The Kaon Bag Parameter at Physical Mass

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We present preliminary results for the calculation of the Kaon Bag parameter B_K in $N_f = 2 + 1$ lattice QCD, using Möbius Domain Wall Fermion ensembles generated by the RBC-UKQCD collaboration. This computation is done directly at physical meson masses, so that we do not have to rely on chiral perturbation theory or any other mass extrapolation. In parallel, the four-quark operator is renormalised through the Rome-Southampton technique. Finally, we compare our value with previous results and draw some conclusions about the remaining dominant contributions in our error budget.

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1. Introduction

The mixing of neutral mesons such as the Kaon plays an important role in *CP*-violation, as the so-called *box-diagrams* involve terms like $V_{xs}^* V_{xd}$. However, those perturbative diagrams cannot deal with the confinement of ingoing and outgoing quarks. Nevertheless, an Operator Product Expansion (OPE) allows a separation between the perturbative part and a non-perturbative QCD part that can be computed on the lattice. Lattice computation have already been very successful on this path (see [1, 2] for a review, and [3, 4] for previous results of our collaboration), and what is at stake is now an improvement of the uncertainties.

In the context of lattice computation, we traditionally express the $K^0 - \bar{K}^0$ matrix element as a ratio normalised by the result of Vacuum Saturation Approximation. On a practical side we therefore have to compute three-point and two-point functions, and fit a ratio on ranges that form a plateau converging to the asymptotic constant.

Then, this result has to be brought to a renormalisation scheme which is common with the perturbative computations. This has been done through the Rome-Southampton method and using several intermediate schemes to evaluate the systematic error coming from this part of the computation. In order to improve the convergence to the perturbative regime, non-exceptional momenta have been used and the matching is done at a scale of 3 GeV.

The present calculation, which is preliminary, has been done on the new $N_f = 2 + 1$ Möbius Domain-Wall Fermion (MDWF) ensembles of the RBC-UKQCD collaboration, generated directly at the physical pion mass. We used two ensembles with roughly the same physical volume ($M_\pi L \simeq$ 3.8) but different lattice spacing: a $48^3 \times 96$ ensemble at $\beta = 2.13$ and a $64^3 \times 128$ ensemble at $\beta = 2.25$. Finite-volume errors are estimated from older ensembles and are not a crucial issue. The bare B_K ratios have been computed with respectively 60 and 21 configurations of those ensembles, while 11 and 6 configurations were sufficient for the Non-Perturbative Renormalisation (NPR) of these preliminary results. All of these measurements will ultimately be performed on larger data sets as part of a forthcoming publication, and as such this work is prelimary.

2. Bare B_K ratio

For the bare ratio, all contractions have been computed using Coulomb gauge fixed wall sources and the All Mode Averaging (AMA) method. The latest combines propagators computed with different precisions (here 10^{-4} and 10^{-8}) in order to improve signals at a minimal cost [5].

The ratio

$$B_K^{bare}(t, \Delta t) = \frac{\langle K^0(\Delta t) \mid \mathcal{O}_{VV+AA}^{\Delta S=2}(t) \mid \bar{K}^0(0) \rangle_{AMA}}{\frac{8}{3} \langle K^0(\Delta t - t) \mid A_0(0) \rangle_{AMA} \langle A_0(t) \mid \bar{K}^0(0) \rangle_{AMA}} \tag{2.1}$$

is fitted to a constant on the two-dimensional space $(t, \Delta t)$, The fit domain is chosen as $\Delta t \in [\Delta t_{min}, \Delta t_{max}]$ and $t \in [t_{margin}, \Delta t - t_{margin}]$ (plus symmetric times).

For the 48^3 ensemble we choose the domain $(t_{margin}, \Delta t_{min}, \Delta t_{max}) = (8, 20, 32)$, giving

$$B_K^{48} = 0.5840(9) \tag{2.2}$$

for an uncorrelated fit with $\chi^2/dof = 0.76$, while still giving an acceptable correlated $\chi^2/dof = 1.68$. On the 64^3 ensemble stabilising the correlated fit becomes impossible and we perform an

uncorrelated fit on the domain $(t_{margin}, \Delta t_{min}, \Delta t_{max}) = (10, 25, 40)$, giving (with $\chi^2/dof = 0.64$, see Fig. 1)

