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Update on the hadronic light-by-light contribution to the muon $g-2$ and inclusion of dynamically charged sea quarks

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We present preliminary results from lattice QCD for part of the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment. While the size of the computed piece is roughly consistent with model calculations, several important systematic errors remain uncontrolled. The most significant is the absence of charged sea quarks, or the use of quenched QED. We outline the next step of using dynamical QED to obtain the full hadronic light-by-light scattering contribution to the muon $g-2$.

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1. Introduction

The muon anomaly a_μ provides one of the most stringent tests of the standard model because it has been measured to great accuracy (0.54 ppm) [1], and calculated to even better precision [2, 3, 4]. At present, the difference observed between the experimentally measured value and the standard model prediction ranges between 249 (87) $\times 10^{-11}$ and 287 (80) $\times 10^{-11}$, or about 2.9 to 3.6 standard deviations [2, 3, 4]. To confirm such a difference, new experiments have been proposed at Fermilab (E989) and J-PARC (E34), aiming for an accuracy of 0.14 ppm. Improvement in the experiments, however, will not make sense unless the uncertainty in the theory is also reduced.

The theory error is dominated by the QCD contributions. The error on the leading-order (LO) hadronic vacuum polarization (HVP) contribution stands at a bit more than half a percent. It is computed from a dispersion relation using the experimental data on the production cross section of hadrons ($+\gamma$) in e^+e^- collisions at low energy, or those from the hadronic decay of the τ-lepton with isospin breaking taken into account. Lattice QCD simulations on this quantity are improving rapidly [5], and will provide an important crosscheck.

Unlike for the HVP, the hadronic light-by-light scattering (HLbL) contribution, $a_\mu(\text{HLbL})$, cannot be computed from experimental data and a dispersion relation. So far, only model calculations have been done. The uncertainty quoted in the "Glasgow consensus" [6] is about 25%, but larger values can be found in the literature [7]. The size of $a_\mu(\text{HLbL})$ is the same order as the current discrepancy between theory and experiment. Thus, a first principles calculation that is systematically improvable is desirable.

2. Non-perturbative QED method

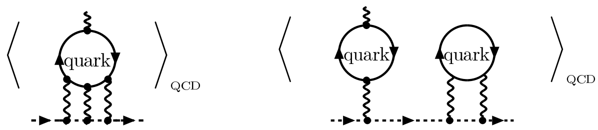

Figure 1: Two classes of diagrams contributing to $a_\mu(\text{HLbL})$. On the left, all QED vertices lie on a single quark loop. The right diagram is one of six where QED vertices are distributed over two to four quark loops.

We start by pointing out the difficulty in computing $a_\mu(\text{HLbL})$ using lattice QCD, and then explain our strategy to overcome it. Figure 1 shows two (of 7) types of diagrams, classified according to how photons are attached to the quark loop(s). In the lattice calculation, the computation of the vacuum expectation value of an operator involving quark fields requires the inversion of the quark Dirac operator. The cost of inversion for all source and sink pairs is prohibitive since it requires solving the linear equation $D_{m_q}\left[U^{\text{QCD}}\right]x = b$ for N_{sites} number of sources b where N_{sites} is the total number of lattice sites. In many problems all of these inversions are not necessary. For our problem, the correlation of four electromagnetic currents must be computed for all possible values of two independent four-momenta. This implies $(3 \times 4 \times N_{\text{sites}})^2$ separate inversions, per QCD configuration and quark species and each given four-momentum of the external photon for the calculation of the quark-line-connected diagram in Fig. 1, which is astronomical.

We have proposed an efficient alternative, a non-perturbative QCD+QED method which treats the photons and muon on the lattice along with the quarks and gluons. To obtain the result for the diagram in Fig. 1 the following quantity is computed [8],

$$\langle \psi(t',\mathbf{p}')\, j^\mu(t_{op},\mathbf{q})\, \overline{\psi}(0,\mathbf{p})\rangle_{\text{HLbL}} = -\sum_{q=u,d,s} (Q_q e)^2 \sum_k \Big\{ \big\langle \gamma_\mu S_q(t_{op},-\mathbf{q};k)\gamma_\nu S_q(k;t_{op},-\mathbf{q}) \frac{\delta_{\nu\rho}}{\hat{k}^2} G(t',\mathbf{p}';-k)\gamma_\rho G(-k;0,-\mathbf{p})\big\rangle_{\text{QCD+QED}} - \big\langle \gamma_\mu S_q(t_{op},-\mathbf{q};k)\gamma_\nu S_q(k;t_{op},-\mathbf{q})\big\rangle_{\text{QCD+QED}} \frac{\delta_{\nu\rho}}{\hat{k}^2} \big\langle G(t',\mathbf{p}';-k)\gamma_\rho G(-k;0,-\mathbf{p})\big\rangle_{\text{QED}} \Big\}, \tag{2.1}$$

