PROCEEDINGS OF SCIENCE

Calculation of $K \to \pi\pi$ decay amplitudes with improved Wilson fermion

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We present results of our trial calculation of the $K \to \pi\pi$ decay amplitudes with the improved Wilson fermion action. Calculations are carried out with $N_f = 2+1$ gauge configurations generated with the Iwasaki gauge action and non-perturbatively $O(a)$-improved Wilson fermion action at $a = 0.091\,\mathrm{fm}$, $m_\pi = 280\,\mathrm{MeV}$ and $m_K = 560\,\mathrm{MeV}(\sim 2m_\pi)$ on a $32^3 \times 64$ ($La = 2.9$fm) lattice.

31st International Symposium on Lattice Field Theory LATTICE 2013
July 29 - August 3, 2013
Mainz, Germany

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PoS(LATTICE 2013)474

1. Introduction

Calculation of the $K \to \pi\pi$ decay amplitudes is very important to quantitatively understand the $\Delta I = 1/2$ rule in the neutral K meson system and the prediction of the direct CP violation parameter $(\varepsilon'/\varepsilon)$ from the standard model. A result for the decay amplitude for the $\Delta I = 3/2$ process at the physical quark mass was reported by RBC-UKQCD Collaboration in Ref. [1]. They also reported a first direct calculation of the amplitude for the $\Delta I = 1/2$ process carried out at $m_\pi = 422\,\mathrm{MeV}$ in Ref. [2]. They employed the domain wall fermion action in these calculations.

In the present work we attempt a direct calculation of the $K \to \pi\pi$ decay amplitudes for both the $\Delta I = 1/2$ and $3/2$ processes with the improved Wilson fermion action. As we discuss below, mixings with four-fermion operators with wrong chirality are absent for the parity odd process even for the Wilson fermion action. A mixing to a lower dimension operator does occur, which gives unphysical contributions to the amplitudes on the lattice. However, it can be non-perturbatively subtracted by imposing a renormalization condition. After the subtraction we can obtain the physical decay amplitudes by the renormalization factor having the same structure as for the continuum. Therefore, by using the Wilson fermion action, statistical improvement is expected with the lattice calculation of the amplitudes for the $\Delta I = 1/2$ process, because calculations with Wilson fermion action are computationally much less expensive than those with the domain wall fermion action.

Our calculations are carried out on a subset of configurations previously generated by PACS-CS Collaboration with the Iwasaki gauge action and non-perturbatively $O(a)$-improved Wilson fermion action at $\beta = 1.9$ on a $32^3 \times 64$ lattice [3]. The subset corresponds to the hopping parameters $\kappa_{ud} = 0.13770$ for the up and down quark and $\kappa_s = 0.13640$ for the strange quark. The parameters determined from the spectrum analysis for this subset are $a = 0.091$ fm and $La = 2.91$ fm, $m_\pi = 280\,\mathrm{MeV}$ and $m_K = 560\,\mathrm{MeV}(\sim 2m_\pi)$. We consider the K meson decay process to the zero momentum two pions on these configurations. We further generate gauge configurations at the same parameters to improve the statistics. The total number of gauge configurations used in the present work is 343 which corresponds to $8,575$ trajectories.

2. Operator mixing

In this section we discuss operator mixing of the $\Delta S = 1$ weak operators for the Wilson fermion action. In the continuum, the effective Hamiltonian of the $K \to \pi\pi$ decay is given by a linear combination of 10 four-fermion operators (Q_j for $j = 1, 2, \cdots 10$) [4], of which 7 operator are linearly independent. They can be classified by the irreducible representation of the flavor $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry group as $(\mathbf{27},\mathbf{1}) + 4\cdot(\mathbf{8},\mathbf{1}) + 2\cdot(\mathbf{8},\mathbf{8})$, whose components are given by

$$
\begin{array}{lll}
(\mathbf{27},\mathbf{1}) & Q_1' = 3Q_1 + 2Q_2 - Q_3\ , & \\
(\mathbf{8},\mathbf{1}) & Q_2' = 2Q_1 - 2Q_2 + Q_3\ , & Q_3' = -3Q_1 + 3Q_2 + Q_3\ , \\
 & Q_5 = (\bar{s}d)(\bar{u}u + \bar{d}d + \bar{s}s)_{LR}\ , & Q_6 = (\bar{s}\times d)(\bar{u}\times u + \bar{d}\times d + \bar{s}\times s)_{LR}\ , \\
(\mathbf{8},\mathbf{8}) & Q_7 = (\bar{s}d)(\bar{u}u - \bar{d}d/2 - \bar{s}s/2)_{LR}\ , & Q_8 = (\bar{s}\times d)(\bar{u}\times u - \bar{d}\times d/2 - \bar{s}\times s/2)_{LR}\ ,
\end{array}
\tag{2.1}
$$