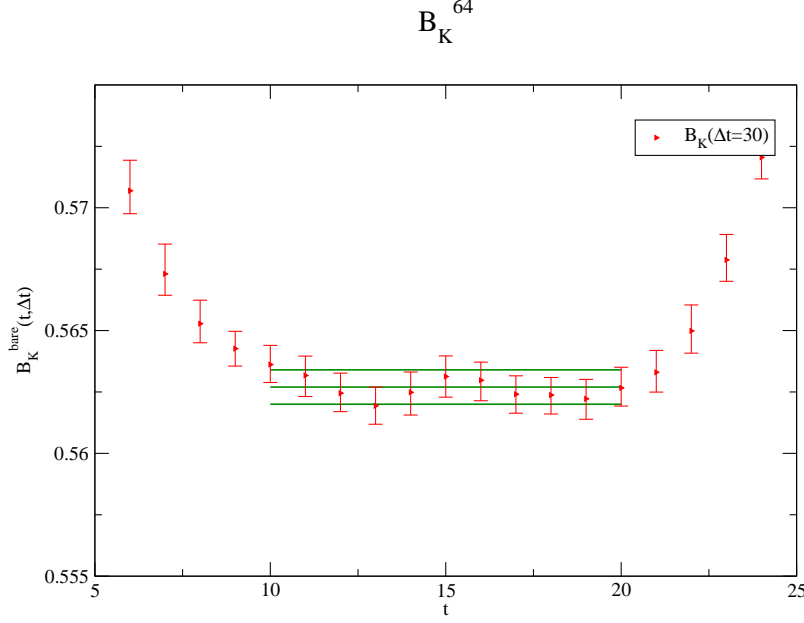$$B_K^{64} = 0.5627(7).$$ (2.3)

$$B_K^{64}$$

Figure 1: We show a part of the fit on the 64^3 ensemble. Green lines represent the result of the fit with its error bars. The fit is actually two-dimensional and we only represent one Δt timeslice for readibility.

3. Non Perturbative Renormalisation

We have to renormalise a four-quark operator, which hopefully renormalise multiplicatively in a Domain-Wall discretisation. In the RI/SMOM scheme this is done through the computation of four-quark Green's functions such as Fig. 2, where ingoing and outgoing quarks use Landau gauge-fixed wall sources with momenta on those walls.

We note Γ_{VV+AA} the associated amputated Green's function. The scheme is completely defined once we choose a projector P allowing us to write the renormalisation condition

$$P\Gamma_{VV+AA}(\mathcal{O}_{VV+AA}^{RI}) = P\Gamma_{VV+AA}(\mathcal{O}_{VV+AA})\mid_{\text{tree}} .$$ (3.1)

We will use the two choices

$$P_\mu^\gamma = \delta_{ij}\delta_{kl}\left[(\gamma_\mu)_{\beta\alpha}(\gamma_\mu)_{\delta\gamma} + (\gamma_\mu\gamma_5)_{\beta\alpha}(\gamma_\mu\gamma_5)_{\delta\gamma}\right]$$ (3.2)

$$P^{\not{q}} = \delta_{ij}\delta_{kl}\left[(\not{q})_{\beta\alpha}(\not{q})_{\delta\gamma} + (\not{q}\gamma_5)_{\beta\alpha}(\not{q}\gamma_5)_{\delta\gamma}\right]/q^2$$ (3.3)

defining respectively the schemes RI/SMOM$_{\gamma_\mu}$ and RI/SMOM$_{\not{q}}$, where i, j, k, l contract with the colour indices of Γ_{VV+AA} and $\alpha, \beta, \gamma, \delta$ with its spin indices.

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Figure 2: Four-quark Green's functions G_{VV+AA} with momenta defining the RI/SMOM non-exceptional scheme (left), and the scheme definition for the bilinears (right). The momenta are chosen such that $p_1^2 = p_2^2 = q^2$, where $q = p_1 - p_2$. An additional momentum of $2q$ (resp. q) leaves the operator as indicated by the arrow. Every momentum or subset of momenta is of order q (never zero), which improves the pertubative running.

β	scheme	This work (48^3 and 64^3)	Previous results (24^3 and 32^3)
2.13	$\overline{\text{MS}}[\text{NDR}](\gamma, 3GeV)$	0.91764(4)	0.91983(10)(51)(180)(3)
2.13	$\overline{\text{MS}}[\text{NDR}](\not{q}, 3GeV)$	0.94611(3)	0.94672(11)(84)(63)(12)
2.25	$\overline{\text{MS}}[\text{NDR}](\gamma, 3GeV)$	0.94346(5)	0.94284(17)(49)(48)(2)
2.25	$\overline{\text{MS}}[\text{NDR}](\not{q}, 3GeV)$	0.96722(2)	0.96698(13)(86)(9)(2)

Table 1: Z_{B_K} results from different ensembles and schemes. On the right column we remind previous results with a very similar action[3], where here the first number is statistical error, then scale setting error, then m_s errors, then $Z_A - Z_V$ errors. Those number agree within systematic errors, and the statistical error has been drastically reduced even with a small number of configurations.

