sin p_\mu \cdot \gamma_\mu + \kappa(1 - \cos p_4) \cdot \gamma_1, \quad (2.2)$$

$$D_3^{(2)}(p) = \sum_{\mu=1,4} i \sin p_\mu \cdot \gamma_\mu + \kappa \sum_{\mu=1,4} (1 - \cos p_\mu) \cdot \gamma_\mu, \quad (2.3)$$

$$D_4^{(2)}(p) = \sum_{\mu=1,4} i \sin p_\mu \cdot \gamma_\mu + \kappa \sum_{\mu,v=1,4} (1 - \cos p_\mu) \cdot \gamma_v, \quad (2.4)$$

$$D_5^{(2)}(p) = \sum_{\mu=1,4} i \sin p_\mu \cdot \gamma_\mu + \kappa \sum_{\mu=1,4} (1 - \cos p_\mu) \cdot \gamma_1, \quad (2.5)$$

where κ is a hopping parameter which is a real number¹. To investigate symmetries of the fermions, we define discrete transformations as

$$\begin{aligned} C &: D(p_1, p_4) \rightarrow -CD^\top(-p_1, -p_4)C^{-1}, \\ P &: D(p_1, p_4) \rightarrow \gamma_4 D(-p_1, p_4) \gamma_4, \\ T &: D(p_1, p_4) \rightarrow \gamma_1 D(p_1, -p_4) \gamma_1, \end{aligned} \quad (2.6)$$

where C is a charge conjugation matrix. In two dimensions, we can choose $C = i\gamma_1$ where $\gamma_1 = \sigma^2$ and $\gamma_4 = \sigma^1$. We also define chiral symmetry and γ_5 hermiticity as,

$$\begin{aligned} \text{chiral} &: D(p_1, p_4) = -\gamma_5 D(p_1, p_4) \gamma_5, \\ \gamma_5 \text{hermiticity} &: D(p_1, p_4) = \gamma_5 D^\dagger(p_1, p_4) \gamma_5, \end{aligned}$$

	C	P	T	CP	CT	PT	CPT	chi	γ_5 h
$D_1^{(2)}$	×	×	○	○	×	×	○	○	×
$D_2^{(2)}$	×	×	○	○	×	×	○	○	×
$D_3^{(2)}$	×	×	×	×	×	×	○	○	×
$D_4^{(2)}$	×	×	×	×	×	×	○	○	×
$D_5^{(2)}$	×	×	○	○	×	×	○	○	×

Table 1: Discrete symmetries, chiral symmetry and γ_5 hermiticity for the Dirac operators $D_{1-5}^{(2)}$.

where γ_5 is a chiral matrix defined as $\gamma_5 = i\gamma_1\gamma_4$. All the Dirac operators preserve exact chiral symmetry and CPT symmetry but break C, P, CT, PT and cubic symmetries. And we can see that $D_1^{(2)}$, $D_2^{(2)}$ and $D_5^{(2)}$ also preserve T and CP symmetries. We summarize symmetries for the Dirac operators in the Table 1.

To understand more about the fermions, we examine the eigenvalue distributions and the number of poles. The Fig.1 shows eigenvalue distribution of the fermions. We clearly see that the eigenvalues of the $D_2^{(2)}$, $D_3^{(2)}$, $D_4^{(2)}$ and $D_5^{(2)}$ spread over the $\text{Re}\lambda\text{-Im}\lambda$ plane entirely in the continuum limit. On the other hand, in the $D_1^{(2)}$ case, the eigenvalues localize along the $\text{Im}\lambda$ axis in the limit. Fig.2 represents the number of poles of the fermions at $\kappa = 0.5, 1$ and 2 . The $D_1^{(2)}$ at $\kappa \neq 1$ generates $p = (0, 0)$ and $(0, \pi)$. However, the other fermions have more than two or odd number of poles.

Using appropriate regularization, such as small mass and anti-boundary condition, the Dirac determinant of $D_{2-5}^{(2)}$ can be evaluated. However, in the principle of lattice gauge theory, any lattice fermion must return the continuum fermion in the continuum limit. The $D_{2-5}^{(2)}$ return the continuum classical action, but the eigenvalues of the Dirac operators do not recover in the limit. Therefore, the Dirac operators should be excluded from a candidate for lattice theory.