where ψ annihilates the state with muon quantum number, and j^μ is the quark electromagnetic current [1]. k is a Euclidean four-momentum, $\mathbf{p}$ is a three-momentum, each quantized in units of $2\pi/L$. $\delta_{\mu\nu}/\hat{k}^2$ ($\hat{k}_\mu \equiv 2\sin(k_\mu/2)$) is the lattice photon propagator in Feynman gauge. S_q and G denote Fourier transformation of $D^{-1}_{m_q}$ and $D^{-1}_{m_\mu}$, respectively, and spin and color indices have been suppressed. One takes $t' \gg t_{op} \gg 0$ to project onto the muon ground state

$$\lim_{t'\gg t_{op}\gg 0} \langle \psi(t',\mathbf{p}')\, j^\mu(t_{op},\mathbf{q})\, \overline{\psi}(0,\mathbf{p})\rangle_{\text{HLbL}} = \frac{\langle 0|\psi(0,\mathbf{p}')|\mathbf{p}',s'\rangle}{2E'V} \langle \mathbf{p}',s'|\Gamma^\mu|\mathbf{p},s\rangle \frac{\langle \mathbf{p},s|\overline{\psi}(0,\mathbf{p})|0\rangle}{2EV} \times e^{-E'(t'-t_{\text{op}})} e^{-E(t_{\text{op}})}, \tag{2.2}$$

where the matrix element of interest is parametrized as

$$\langle \mathbf{p}',s'|\Gamma^\mu|\mathbf{p},s\rangle \;\equiv\; \bar{u}(\mathbf{p}',s') \left(F_1(q^2)\gamma^\mu + i\frac{F_2(q^2)}{2m_\mu}[\gamma^\mu,\gamma^\nu]q^\nu \right) u(\mathbf{p},s). \tag{2.3}$$

$u(\mathbf{p},s)$ is a Dirac spinor, and $q = p' - p$ is the space-like four-momentum transferred by the photon. The contribution to the anomaly is found from $a_\mu \equiv (g_\mu - 2)/2 = F_2(0)$.

For now quenched QED (q-QED) is used for the average in (2.1), implying no fermion-antifermion pair creation/annihilation via the photon. Note that only the sea quarks need to be charged under U(1); the lepton vacuum polarization corresponds to higher order contributions which we ignore. This approximation was chosen to make this first calculation computationally easier, even though it is incomplete. We can remove it to get the complete physical result, as discussed below in the final section. The first term, expanded in q-QED, can be reorganized according to the number of photons exchanged between the quark loop and the muon line. If the second term in Eq. (2.1) is subtracted, the connected diagram in Fig. 1 emerges as the leading-order contribution ($\times 3$ since two photons attach to the left, right, and on both sides of the one attached "by hand" in our method).

The main challenge in the non-perturbative method is the subtraction of the leading, unwanted components (α for the electric part and α^2 for the magnetic part). Note that the two terms in Eq. (2.1) differ only by way of averaging. For finite statistics, the delicate cancellation between them is only realized because they are highly correlated with respect to the QCD and QED configurations used in the averaging. We first test this point by asking if the nonperturbative QED method

[1] The point-split, exactly conserved, lattice current is used for the internal vertices while the local current is inserted at the external vertex.

applied to leptons only reproduces the known value of the sixth-order leptonic light-by-light scattering contribution [9], which is given exactly by the counterpart of the connected diagram in Fig. 1.

The test calculation was done in quenched [2] non-compact QED, in the Feynman gauge, using domain wall fermions (DWF) [12]. To take advantage of the logarithmic enhancement due to the lighter electron mass, $\ln(m_\mu/m_e)$ [9], the mass of the lepton in the loop was set to 0.01. The form factor F_2 was computed only at the lowest non-trivial momenta and was not extrapolated to zero. The result is $F_2 = 3.96(70) \times 10^{-4} = 24.4(4.3)(\alpha/\pi)^3$ while perturbation theory gives about $10(\alpha/\pi)^3$ for $F_2(0)$. The calculation repeated on a larger $24^3 \times 32$ lattice, again for the lowest non-zero momentum, yields $F_2 = 1.19\ (32) \times 10^{-4} = 7.32\ (1.97)(\alpha/\pi)^3$, roughly consistent with perturbation theory and indicating large finite volume effects.