This condition gives us the factor

$$\frac{Z_{VV+AA}}{Z_q^2} = \frac{P\Gamma_{VV+AA}(\mathscr{O}_{VV+AA})\,|_{\text{tree}}}{P\Gamma_{VV+AA}(\mathscr{O}_{VV+AA}^{RI})} \tag{3.4}$$

and we do the same for bilinear operators to get

$$\frac{Z_A}{Z_q} = \frac{P\Gamma_A(\mathscr{O}_A)\,|_{\text{tree}}}{P\Gamma_A(\mathscr{O}_A^{RI})} \tag{3.5}$$

so that Z_q cancels in the ratio

$$Z_{B_K} = \frac{Z_{VV+AA}}{Z_A^2}. \tag{3.6}$$

Those renormalisation factors, computed on the momenta $[p,0,p,0]$ with $p = 9.25$ and $p = 9.5$ thanks to boundary twisting, are run and matched to the pertubatively-defined $\overline{\text{MS}}[\text{NDR}]$ at scale 3 GeV, as shown in Fig. 3 with other results summarized in Tab. 1.

4. Preliminary Continuum Extrapolation

After multiplying our bare B_K by those values for the two schemes (γ, γ) and $(\not{q}, \not{q})$, we can
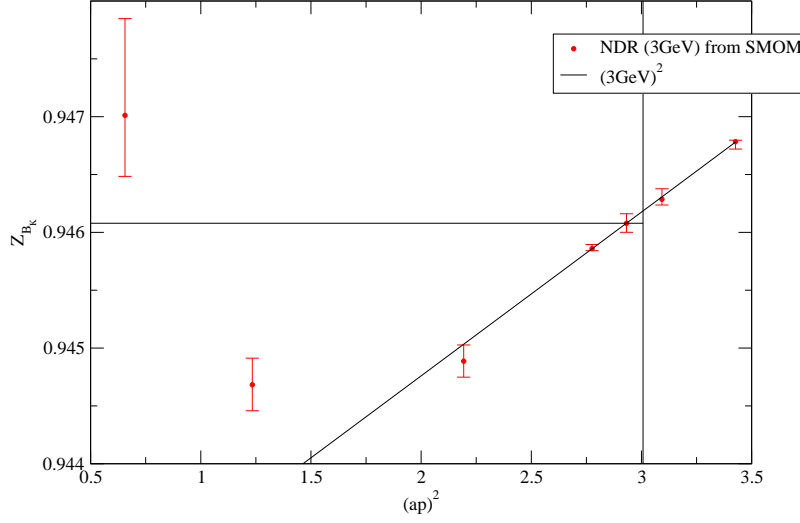
Figure 3: Renormalisation factor Z_{B_K} in $\overline{\text{MS}}$[NDR](3 GeV) from intermediate scheme RI/SMOM$_{\not{q}}$, with 48^3 ensemble. The vertical line shows the $(ap)^2$ value corresponding to the scale $|p| = 3$ GeV, while the horizontal line show the value for the closest simulated point relative to this scale. The slope of the linear fit is one of the ways to evaluate the systematic error due to not having an infinitely large Rome-Southampton window, although we prefer using the difference between different intermediate schemes at the end of the computation. Only the two points close to 3 GeV are actually used, the other momenta having very low statistics.

extrapolate them to the continuum, removing $\mathcal{O}(a^2)$ effects :

$$B_K^{\gamma,\gamma} = 0.5241(11)_{stat48}(17)_{stat64}(4)_{a48}(7)_{a64} \tag{4.1}$$

$$B_K^{\not{q},\not{q}} = 0.5332(12)_{stat48}(17)_{stat64}(7)_{a48}(12)_{a64}, \tag{4.2}$$

where we have separated errors coming from first the B_K value on the 48^3 lattice, then on the 64^2 lattice, and then the errors propagated from each scale setting. The difference between those two values is an estimate of the systematic error due to the perturbation conversion from intermediate MOM schemes to $\overline{\text{MS}}$. Concerning the central value any combination of these figures could theoretically be justified, however we choose to use the $(\not{q},\not{q})$ scheme as the reference as we observe a better perturbative matching from it. Including the estimation of finite-volume errors from our earlier ensembles[1], we then quote the following *preliminary* result:

$$B_K^{\overline{\text{MS}}}(3 \text{ GeV}) = 0.533(3)_{stat}(0)_\chi(3)_{FV}(11)_{NPR} \tag{4.3}$$