3. Gross-Neveu model in two dimensions

We apply the non- γ_5 hermiticity fermion to the two dimensional Gross-Neveu model. The model can be solved exactly in large- N limit using the saddle point approximation. In this limit, the parity broken phase diagram, called Aoki phase[4], can be obtained. We also focus on the model with an imaginary chemical potential and the chiral symmetry broken phase diagram [6]. In this work, we apply only the Dirac operator $D_1^{(2)}$ defined in Eq.(3.4). At a first sight, the fermion seems to be improper for a practical calculation because it does not preserve γ_5 hermiticity. The analyzes mentioned below are verification for application of the fermion to simple cases.

¹In this paper, the indices “1” and “4” denote temporal and spacial components, respectively.

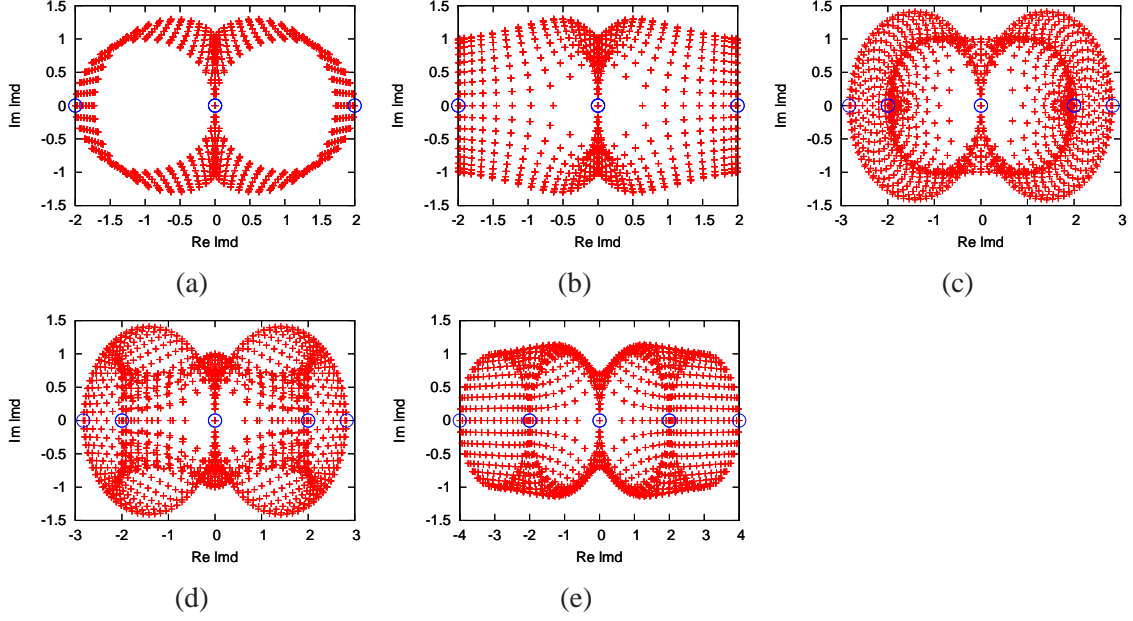


Figure 1: The eigenvalue distribution of Dirac operators, (a) $D_1^{(2)}$, (b) $D_2^{(2)}$, (c) $D_3^{(2)}$, (d) $D_4^{(2)}$, and (e) $D_5^{(2)}$. The horizontal and vertical axes denote the real part and the imaginary part of eigenvalues, respectively. The hopping parameter, the fermion mass and lattice size are fixed at $\kappa = 1$, $m = 0$ and $L^2 = 36 \times 36$. The blue circle points denote eigenvalues for momenta $p = (0, 0)$, $(0, \pi)$, $(\pi, 0)$, and (π, π) .

We define the lattice action of the Gross-Neveu model as

$$S_{\text{lat,aux}}^{\text{GN}} = \sum_{n,m} \bar{\psi}_n \left[\bar{D}_{1nm}^{(2)} + (m + \sigma_n + \pi_n i \gamma_5) \delta_{n,m} \right] \psi_m + \frac{N}{2} \sum_n \left[\frac{\sigma_n^2}{g_\sigma^2} + \frac{\pi_n^2}{g_\pi^2} \right], \quad (3.1)$$

