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The static quark self-energy at $\mathcal{O}(\alpha^{20})$ in perturbation theory

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In Refs. [1, 2] we determined the infinite volume coefficients of the perturbative expansions of the self-energies of static sources in the fundamental and adjoint representations in SU(3) gluodynamics to order α^{20}. We used numerical stochastic perturbation theory [3], where we employed a new second order integrator and twisted boundary conditions. The expansions were obtained in lattice regularization with the Wilson action and two different discretizations of the covariant time derivative within the Polyakov loop. Overall, we obtained four different perturbative series. For all of them the high order coefficients displayed the factorial growth predicted by the conjectured renormalon picture, based on the operator product expansion. This enabled us to determine the normalization constants of the leading infrared renormalons of heavy quark and heavy gluino pole masses. Here we present improved determinations of the normalization constants and the perturbative coefficients by incorporating the four-loop β-function coefficient (which we also determine) in the fit function.

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PoS(LATTICE 2013)371

In Refs. [1, 2], the existence of renormalons ($c_n \sim n!$) in quantum gluodynamics has been unambiguously established. The quantities studied were δm and $\delta m_{\tilde{g}}$, the self-energies of static sources in the fundamental ($R=3$) and adjoint ($R=8$) representations. Using Numerical Stochastic Perturbation Theory [3], they were computed up to $\mathcal{O}(\alpha^{20})$ and extrapolated to infinite volume in the Wilson action lattice scheme:

$$\delta m = \frac{1}{a}\sum_{n=0}^{19} c_n^{(3,\rho)}\alpha^{n+1}(1/a)\,(\text{fundamental}), \qquad \delta m_{\tilde{g}} = \frac{1}{a}\sum_{n=0}^{19} c_n^{(8,\rho)}\alpha^{n+1}(1/a)\,(\text{adjoint})\,, \tag{1}$$

where a is the lattice spacing. $\rho = 0$ and $\rho = 1/6$ stand for un-smeared and smeared temporal links, respectively, within the Polyakov line,

$$L^{(R)}(N_S,N_T) = \frac{1}{N_S^3}\sum_{\mathbf{n}}\frac{1}{d_R}\text{tr}\left[\prod_{n_4=0}^{N_T-1} U_4^R(n)\right]\,, \tag{2}$$

used to determine δm through the relation

$$\delta m = -\lim_{N_S,N_T\to\infty}\frac{\ln\langle L^{(3,\rho)}(N_S,N_T)\rangle}{aN_T}\,, \qquad \delta m_{\tilde{g}} = -\lim_{N_S,N_T\to\infty}\frac{\ln\langle L^{(8,\rho)}(N_S,N_T)\rangle}{aN_T}\,. \tag{3}$$

Renormalon dominance predicts that the large n dependence of $c_n^{(R,\rho)}$ should be (see Ref. [2] for notation and definitions)

$$c_n^{(3/8,\rho)} \overset{n\to\infty}{=} N_{m/m_{\tilde{g}}}\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}s_1+\frac{b(b-1)}{(n+b)(n+b-1)}s_2+\cdots\right). \tag{4}$$

One of the major results of this analysis was the confirmation of this behavior and the determination of the normalization of the renormalon of the quark and gluelump ($N_{m_{\tilde{g}}} = -N_\Lambda$) pole mass:

$$N_m^{\text{latt}} = 19.0\pm 1.6\,, \quad (C_F/C_A)\,N_\Lambda^{\text{latt}} = -18.7\pm 1.8\,, \tag{5}$$

$$N_m^{\overline{\text{MS}}} = 0.660\pm 0.056\,, \quad (C_F/C_A)\,N_\Lambda^{\overline{\text{MS}}} = -0.649\pm 0.062\,. \tag{6}$$