5. Conclusion

This value has to be compared to the previous results of our collaboration in [4]

$$B_K^{\overline{\text{MS}}}(3 \text{ GeV}) = 0.535(8)_{stat}(7)_\chi(3)_{FV}(11)_{NPR} \tag{5.1}$$

[1]This is a conservative estimation, since the application of ChPT[6] suggests that this error is now much reduced.
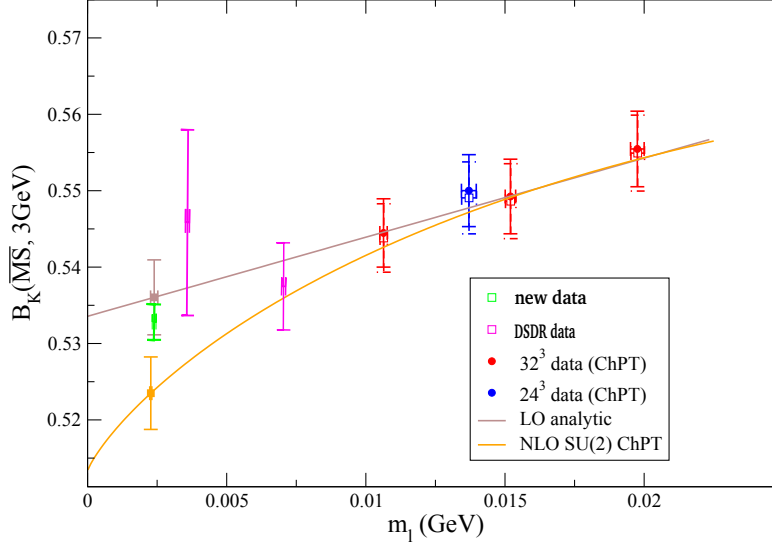
Figure 4: We present the results of this study, in green, compared to previous results used in a chiral extrapolation. The fits we show are only based on blue and red points, so that the systematic error separating the two formulae is very visible, although the purple data already suppressed much of this problem[4]. In the present study, on top of improving the statistical error, those systematics are completely removed.

Collaboration	$\hat{B}_K$
This work (preliminary)	0.755(4)(15)
RBC-UKQCD'12	0.758(11)(19)
RBC-UKQCD'10	0.749(7)(26)
BMW'11	0.773(8)(9)
SWME'11	0.716(10)(35)
Aubin'09	0.724(8)(29)
ETM'10 ($N_f = 2$)	0.729(25)(17)

Table 2: RGI value for B_K derived from different previous works. Other values might be found in [1, 2] .

and the older [3]

$$B_K^{\overline{\text{MS}}}(3 \text{ GeV}) = 0.529(5)_{stat}(15)_\chi(2)_{FV}(11)_{NPR}. \tag{5.2}$$

Although those result are perfectly compatible within error bars, we saw a slight increase of the central value with the inclusion of the near-physical mass ensembles, which reduced the gap between Taylor and ChPT fits and discarded large chiral logarithmic effects. The current analysis clearly confirms this phenomenum seen in [4].

We also provide a value in the Renormalisation Group Independent (RGI) scheme in Tab. 2 in order to allow for a comparison with results of some other collaborations.

We note that most determinations have a central slightly lower than the value we present. We suspect this could be related to ChPT fits outside their range of validity, and also the use of only Z

factors from schemes whose running is less well described by perturbation theory than $\text{RI}/\text{SMOM}_{\slashed{q}}$.

Most of the works also are, like this one, dominated by systematic errors. In our case, now that data down to the physical pion mass has eliminated the main systematic error, mass extrapolation, nearly all of the error is due to NPR.

Therefore we have shown that our new data and techniques provide an important improvement of the B_K value, leading to $B_K^{\overline{\text{MS}}}(3 \text{ GeV}) = 0.533(3)(11)$. This is still preliminary, and in particular would need more statistics on NPR to be confident of the currently negligible statistical error. The direct simulation at physical pion mass has been a very important ingredient, eliminating the dominant systematic error. The very obvious conclusion is now that what is important is improving the non-perturbative renormalisation. This would in particular imply determining Z_{B_K} at higher energy scales , which in turn would imply simulating on finer and finer lattices. Renormalisation on finer lattices could be achieved on small volumes with step scaling. This includes running over a charm threshold since it is important to match to perturbation theory at scales above those where the real world charm becomes relevant. This will be a major objective for the next simulations. Additionally, the determination of Z factors can also be improved by designing schemes with a better matching, a direction that has already been intensively studied by our collaboration, with improvements of the RI-MOM scheme already used in this work.

Acknowledgements

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