$$S_{\text{lat,aux}}^{\text{GNim}} = \sum_{n,m} \bar{\psi}_n \left[D_{1nm}^{(2)} + (\sigma_n + \pi_{4n} i \gamma_4) \delta_{n,m} \right] \psi_m + \frac{N}{2} \sum_n \left[\frac{1}{g_\sigma^2} (\sigma_n - m)^2 + \frac{1}{g_\pi^2} (\pi_{4n} - \mu)^2 \right], \quad (3.2)$$

where μ is a chemical potential, and σ , π , and π_4 are an auxiliary scalar field, an auxiliary pseudo-scalar, and an pseudo-vector field, respectively. The lower indices in these actions denote a coordinate in the lattice space. We omit the flavor indices, $\bar{\psi}\psi \equiv \sum_{i=1}^N \bar{\psi}^i \psi^i$. We can obtain original actions including four-fermi couplings by integrating out the auxiliary fields σ , π , and π_4 . These actions preserve not only $U_V(1)$ symmetry but also chiral \mathbf{Z}_4 symmetry in the massless case. In the (3.2) case, chiral \mathbf{Z}_4 symmetry is enhanced to $U_A(1)$ symmetry at $g_\sigma^2 = g_\pi^2$.

Now, we choose two Dirac operators for the Gross-Neveu model with or without an imaginary chemical potential as follows:

$$\bar{D}_{1nm}^{(2)} = \sum_{\mu=1,4} \frac{1}{2} (\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2} (2\delta_{n,m} - \delta_{n+\hat{4},m} - \delta_{n-\hat{4},m}) \cdot \gamma_4, \quad (3.3)$$

$$D_{1nm}^{(2)} = \sum_{\mu=1,4} \frac{1}{2} (\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}) \cdot \gamma_\mu + \frac{\kappa}{2} (2\delta_{n,m} - \delta_{n+\hat{1},m} - \delta_{n-\hat{1},m}) \cdot \gamma_1, \quad (3.4)$$

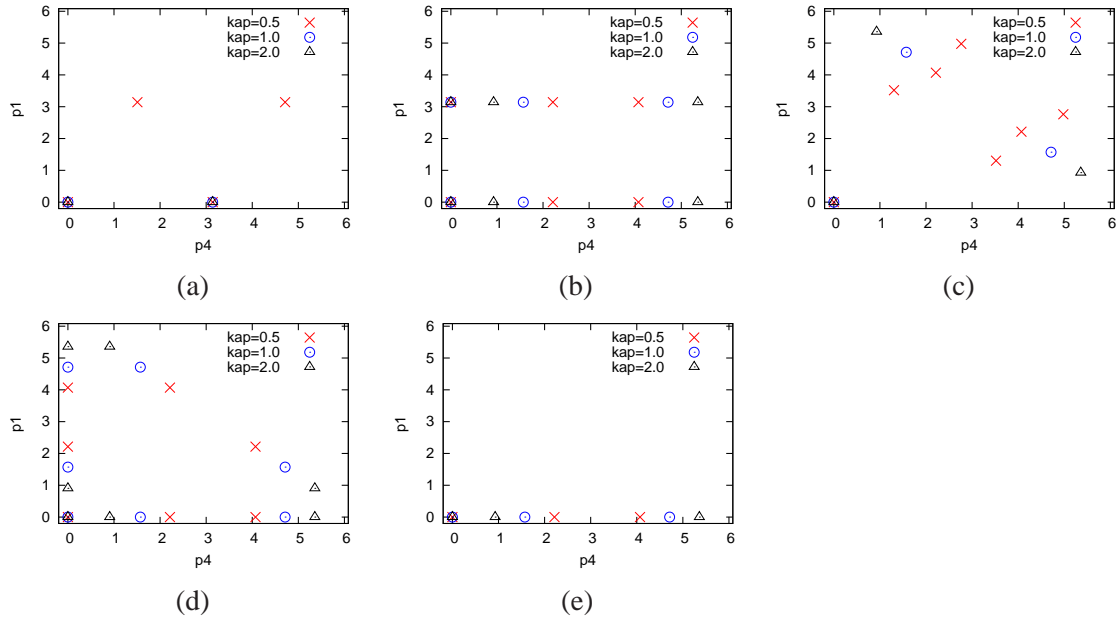


Figure 2: The poles of five Dirac operators, (a) $D_1^{(2)}$, (b) $D_2^{(2)}$, (c) $D_3^{(2)}$, (d) $D_4^{(2)}$, and (e) $D_5^{(2)}$. The horizontal and vertical axes denote momenta p_4 and p_1 , respectively. The cross points, circle points, and triangle points represent poles at $\kappa = 0.5, 1$, and 2 , respectively.

