# Effective $\theta$ term by CP-odd electromagnetic background fields

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We discuss our study of QCD in the presence of CP-odd electromagnetic (em) background fields. We investigate the propagation of the CP-odd term from the e.m. sector to the strong sector, inducing an effective $\theta$ term. We discuss the method we have used in our lattice QCD simulations, and the results of our analysis, which are relevant to the determination of the effective pseudoscalar QED-QCD interactions. We also explore how these CP-odd e.m. background fields influence the number of the Dirac zero modes in our configurations.

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## 1. Introduction

Theoretically one can add to the Euclidean action of QCD the additional term $-i\theta Q$, where

$$Q = \int d^4x\, q(x) = \int d^4x \frac{g^2}{64\pi^2} G^a_{\mu\nu}(x)\tilde{G}^a_{\mu\nu}(x) \tag{1.1}$$

is the topological charge operator, $G^a_{\mu\nu}$ is the non-Abelian gauge field strength and $\tilde{G}^a_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} G^a_{\rho\sigma}$.

However, this term violates explicitly *CP*, while experiments tell us that QCD is invariant under *P* and *CP*, setting a quite stringent upper bound on the parameter $\theta$, which is expected to be $|\theta| \lesssim 10^{-10}$ [1, 2]. Nevertheless, $\theta$ represents an important parameter in strong interactions, both from the theoretical and phenomenological point of view.

Recently, it was hypothesized that local fluctuations of the topological charge may induce measurable phenomena in the presence of extremely intense magnetic fields. This scenario can be realized in non-central heavy ion collisions, where one expects magnetic fields up to $10^{15}$ Tesla at LHC. According to the chiral magnetic effect [3, 4], for a magnetic field strong enough to align the magnetic moments of quarks, these local fluctuations of the topological charge would induce a net unbalance of chirality, leading to a net separation of electric charge along the direction of the magnetic field.

Albeit e.m. background fields couple directly only with charged particles, recent lattice studies with dynamical fermions have shown that these fields, via quark loop effects, can influence also the gluonic sector [5, 6, 7, 8, 9, 10]. In an attempt to better clarify such issue, we investigate how the explicit breaking of the CP symmetry in the electromagnetic sector propagates to the gluon fields.

We will consider QCD in the presence of constant and uniform e.m. fields such that $F_{\mu\nu}\tilde{F}_{\mu\nu} \propto \vec{E} \cdot \vec{B} \neq 0$, which are expected to induce an effective CP-violating interaction in the gluon sector, $\theta_{\text{eff}}\frac{g^2}{64\pi^2} G^a_{\mu\nu}(x)\tilde{G}^a_{\mu\nu}(x)$, where $\theta_{\text{eff}}$ must be an odd function of $\vec{E} \cdot \vec{B}$. At the lowest order, we can write

$$\theta_{\text{eff}} \simeq \chi_{CP}\, e^2 \vec{E} \cdot \vec{B} + O((\vec{E} \cdot \vec{B})^3) \tag{1.2}$$

where $\chi_{CP}$ can be seen as the susceptibility of the QCD vacuum to CP-breaking e.m. fields, and is directly related to the strength of the effective pseudoscalar QED-QCD interaction, $\chi_{CP}\, q(x)\, e^2 \vec{E} \cdot \vec{B}$ [11, 12].

In [13] we measured $\chi_{CP}$ on the lattice performing lattice simulation of QCD in the presence of e.m. background fields such that $\vec{E} \cdot \vec{B} \neq 0$, and we have determined the induced $\theta_{\text{eff}}$ by studying distribution of the topological charge. In this work we also report our study of the Dirac zero modes in the presence of such CP-odd e.m. background fields.