These numbers are by more than ten standard deviations separated from zero, consolidating, with this significance, the existence of the $d=1$ renormalon in gluodynamics for two different quantities. The above numbers are in agreement, within errors, with determinations from continuum-like computations [4, 5, 6, 7], but they have been obtained using completely independent methods. This is highly nontrivial given the factor $\simeq 29$ between the values of N_m and N_Λ in both schemes. Moreover, in the $\overline{\text{MS}}$ scheme the normalization was determined from the first few terms of the perturbative series only, while in the lattice scheme $n \geq 9$ was required. We remark that there has always been some doubt about the reliability of determinations of $N_m^{\overline{\text{MS}}}$ and $N_\Lambda^{\overline{\text{MS}}}$ from just very few orders of perturbation theory. We have now provided an entirely independent determination of these objects based on many orders of the expansion that can systematically be improved upon. Moreover, for the first time, it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

We expect that the renormalon dominance of perturbative expansions sets in at much lower orders in the $\overline{\text{MS}}$ scheme than in the lattice scheme. This is supported by the consistency of our N_m-determination with continuum estimates mentioned above. Also the earlier onset of the asymptotics

PoS(LATTICE 2013)371

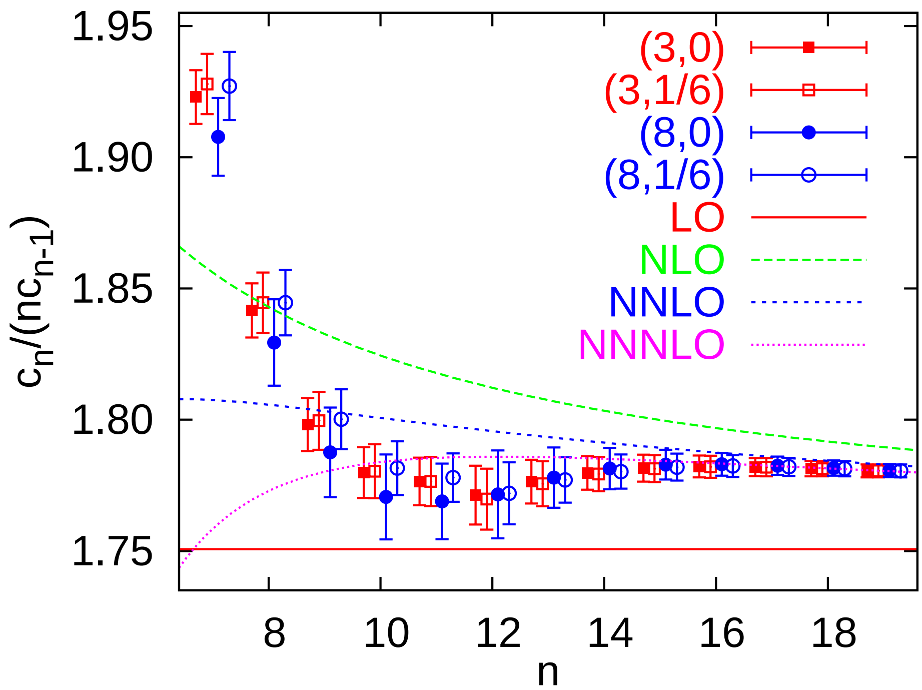

Figure 1: *The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet fundamental static self-energies, compared to the prediction Eq. (59) of Ref. [2] for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion. For clarity, the data sets are slightly shifted horizontally.*

in the $\overline{\text{MS}}$-like schemes devised in Ref. [2] is coherent with this assumption. In Ref. [2] we turned this argument around to estimate β_3^{latt} from the lattice-to-$\overline{\text{MS}}$ scheme conversion, assuming that $c_{3,\overline{\text{MS}}}$ was dominated by the renormalon

$$c_{3,\overline{\text{MS}}} \simeq N_m^{\overline{\text{MS}}} \left(\frac{\beta_0}{2\pi}\right)^3 \frac{\Gamma(4+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(3+b)} s_1 + \frac{b(b-1)}{(3+b)(2+b)} s_2 + \cdots \right). \tag{7}$$