In the large N limit, we can derive the following gap equations for the critical line of Aoki phase using the saddle point approximation,

$$\frac{\sigma_0}{g_\sigma^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2\sigma_{m_c}}{\sigma_{m_c}^2 + H(k)}, \quad \frac{1}{g_\pi^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2}{\sigma_{m_c}^2 + H(k)}, \quad (3.5)$$

where

$$H(k) = \sin^2 k_1 + [\sin k_4 - i\kappa(1 - \cos k_4)]^2, \quad (3.6)$$

m_c is a critical mass, and $\sigma_{m_c} = m_c + \sigma_0$. And the lower index 0 denotes the solutions for the saddle point approximation. In the similar way, gap equations for the chiral broken phase are obtained as,

$$\frac{1}{g_\sigma^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2}{\tilde{H}(k)}, \quad \frac{\pi_{40} - \mu_c}{g_\pi^2} = \int \frac{d^2k}{(2\pi)^2} \frac{2(\pi_{40} + \sin k_4)}{\tilde{H}(k)}, \quad (3.7)$$

where

$$\tilde{H}(k) = [\sin k_1 - i\kappa(1 - \cos k_1)]^2 + (\sin k_4 + \pi_{40})^2, \quad (3.8)$$

and μ_c is a critical chemical potential. We also obtain the Aoki phase of the fermion adding flavored mass [5] defined as

$$m \rightarrow m + m_f(p),$$

$$m_f(p) = \begin{cases} m_f \cdot \cos p_1 \cos p_4 & \text{for the naive fermion} \\ m_f \cdot \cos p_1 & \text{for the non- } \gamma_5 \text{hermiticity fermion} \end{cases}, \quad (3.9)$$

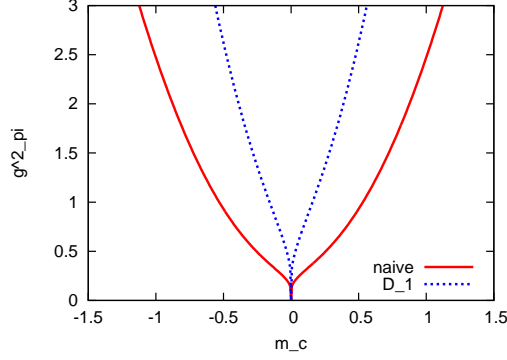


Figure 3: The Aoki phase using the naive fermion (Red line) and the non- γ_5 hermiticity fermion $\bar{D}_1^{(2)}$ (Blue line) at $g_\sigma^2 = g_\pi^2/2$. We fixed the hopping parameter of the non- γ_5 hermiticity fermion at $\kappa = 1$. The horizontal and vertical axes denote the critical mass m_c and the squared four-fermi coupling constant g_π^2 , respectively.

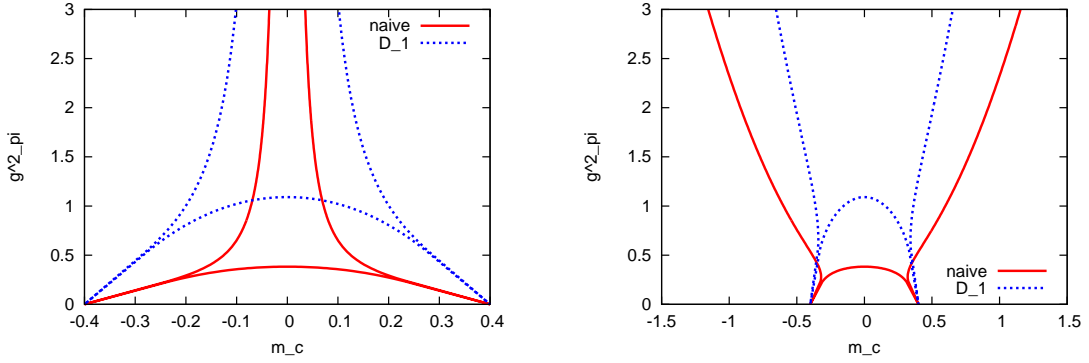


Figure 4: The Aoki phase using the naive fermion (Red line) and the non- γ_5 hermiticity fermion $\bar{D}_1^{(2)}$ (Blue line) adding the flavored mass term at (Left) $g_\sigma^2 = g_\pi^2$ and (Right) $g_\sigma^2 = g_\pi^2/2$. We fix the factor of flavored mass and the hopping parameter of the non- γ_5 hermiticity fermion at $m_f = 0.4$ and $\kappa = 1$, respectively. The horizontal and vertical axes denote the critical mass m_c and the squared four-fermi coupling constant g_π^2 , respectively.

where the m_f in the r.h.s is a constant.