2.1 QCD contribution

The inclusion of QCD into the light-by-light amplitude is straightforward: simply multiply the U(1) gauge links with SU(3) links to create a combined photon and gluon configuration [13], and follow exactly the same steps, using the same code, as described in the previous sub-section. We use one quenched QED configuration per QCD configuration, though different numbers of each could be beneficial and should be explored.

Our main result is computed on a lattice of size $24^3 \times 64$ with spacing $a = 0.114$ fm ($a^{-1} = 1.73$ GeV) and light quark mass 0.005 ($m_\pi = 329$ MeV) (an RBC/UKQCD collaboration 2+1 flavor, DWF+Iwasaki ensemble [10, 11]). The muon mass is set, somewhat arbitrarily, to $m_\mu = 0.1$ (190 MeV), and $e = 1$ as before. The all mode averaging (AMA) technique [14] is used to achieve large statistics at an affordable cost. Besides the exact part of the AMA calculation, which is done using a single point source on 20 configurations, the approximation was computed using 400 low-modes of the even-odd preconditioned Dirac operator and 216 point sources (for the external vertex) computed with stopping residual 10^{-4} on 505 configurations. The external vertex is inserted on time slice $t_{\rm op} = 5$ with incoming and outgoing muons at $t = 0$ and 10, respectively. While the incoming muon is at rest, the three-momenta in units of $2\pi/L$ of the outgoing muon are taken to be $(\pm 1, 0, 0)$, $(\pm 1, \pm 1, 0)$, $(\pm 1, \pm 1, \pm 1)$, $(\pm 2, 0, 0)$, and $(\pm 1, \pm 2, 0)$ and permutations. We also include the vector current renormalization in pure QCD from [11] at the external vertex.

In the AMA procedure the expectation value of an operator is given by $\langle \mathscr{O} \rangle = \langle \mathscr{O}_{\rm rest} \rangle + \frac{1}{N_G} \sum_g \langle \mathscr{O}_{\rm approx,g} \rangle$ [14], where N_G is the number of measurements of the approximate observable, and “rest" refers to the contribution of the exact observable minus the approximation, evaluated for the same conditions. In the present calculation, the statistical errors are completely dominated by the second term, and $N_G = 216$.

In Fig. 2 we show $F_2(Q^2)$ for all of the momenta used in our study. The signals for the second and third lowest Q^2 values are more than four standard deviations (σ) from zero, the largest, 2.5 σ, while the lowest and second largest Q^2 points show no signal outside of statistical errors. The former have many more momenta+projector combinations to average over, which is reflected in the size of the errors (in order of increasing Q^2, there are 12, 48, 48, 12, and 96 combinations). The

[2] In the pure QED case, quenching is not an approximation since the neglected vacuum polarization contributions give higher order corrections to the light-by-light scattering diagram.

central values are comparable in size and sign to model estimates, with statistical errors roughly the same size as quoted model errors.

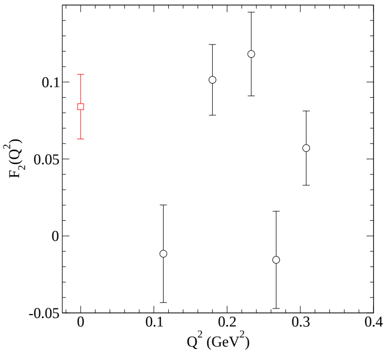

Figure 2: The magnetic form factor (in units of $(\alpha/\pi)^3$) for the muon from HLbL scattering. The square shows the model estimate [6].

Finite size and quark mass dependence were checked on another DWF+Iwasaki ensemble [15] with size $16^3 \times 32$ and light quark mass $m_q = 0.01$ ($m_\pi = 422$ MeV). Two muon masses, $m_\mu = 0.4$ and 0.1, were used. The external electromagnetic vertex is inserted on time slice $t_{\rm op} = 6$ and the incoming and outing muons are created and destroyed at $t = 0$ and 12, respectively. The outgoing muon has one unit of momentum, and results were averaged over all directions and reflections $\pm\vec{p}$. Following the same procedure (except that we did not use AMA), for $m_\mu = 0.4$ (6.5 times the physical muon mass), $F_2(Q^2 = 0.38\,\text{GeV}^2) = (-15.7 \pm 2.3) \times 10^{-5} = (-9.66 \pm 1.42) \times (\alpha/\pi)^3$. The magnitude is roughly 10 times larger than the model estimates for a_μ(HLbL), and even has the opposite sign. Though one should not place too much trust in comparisons with models since the pion mass is heavy here, the pion pole term is probably not dominant and the sub-leading terms can be important and have the opposite sign [6]. Further, the sub-leading terms are proportional to m_μ^2 [6], and setting $m_\mu = 0.1$, the magnitude of the form factor decreases significantly, $F_2(q^2 = 0.19\,\text{GeV}^2) = (-2.21 \pm 0.75) \times 10^{-5} = (-1.36 \pm 0.46) \times (\alpha/\pi)^3$, but still has the opposite sign.