Using our central value $c_{3,\text{latt}}^{(3,0)} = 794.5$, we obtained[1]

$$\beta_3^{\text{latt}} \simeq -1.12 \times 10^6 . \tag{8}$$

Crucial for the accurate determination of the coefficients $c_n^{(R,\rho)}$ (and the normalizations $N_{m,m_{\tilde{g}}}$) was the good theoretical control of the infinite volume extrapolation. Nevertheless, the final errors of the coefficients were still dominated by the systematics of this, due to the unknown higher order coefficients of the β function: in our fits we used the known values of $\beta_{0,1,2}$ and set $\beta_i = 0$ from β_3 onwards.

We repeat the analysis of Ref. [2] including the running due to β_3^{latt} in Eqs. (68) and (70) of Ref. [2], and also its effect on the asymptotic analytic form of the renormalon in Eq. (4). As β_3^{latt} is a free parameter, N_m and the coefficients c_n for $n \geq 3$ become functions of β_3^{latt}. Selfconsistency with Eq. (7) (assuming renormalon dominance at early orders in the $\overline{\text{MS}}$ scheme) fixes β_3^{latt} and we obtain $\beta_3^{\text{latt}} = -1.16 \times 10^6$ (and $d_3 = 352$). This value is almost identical to Eq. (8), illustrating

[1] This number and $d_3 = 351$ correct Eq. (103) of the published version of Ref. [2].

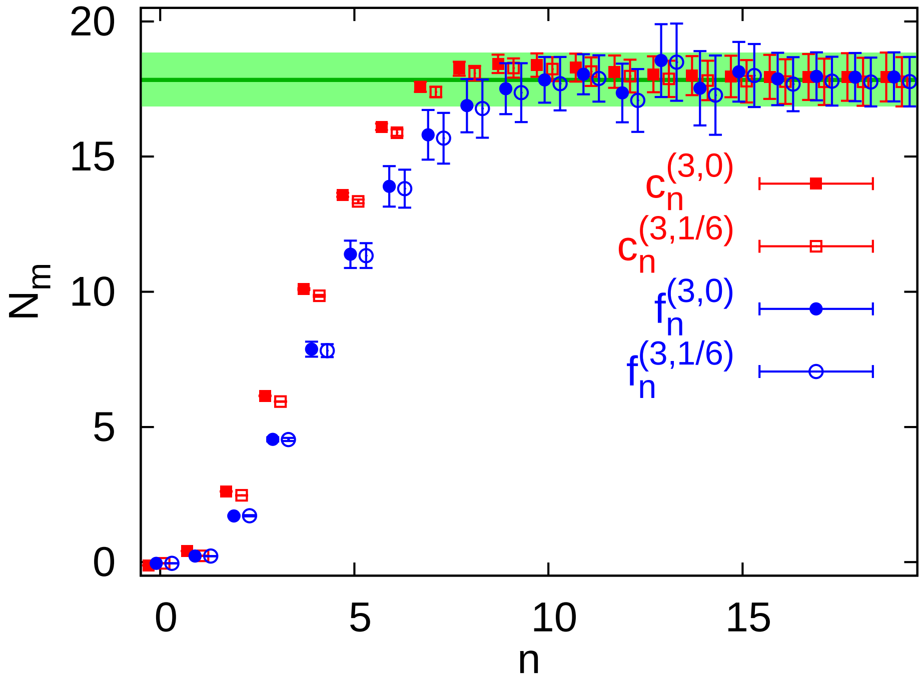

Figure 2: *N_m in the lattice scheme, determined via Eq. (4) truncated at NNLO from the coefficients $c_n^{(3,0)}, c_n^{(3,1/6)}, f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result quoted in Eq. (10).*

the stability of the result to this procedure. Having gained this confidence, we will use this value to improve upon our analysis of Ref. [2].