## 2. The method

We can introduce the external e.m. fields in the QCD lagrangean by modifying the covariant derivative of quarks, $D_\mu = \partial_\mu + igA^a_\mu T^a + iqA_\mu$, where $A_\mu$ is the e.m. gauge potential and $q$ is the quark electric charge. That can be implemented on the lattice by adding proper $U(1)$ phases $u_\mu(n)$ to the usual $SU(3)$ parallel transports, $U_\mu(n) \to u_\mu(n)U_\mu(n)$, where $n$ is a lattice site. Because

of the periodic boundary conditions used in our lattice simulations, the possible value of the e.m. fields must be integer multiple of a minimum quantum:

$$f = 2\pi/(qa^2 L_\mu L_\nu), \tag{2.1}$$

where $L_\mu$, $L_\nu$ are the lattice extensions along the directions orthogonal to the field (for more detail see [13]). We have considered two flavour QCD with dynamical fermions, using standard charges for the *up* and *down* quarks, namely $q_u = 2|e|/3$ and $q_d = -|e|/3$, therefore the quantization is given in units of $f = 6\pi/(|e|a^2 L_\mu L_\nu)$.

To guarantee the feasibility of numerical simulations, we must preserve the positivity of the fermion determinant after the addition of the $U(1)$ phases to the $SU(3)$ link variables. This require that the spectrum of the Dirac matrix in the path integral remains purely imaginary.

However, such condition is not verified if we try to introduce a real electric field in Minkowski space: it is easy to verify that this would require an imaginary value of the electric field in Euclidean space, which takes the $u_\mu$ variables out of the $U(1)$ group, making the fermion determinant complex: this sign problem would hinder numerical simulations.

To circumvent this problem we adopt the following strategy, used also in lattice studies of the electric polarizability of hadrons [14, 15]: we simulate real magnetic fields $\vec{B}$ and imaginary electric fields $\vec{E} = i\vec{E}_I$ in Minkowski space, and then exploit analytic continuation. As a consequence, we expect to produce a purely imaginary effective parameter $\theta_{\mathrm{eff}} = i\,\theta_{I\mathrm{eff}}$.

The presence of an imaginary $\theta_I$ adds a factor $\exp(\theta_I Q)$ to the probability distribution of gauge fields in the path integral. That will shift the distribution of the topological charge by an amount which, at the linear order in $\theta_I$, is given by the topological susceptibility $\chi$ at $\theta_I = 0$:

$$\langle Q \rangle_{\theta_I} \simeq V \chi \, \theta_I = \langle Q^2 \rangle_{\theta=0} \, \theta_I \tag{2.2}$$

where $V$ is the spacetime volume. That gives us the opportunity of determining the effective $\theta_{I\mathrm{eff}}$ produced by a given e.m. field as

$$\theta_{I\mathrm{eff}} \simeq \frac{\langle Q \rangle (\vec{E}_I, \vec{B})}{\langle Q^2 \rangle_0} + O((\vec{E}_I \cdot \vec{B})^3) \tag{2.3}$$

where $\langle \cdot \rangle_0$ is defined as the average taken at zero e.m. field. In the region of small $\theta_{I\mathrm{eff}}$, which is the one relevant to Eq. (1.2), we expect negligible corrections to Eq. (2.3).

## 3. Results

We performed simulation of $N_f = 2$ QCD at $T = 0$ for a fixed pseudo-Goldstone pion mass $m_\pi \simeq 480$ MeV. Different lattice spacings have been explored by tuning the inverse gauge coupling $\beta$ and $am$ as described in Ref. [13]. We also used different lattice volumes to check for finite size corrections (see Fig. 2).