The results are in Fig.3, Fig.4, and Fig.5 ². We can see that the phase structures using the non- γ_5 hermiticity fermion are very similar to the naive fermion cases. All the coupling constants are real numbers despite of using fermions without γ_5 hermiticity.

4. Summary and discussion

We formulated new two dimensional fermions without γ_5 hermiticity (non- γ_5 hermiticity fermion), based on the minimal doubling fermion, and investigated some properties of the fermions. The fermions preserve translation invariance, exact chiral symmetry and locality, but break cubic symmetry and some discrete symmetries. The eigenvalues of $D_1^{(2)}$, defined in Eq.(2.1), distributes along

²In the phase diagrams using the fermions adding the flavored mass, there is first-order phase transition at the bottom of the diagrams enclosed the critical line, so that the gap equations defined in Eqs.(3.5), are not applicable in this region. However, we take leave not to correct to emphasize that we can obtain solutions for the gap equations.

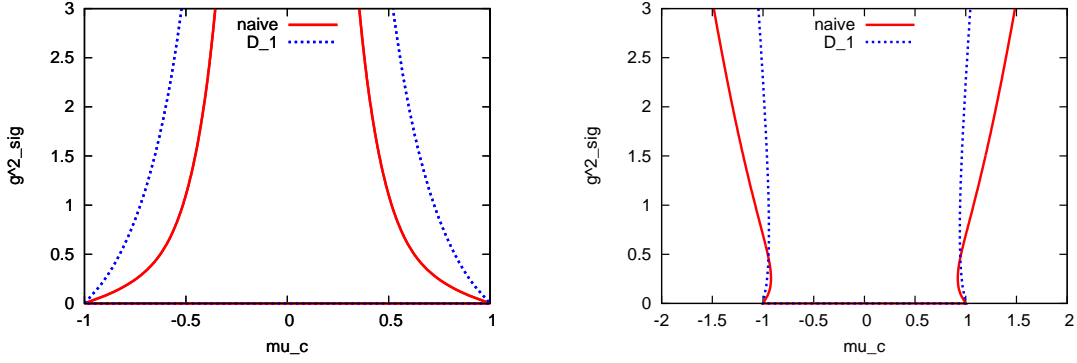


Figure 5: The chiral broken phase diagrams using the naive fermion (Red line) and the non- γ_5 hermiticity fermion $D_1^{(2)}$ (Blue line) at (Left) $g_\sigma^2 = g_\pi^2$ and (Right) $g_\sigma^2 = 2g_\pi^2$. We fix the hopping parameter in the non- γ_5 hermiticity fermion at $\kappa = 2$. The horizontal and vertical axes denote the critical chemical potential and the squared four-fermi coupling constant g_σ^2 , respectively.

the imaginary axis in the continuum limit, but the others do not. In general, the fermions either have at least four poles, or else an odd number poles, but $D_1^{(2)}$ at $\kappa \geq 1$ generates two poles.

We also applied the fermion, $D_1^{(2)}$, to the two dimensional Gross-Neveu model with or without imaginary chemical potential. In the application, the parity broken phase diagram which is called Aoki phase and the chiral symmetry broken phase diagram were obtained. The phase structures are similar to the ones obtained with the naive fermion, and all the coupling constants are real numbers despite of γ_5 hermiticity breaking.

In the continuum theory, the eigenvalues of Dirac operator without γ_5 hermiticity spread over the whole $\text{Re}\lambda - \text{Im}\lambda$ plane. Therefore, the behavior of eigenvalues of $D_1^{(2)}$ can be interpreted as an effect of the lattice spacing. In the case of the non- γ_5 hermiticity fermions, each spectrum is paired with complex conjugate, so that the Dirac determinant is real-valued.

We will investigate gauge theory using the non- γ_5 hermiticity fermion and higher dimensional extension in future work.

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