In Table 1 we display the infinite volume coefficients $c_n^{(R,\rho)}$, including all systematic errors. The unsmeared c_0-values are fixed using diagrammatic lattice perturbation theory. The central values are obtained as in Ref. [2] but including the running due to $\beta_3^{\rm latt}$ into Eqs. (68) and (70) of this reference. We will take the errors of this fit as statistical ($\sigma^2_{\rm stat.}$). The quoted errors in table 1 have been computed as in Ref. [2]. They result from summing statistical and theoretical uncertainties in quadrature. Schematically, we have at each order n

$$\sigma_{\rm final} = \sqrt{\sigma^2_{\rm stat.} + \sigma^2_{\beta} + \sigma^2_{T}}\,, \tag{9}$$

where σ_T is the difference between central values of the fit with $\nu_T = 11$ (our central value) and $\nu_T = 9$ (see Ref. [2] for details). σ_β is the difference between setting $\beta_3^{\rm latt} = 0$ or not. We find $\sigma_\beta \gg \sigma_T, \sigma_{\rm stat.}$, so that the dominant error still stems from logarithmic $N_S^{-1}\ln^i(N_S)$-corrections, due to our lack of knowledge of $\beta_4^{\rm latt}$ etc..

The same analysis yields the $1/N_S$ correction coefficients $f_n^{(R,\rho)}$, where we determine the errors in the same way as for $c_n^{(R,\rho)}$. We display these results in Table 2. The renormalon picture predicts that $c_n \simeq f_n$ for large n. This equality is achieved with a high degree of accuracy from $n = 9$ onwards in all four cases (compare Tables 1 and 2).

In Table 3 we display the infinite volume $c_n^{(R,\rho)}/(nc_{n-1}^{(R,\rho)})$-ratios. The central values are trivially deduced from Table 1. The statistical errors are obtained from the global fit to the volume dependence, and include the statistical correlations between the different n-value $c_n^{(R,\rho)}$ coefficients. The total error is obtained as before, using Eq. (9). In Fig. 1 we display these ratios and compare them

Table 1: *The infinite volume coefficients $c_n^{(R,\rho)}$, including all systematic errors. The unsmeared c_0-values are fixed using diagrammatic lattice perturbation theory.*

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)}C_F/C_A$	$c_n^{(8,1/6)}C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(15)	7.671(11)	7.917(19)	7.682(13)
$c_4/10^3$	8.208(30)	8.009(28)	8.191(39)	8.010(32)
$c_5/10^4$	9.299(49)	9.135(49)	9.273(71)	9.116(55)
$c_6/10^6$	1.1478(86)	1.1324(87)	1.139(12)	1.1287(97)
$c_7/10^7$	1.545(16)	1.528(17)	1.521(21)	1.523(18)
$c_8/10^8$	2.276(32)	2.255(33)	2.225(42)	2.247(35)
$c_9/10^9$	3.684(68)	3.653(71)	3.580(90)	3.640(72)
$c_{10}/10^{10}$	6.56(15)	6.50(16)	6.34(20)	6.49(16)
$c_{11}/10^{12}$	1.281(36)	1.271(37)	1.234(45)	1.268(37)
$c_{12}/10^{13}$	2.723(89)	2.699(91)	2.62(11)	2.697(92)
$c_{13}/10^{14}$	6.29(23)	6.23(24)	6.06(27)	6.23(24)
$c_{14}/10^{16}$	1.567(63)	1.552(64)	1.512(70)	1.553(64)
$c_{15}/10^{17}$	4.19(18)	4.15(18)	4.04(20)	4.15(18)
$c_{16}/10^{19}$	1.194(54)	1.182(55)	1.153(59)	1.184(55)
$c_{17}/10^{20}$	3.62(17)	3.58(17)	3.49(18)	3.59(17)
$c_{18}/10^{22}$	1.160(57)	1.148(57)	1.121(61)	1.150(57)
$c_{19}/10^{23}$	3.92(20)	3.88(20)	3.79(21)	3.89(20)