with $Q_1 = (\bar{s}d)(\bar{u}u)_{LL}$, $Q_2 = (\bar{s}\times d)(\bar{u}\times u)_{LL}$ and $Q_3 = (\bar{s}d)(\bar{u}u + \bar{d}d + \bar{s}s)_{LL}$, where $(\bar{s}d)(\bar{u}u)_{L,R/L} = (\bar{s}\gamma_\mu(1-\gamma_5)d)(\bar{u}\gamma_\mu(1\pm\gamma_5)u)$ and $\times$ means contraction of the color indices : $(\bar{s}\times d)_L(\bar{u}\times d)_L = (\bar{s}_a d_b)_L(\bar{u}_b d_a)_L$.

In the continuum, mixings between operators in different representations are forbidden. For the Wilson fermion action, however, chiral symmetry is broken to the vector subgroup, $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \to \mathrm{SU}(3)_V$. Hence mixings among different representations are in general allowed, and new operators arise through radiative corrections. However, these problems are absent for the parity odd part of the operators in (2.1), which are the operators considered for the direct calculation of the $K \to \pi\pi$ decay amplitudes in the present work.

To investigate the operator mixing, we exploit the full set of unbroken symmetries for the Wilson fermion, namely flavor $\mathrm{SU}(3)_V$, parity P, charge conjugation C, and CPS which is the symmetry under CP transformation followed by the exchange of the d and s quarks. All operators in (2.1) are $CPS = +1$ operators. We know that the following operators also have the same quantum numbers as the operators in (2.1). We therefore have to consider operator mixing with them,

$$Q_X = (\bar{s}d)(\bar{d}d - \bar{s}s)_{SP+PS} \quad , \quad Q_Y = (\bar{s} \times d)(\bar{d} \times d - \bar{s} \times s)_{SP+PS} \, , \tag{2.2}$$

where $(\bar{s}d)(\bar{d}d)_{SP+PS} = (\bar{s}d)_S(\bar{d}d)_P + (\bar{s}d)_P(\bar{d}d)_S$, $(\bar{s}d)_S = \bar{s}d$ and $(\bar{s}d)_P = \bar{s}\gamma_5 d$.

It was shown in Ref. [5] that the parity odd part of the LL and LR type operators, and the $SP+PS$ type operators do not mix with each other by the gluon exchange diagrams due to the CPS, CPS' and CPS'' symmetry, where S' is defined as $(\overline{\psi_1}\psi_2)(\overline{\psi_3}\psi_4) \to (\overline{\psi_2}\psi_1)(\overline{\psi_4}\psi_3)$ and S'' by $(\overline{\psi_1}\psi_2)(\overline{\psi_3}\psi_4) \to (\overline{\psi_4}\psi_3)(\overline{\psi_2}\psi_1)$. Thus the operators $Q_{X,Y}$ (the $SP+PS$ type) do not mix with those in (2.1) (the LL and LR type). The operators $Q_{7,8} \in (\mathbf{8},\mathbf{8})$ (the LR type) do not mix with the LL type operators ($Q_{1,2,3} \in (\mathbf{27},\mathbf{1}),(\mathbf{8},\mathbf{1})$), and also with $Q_{5,6} \in (\mathbf{8},\mathbf{1})$ (the LR type) because the gluon exchange diagrams do not change the flavor structure and these operators have different structures. Further the mixing between the $(\mathbf{27},\mathbf{1})$ and the $(\mathbf{8},\mathbf{1})$ representation is forbidden by the flavor $\mathrm{SU}(3)_V$ symmetry. To sum up, the renormalization factor for the gluon exchanging diagrams has the same form as for the continuum.

Next let us investigate the possibility of unwanted mixings though the penguin diagrams. In the penguin diagrams for $Q_{7,8} \in (\mathbf{8},\mathbf{8})$, cancellation of the quark loops at the weak operators occurs as $d-s$. This means that the renormalization due to the penguin diagram is proportional to the quark mass difference and mixing to four-fermion operators is absent due to the dimensional reason. In addition the operator arising from the penguin diagrams should have the flavor structure $(\bar{s}d)(\bar{u}u + \bar{d}d + \bar{s}s)$, which is different from that of $Q_{7,8}$. Thus operator mixing from $Q_{7,8} \in (\mathbf{8},\mathbf{8})$ to the other representations and its reverse are absent. These statements also hold for $Q_{X,Y}$ in (2.2) for the same reason, and the operators $Q_{X,Y}$ are fully isolated in the theory.