For the determination of $Q$ on gauge configurations, we adopted the standard discretized gluonic definition, measured after cooling [16], i.e. recursive minimization of the pure gauge action to reduce ultraviolet (UV) artifacts. We then rescaled the charge by a constant factor, so that its distribution gets peaked around an integer values, (see, e.g., Fig 1), and we finally fix $Q$ to the closest integer (for details and discussions on the used procedure see [13]).
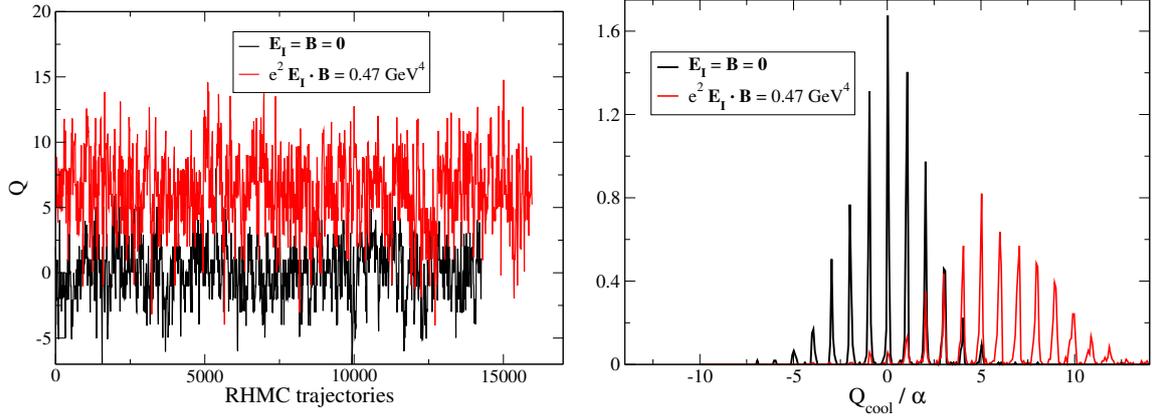
**Figure 1:** Observed shift of the topological charge for $e^2 \vec{E}_I \cdot \vec{B} = 0.47 \text{GeV}^4$ on a $16^4$ lattice, for $m_\pi \simeq 480$ MeV and $a \simeq 0.113$ fm, for two values of the e.m. background $\vec{E}_I \cdot \vec{B}$. *Left panel*: Monte Carlo histories. *Right panel*: topological charge distribution.

In the left panel of Fig. 1 we show the Monte-Carlo history of $Q$ for two numerical simulations performed respectively at $\vec{E}_I \cdot \vec{B} = 0$ and $e^2 \vec{E}_I \cdot \vec{B} \simeq 0.47$ GeV$^4$. We see that, when we switch on the external fields $\vec{E}_I \cdot \vec{B} \neq 0$, the fluctuations of the topological charge $Q$ shifts from zero toward positive values, as clearly appears also from the right panel of Fig. 1, where we plot the corresponding distribution of $Q$.

To better investigate the effective dependence of $\langle Q \rangle (\vec{E}_I, \vec{B})$ on the background field combination $\vec{E}_I \cdot \vec{B}$, in Fig. 2 we show $\langle Q \rangle (\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$, where data are obtained for a variety of combinations of $\vec{E}_I$ and $\vec{B}$, mostly taken parallel to the $z$ axis, and then plotted versus $\vec{E}_I \cdot \vec{B}$. The fact that all data fall on the same curve, even when $\vec{E}_I$ and $\vec{B}$ are not parallel, is a nice demonstration that $\theta_{I\text{eff}}$ is indeed a function of $\vec{E}_I \cdot \vec{B}$ alone, as expected. We have also considered different combinations of the fields having the same or opposite values for $\vec{E}_I \cdot \vec{B}$, to verify explicitly that $\theta_{I\text{eff}}$ is odd in $\vec{E}_I \cdot \vec{B}$.

For small fields we observe a linear dependence in $\vec{E}_I \cdot \vec{B}$, while for larger fields the observable shows saturation effects, as common to many systems with a linear response to external stimulation. All data can be nicely fitted by the function

$$\langle Q \rangle (\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0 = a_0 \, \text{atan}(a_1 \vec{E} \cdot \vec{B}) \, , \tag{3.1}$$

the best fit curve, corresponding to $\chi^2/\text{d.o.f.} = 0.74$, is shown in Fig. 2.

We expect $\langle Q \rangle (\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$ to be $V$ independent, because $\langle Q \rangle$ and $\langle Q^2 \rangle_0$ are both derivatives of the free energy with respect of $\theta$, so they are proportional to $V$, and their ratio should be volume independent. In Fig. 2 we show $\langle Q \rangle (\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$ for $m_\pi \simeq 480$ MeV and different spacings $a$ and lattice volumes $L^4$. From the right panel in Fig. 2 we can exclude relevant finite size effects, even on the smallest volumes explored, corresponding to $am_\pi L \sim 4$.