Finally, to check the subtraction is working in the (QCD+QED) case e is varied as follows. The same non-compact QED configurations are used in each case; e is varied only when constructing the exponentiated gauge-link, $U_\mu(x) = \exp\left(ieA_\mu(x)\right)$. Thus the ratio of form factors, and hence the α-dependence, can be determined accurately. Since one photon is inserted explicitly, and the charges at the associated vertices are not included in the lattice calculation, the subtracted amplitude should behave like e^4. Using $e = 0.84$ and 1.19, the changes in the subtracted correlation function relative to $e = 1$ should be 0.5 and 2.0, respectively. This is what is observed numerically.

3. Discussion and Summary

In this paper, we presented preliminary lattice QCD results for part of the contribution to a_μ(HLbL). The calculation is done using a nonperturbative QED method whose feasibility was tested in the pure QED case. As shown in Fig. 2, both the sign and size of F_2 for relatively heavy pions ($m_\pi = 329$ MeV) are found to be consistent with model values, except for the lowest and second largest values of Q^2 where it is zero within statistical errors. Even heavier quarks and muons lead to large values with the opposite sign compared to models, though this is consistent with expected non-leading contributions.

A significant shortcoming in the current calculation is the absence of diagrams with two or more quark loops coupled to photons as the one shown on the right in Fig. 1. They are next-to-leading order in the number of colors, $1/N_C$, and 5 diagrams other than the ones in Fig 1 vanish in the SU(3)-flavor symmetry limit. They may be significant and could correct the odd Q^2 behavior of F_2 in Fig. 2. This behavior may also be due to poor statistics and prevents us from making a reliable $Q^2 \to 0$ extrapolation.

We are currently working to include the contribution of the missing diagrams by using dynamical QCD+QED configurations, or equivalently, by re-weighting the quenched QED configurations [16]. In Fig. 3 we show all possible quark-line disconnected diagrams. The corresponding subtracted correlation functions are shown in Fig. 4. Since the diagrams are not computed separately, but arise from hadronic vacuum polarization in the dynamical QED configurations, it is important that they occur with the same multiplicity. A careful accounting of the contributions shown in Fig. 4 shows this is true. A new complication arises in the last diagram where the quark loops containing the external vertex and the manually inserted virtual photon are different. In the latter, a random 4-volume source is required.

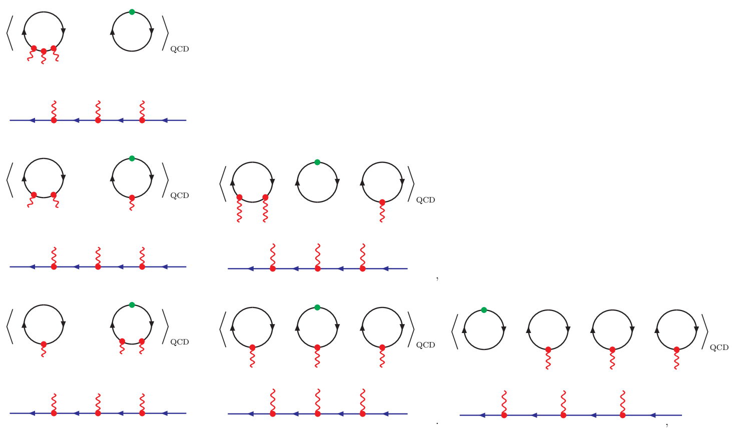

Figure 3: Disconnected quark-line diagrams in HLbL scattering. The photons connect the quark loops with the muon line in all possible combinations.

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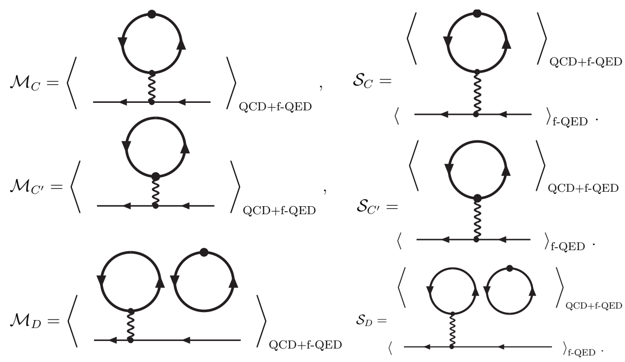

Figure 4: The complete HLbL contribution in the non-perturbative QED method.

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