with the renormalon expectations (Eq. (59) of Ref. [2]). We see that the agreement is fantastic. This means that, for large n, the coefficients are very well approximated by Eq. (4), which we can use to fix $N_{m,\Lambda}(n)$. For large n the result should be independent of n. We confirm this behavior in Fig. 2. Working as in Ref. [2] we obtain accurate determinations of the normalization constants of the renormalon. They read

$$N_m^{\rm latt} = 17.9 \pm 1.0\,, \quad (C_F/C_A)N_\Lambda^{\rm latt} = -17.6 \pm 1.2\,, \tag{10}$$

$$N_m^{\overline{\rm MS}} = 0.620 \pm 0.035\,, \quad (C_F/C_A)N_\Lambda^{\overline{\rm MS}} = -0.610 \pm 0.041\,. \tag{11}$$

We stress that the N_m-value is by 18 standard deviations different from zero! Other combinations of interest are (see Eqs. (56) and (58) of Ref. [2])

$$N_{V_s}^{\overline{\rm MS}} = -1.240 \pm 0.069\,, \quad N_{V_o}^{\overline{\rm MS}} = 0.13 \pm 0.12\,. \tag{12}$$

The errors of the coefficients, ratios and $N_{m,\Lambda}$ are still dominated by the systematics, though now they are reduced, relative to Ref. [2]. Our previous central values (setting $\beta_3^{\rm latt} = 0$) agree within one standard deviation with the new, improved numbers above.

We are now in the position to predict the four-loop relation between the pole and the $\overline{\rm MS}$ mass in the limit of zero flavours, $r_3/m_{\overline{\rm MS}}$ (see Eq. (43) of Ref. [2] for notation) using Eq. (7). We obtain

$$c_3^{\overline{\rm MS}} = r_3/m_{\overline{\rm MS}} = 37.9(2.2)\,, \tag{13}$$

Table 2: *The $1/N_S$ correction coefficients $f_n^{(R,\rho)}$, including all systematic errors. The unsmeared f_0-values are fixed using diagrammatic lattice perturbation theory.*

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7811(69)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(68)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(73)	5.60(11)	5.691(78)
$f_3/10^2$	5.867(99)	5.858(73)	6.00(17)	5.946(81)
$f_4/10^3$	6.40(23)	6.36(20)	6.65(40)	6.33(24)
$f_5/10^4$	7.79(35)	7.76(31)	7.73(67)	7.84(42)
$f_6/10^5$	9.91(53)	9.85(50)	9.73(99)	9.85(69)
$f_7/10^7$	1.389(81)	1.378(82)	1.35(15)	1.38(11)
$f_8/10^8$	2.11(12)	2.09(13)	2.05(22)	2.09(17)
$f_9/10^9$	3.50(19)	3.47(22)	3.35(36)	3.47(26)
$f_{10}/10^{10}$	6.36(30)	6.31(35)	6.10(65)	6.31(41)
$f_{11}/10^{12}$	1.264(52)	1.253(60)	1.21(12)	1.253(65)
$f_{12}/10^{13}$	2.61(16)	2.56(17)	2.57(32)	2.58(18)
$f_{13}/10^{14}$	6.47(47)	6.44(50)	6.13(86)	6.43(51)
$f_{14}/10^{16}$	1.53(12)	1.50(13)	1.49(21)	1.51(13)
$f_{15}/10^{17}$	4.23(26)	4.20(27)	4.07(42)	4.20(28)
$f_{16}/10^{19}$	1.189(64)	1.176(66)	1.151(89)	1.178(67)
$f_{17}/10^{20}$	3.62(18)	3.59(18)	3.50(21)	3.59(18)
$f_{18}/10^{22}$	1.159(58)	1.148(58)	1.120(64)	1.149(59)
$f_{19}/10^{23}$	3.92(20)	3.89(20)	3.79(21)	3.89(20)