Instead, we observed a significant dependence on the UV cutoff until $a \lesssim 0.15$ fm. Apart from standard lattice artifacts related to the path integral discretization, additional systematic effects may be related to the method used to determine $Q$: if $a$ is coarse enough that part of the induced topological background lives close to the UV scale, then the cooling procedure is expected to destroy part of such background. However data obtained for $a \lesssim 0.15$ fm are in good agreement
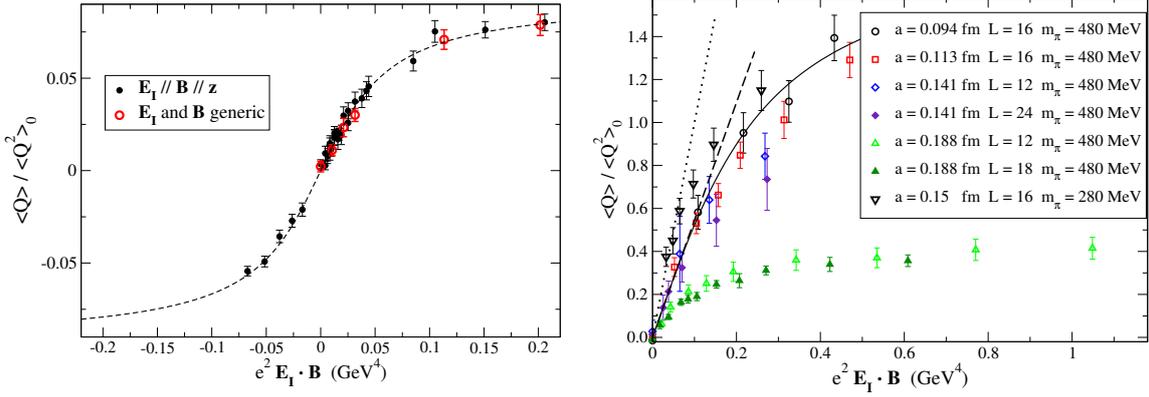
**Figure 2:** *Left panel*: $\langle Q \rangle (\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$ for various $(\vec{E}_I, \vec{B})$ on a $16^4$ lattice for $m_\pi \simeq 480$ MeV and $a \simeq 0.28$ fm, where open circles corresponds to non parallel configurations of $\vec{E}_I \cdot \vec{B} / f^2$, while the dashed line is a best fit to Eq. (3.1). *Right Panel*: $\langle Q \rangle (\vec{E}_I, \vec{B}) / \langle Q^2 \rangle_0$ as a function of $\vec{E}_I \cdot \vec{B}$ for different spacings $a$ and lattice volumes $L^4$, and $m_\pi \simeq 480$ MeV. The continuous line corresponds to a best fit to Eq. (3.1) at the smallest value of $a$, the dashed line is the corresponding slope at $\vec{E}_I \cdot \vec{B} = 0$. We also plot our preliminary results for $m_\pi \simeq 280$ MeV.

with each other, especially in the region of small values of $\vec{E}_I \cdot \vec{B}$, where corrections to Eq. (2.3) should be negligible.

We determined $\chi_{CP}$ performing best fits of the data in Fig. 2 to the function in Eq. (3.1), in a range of fields such that $e^2 \vec{E}_I \cdot \vec{B} < 0.8$ GeV$^4$, then considering its slope at $\vec{E}_I \cdot \vec{B} = 0$ and exploiting Eqs. (1.2) and (2.3). For each slope we obtained a good agreement with a direct linear fit performed on a narrow region of small $\vec{E}_I \cdot \vec{B}$. Because of the large artifacts at coarse lattice spacing, we consider only data up to $a < 0.15$ to extrapolate our result, finding $\chi_{CP} = 5.47(78)$ GeV$^{-4}$ ($\chi^2$/dof $\simeq 0.1$). We also expect an additional $\sim 20\%$ uncertainty on $\chi_{CP}$ coming out from a 5% systematic uncertainty in our knowledge of $a$. Preliminary results obtained on a $16^4$ lattice and for $a \simeq 0.15$ fm indicate instead $\chi_{CP} \sim 10$ GeV$^{-4}$ if $m_\pi \simeq 280$ MeV, suggesting that $\chi_{CP}$ tends to increase when approaching the chiral limit.