where the error is dominated by the uncertainty of N_m (the effect due to $1/n$ effects in the asympotic formula is subleading). This number is in perfect agreement with (-1/2 times) the number quoted in Eq. (4) of Ref. [4]. Once we have this value for $c_3^{\overline{\rm MS}}$ we can determine $\beta_3^{\rm latt}$ as discussed around Eq. (7). We obtain[2] ($d_3 = 352(3)$)

$$\beta_3^{\rm latt} = -1.16(12) \times 10^6 \,. \tag{14}$$

The error is (conservatively) determined by linearly adding the errors due to N_m, $c_2^{\rm latt}$ and $c_3^{\rm latt}$ (again the $1/n$ corrections are negligible in comparison), even though they are correlated.

Eq. (14) is consistent with the value $\beta_3^{\rm latt} = -1.55(19) \times 10^6$ obtained in Ref. [8], which we had been unaware of at the time we wrote Ref. [2]. This number was found from a non-perturbatively determined step-scaling function which allowed to compute $\alpha(a^{-1})$ for inverse lattice spacings up to $a^{-1} \lesssim 50$ GeV. Note though that such a high value for $-\beta_3^{\rm latt}$ would be in tension with the renormalon dominance of $c_3^{\overline{\rm MS}}$. Therefore, we cannot avoid to remark that smaller values for $-\beta_3^{\rm latt}$ are obtained in Ref. [8] by restricting the fit range to the points at smaller lattice spacings (but then less points are available).

[2]Note that the impact of the $\beta_3^{\rm latt}$-error on the infinite volume coefficients is clearly negligible compared with σ_β, the difference between the evaluations setting $\beta_3^{\rm latt} = 0$ or not.

PoS(LATTICE 2013)371

Table 3: *The infinite volume ratios $c_n^{(R,\rho)}/\left(nc_{n-1}^{(R,\rho)}\right)$, including all systematic errors. Note that $\beta_0/(2\pi) \approx$ 1.7507.*

n	$c_n^{(3,0)}/\left(nc_{n-1}^{(3,0)}\right)$	$c_n^{(3,1/6)}/\left(nc_{n-1}^{(3,1/6)}\right)$	$c_n^{(8,0)}/\left(nc_{n-1}^{(8,0)}\right)$	$c_n^{(8,1/6)}/\left(nc_{n-1}^{(8,1/6)}\right)$
1	5.2594(53)	8.846(18)	5.2616(56)	8.848(18)
2	3.8662(61)	6.361(12)	3.8539(65)	6.364(12)
3	3.0756(55)	3.1474(47)	3.0735(75)	3.1501(53)
4	2.5827(89)	2.6104(89)	2.586(12)	2.6067(99)
5	2.2659(95)	2.2812(98)	2.264(15)	2.276(12)
6	2.057(10)	2.066(11)	2.046(15)	2.064(13)
7	1.923(10)	1.928(11)	1.908(15)	1.927(13)
8	1.842(10)	1.845(11)	1.829(16)	1.845(12)
9	1.798(10)	1.780(11)	1.788(17)	1.800(11)
10	1.7798(97)	1.780(10)	1.771(16)	1.782(10)
11	1.7765(91)	1.7765(94)	1.769(14)	1.7780(92)
12	1.771 (11)	1.770(12)	1.772(17)	1.772(12)
13	1.7764(83)	1.7756(86)	1.778(11)	1.7770(86)
14	1.7797(64)	1.7793(65)	1.7814(79)	1.7802(65)
15	1.7816(51)	1.7814(52)	1.7829(57)	1.7819(51)
16	1.7822(42)	1.7821(42)	1.7830(43)	1.7824(42)
17	1.7820(35)	1.7819(35)	1.7825(35)	1.7821(35)
18	1.7813(29)	1.7813(29)	1.7816(29)	1.7814(29)
19	1.7805(25)	1.7805(25)	1.7806(25)	1.7805(25)

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