## 4. Zero modes

From our lattice results it clearly appears that CP-odd e.m. background have a non trivial influence on the gluon fields, shifting the total distribution of the topological charge to finite values. From a naive consideration, these configurations should be suppressed in the path integral, because of their higher value of the action. Moreover, the axial anomaly equation tell us that a non zero value of $Q$ is associated with the presence of zero modes in the fermion matrix, which should drop in the chiral limit, the contribution of these configurations.

To explain the observed phenomena (and the apparent increase of its strength for smaller masses) one has to consider the full axial anomaly equation, with the inclusion of the $U(1)$ term brought from the external e.m. fields, $Q_{U(1)} = \vec{E} \cdot \vec{B}$. Then the full axial anomaly equation become:

$$n_- - n_+ = Q_{tot} \equiv Q_{SU(3)} + N_C \, Q_{U(1)} \tag{4.1}$$

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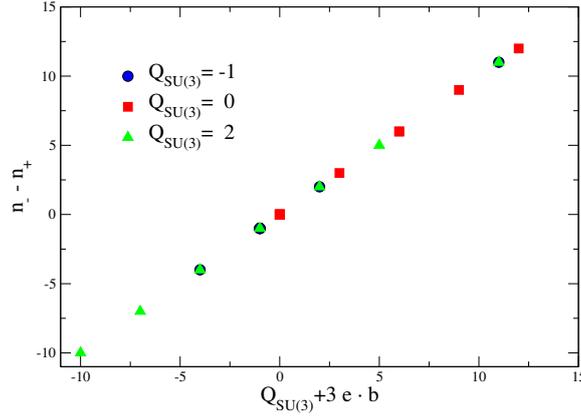
**Figure 3:** $n_- - n_+$ for three different values of the topological charge $Q_{SU(3)}$ and for various combination of the electric and magnetic fields (here given in units of their quanta).

where $Q_{SU(3)}$ and $Q_{U(1)}$ are respectively the non-abelian and the abelian contributions to $Q_{tot}$, which is the difference of left handed and right handed zero modes.

To verify explicitly this relation, we measured the number of zero modes in a set of $O(40)$ configurations, where we have fixed both the topological content and the external fields. To measure the zero modes we used overlap fermions, which, as well known, can correctly distinguish the chirality of fermions on the lattice. We explicitly verified the relation (4.1), with $N_C = 3$, for various external e.m. fields and three different values of the topological charge $Q_{SU(3)}$ (see Fig. 3). So, at least in the chiral limit, the relevant gluonic configurations in the path integral must have nontrivial $Q_{SU(3)}$ in such a way to balance the contributions carried by the electromagnetic part of the anomaly.

## 5. Discussion

Our results can be compared with the phenomenological estimate given in Ref. [12], which is based on the effective couplings of the $\eta$ and $\eta'$ mesons to two photons and to two gluons, where the authors found $\chi_{CP} \approx 0.73/(\pi^2 f_\eta^2 m_{\eta'}^2) \sim 3$ GeV$^{-4}$. Our measurements suggest that the lattice QCD result for the effective pseudoscalar QED-QCD interaction is larger, even if of the same order of magnitude, but one has to consider the different systematics (the phenomenological estimate is based on a theory with 2+1 light flavors), and the unphysical value of the quark mass used in our simulations.

Concerning the validity of analytic continuation from imaginary to real electric fields in Minkowski space, a real, non-zero and constant electric field, even if infinitesimal, will induce vacuum instabilities in the thermodynamical limit . On the other hand this should not be true in presence of an infrared cutoff, i.e. if electric fields are limited in space. Therefore our result should be useful for the determination of a local effective $\theta$ parameter produced by smooth and limited in space CP-odd e.m. fields. It would be interesting in the future to repeat this analysis with smoothly varying fields, as well as with physical quark masses and at finite temperature.

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