Up to now, we have shown that the renormalization factor for the parity odd part of the four-fermion operators in (2.1) have the same form as that in the continuum. Here we consider the mixing to lower dimensional operators. From the CPS symmetry and the equation of motion of the quark, there is only one operator with $\dim < 6$, which is

$$Q_P = (m_d - m_s) \cdot P = (m_d - m_s) \cdot \bar{s}\gamma_5 d \, . \tag{2.3}$$

This operator also appears in the continuum, but does not give a finite contribution to the physical decay amplitude, since it is a total derivative operator. But this is not valid for the Wilson fermion due to chiral symmetry breaking by the Wilson term, and the operator (2.3) gives a non-zero unphysical contribution to the amplitudes on the lattice. This contribution should be subtracted non-perturbatively, because the mixing coefficient includes a power divergence due to the lattice cutoff growing as $1/a^2$. In the present work we subtract it by imposing the following relation [6],

$$\langle 0|\,\overline{Q}\,|K\rangle = \langle 0|\,Q - \alpha(Q)\cdot P\,|K\rangle = 0\ , \tag{2.4}$$

for each operator in (2.1). The subtracted operators $\overline{Q}$ are renormalized by the renormalization factor having same form as in the continuum.

3. Calculation

We extract the decay amplitude from the time correlation function of the $K \to \pi\pi$ process,

$$G(Q^I)(t) = \frac{1}{T}\sum_{\delta=0}^{T-1} \langle 0|\, W_K(t_K+\delta)\ \overline{Q}(t+\delta)\ W^I_{\pi\pi}(t_\pi+\delta)\,|0\rangle\ , \tag{3.1}$$

where $W_K(t)$ is the wall source for the K^0 meson and $W^I_{\pi\pi}(t)$ is that for the isospin I two-pion system. $\overline{Q}(t)$ is the subtracted weak operator defined by (2.4). We impose the periodic boundary condition in all directions. The summation over δ, where $T = 64$ denotes the temporal size of the lattice, is taken to improve the statistics. We set $t_K = 26$ and $t_\pi = 0$ in the present work. The gauge configurations are fixed to the Coulomb gauge at the time slice of the wall source $t = t_K + \delta$ and $t_\pi + \delta$ for each δ. The mixing coefficient of the lower dimensional operator $\alpha(Q)$ is evaluated from the ratio,

$$\alpha(Q) = \sum_{\delta_1=0}^{T-1} \langle 0|\, W_K(t_K+\delta_1)\, Q(t+\delta_1)\,|0\rangle \Big/ \sum_{\delta_2=0}^{T-1} \langle 0|\, W_K(t_K+\delta_2)\, P(t+\delta_2)\,|0\rangle\ , \tag{3.2}$$

in the large t region.

For the calculation of quark loop at the weak operator $Q(x,x)$, *i.e.*, the quark propagator starting from the weak operator and ending at the same position, we use the stochastic method with the hopping parameter expansion technique (HPE) and the truncated solver method (TSM) proposed in Ref. [7]. The action of the Wilson fermion can be written as

$$S^W = \bar\psi\, W\, \psi = \bar\psi\,(M - D)\,\psi = \bar\psi\, M(1-\bar D)\,\psi \quad , \qquad (\,\bar D = M^{-1}D\,) \tag{3.3}$$

$$(M\psi)(x) = (1 - \kappa C_{SW}(\sigma\cdot F(x))/2)\psi(x)\ , \tag{3.4}$$

$$(D\psi)(x) = \kappa\sum_\mu \left(p^-_\mu U_\mu(x)\psi(x+\mu) + p^+_\mu U^\dagger_\mu(x-\mu)\psi(x-\mu)\right)\ , \tag{3.5}$$

where $p^\pm_\mu = 1 \pm \gamma_\mu$. From (3.3) the quark propagator Q can be written by a hopping parameter expansion form as $Q = W^{-1} = \sum_{n=0}^{k-1} \bar D^n M^{-1} + \bar D^k W^{-1}$ for any integer value of k. We use this to calculate the quark loop $Q(x,x)$ at the weak operator. In this case, the terms with the odd power of $\bar D$ do not contribute, thus $Q(x,x) = (M^{-1} + \bar D^2 M^{-1} + \bar D^4 W^{-1})(x,x)$ for $k = 4$. Using this, we calculate the quark loop by the stochastic method as,

$$Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{N_R}\sum_{j=1}^{N_R} \xi^*_j(\mathbf{x},t)\ S_j(\mathbf{x},t)\ , \tag{3.6}$$

$$S_j(\mathbf{x},t) = \sum_{\mathbf{y}} \left(M^{-1} + \bar D^2 M^{-1} + \bar D^4 W^{-1}\right)(\mathbf{x},t;\mathbf{y},t)\ \xi_j(\mathbf{y},t)\ , \tag{3.7}$$

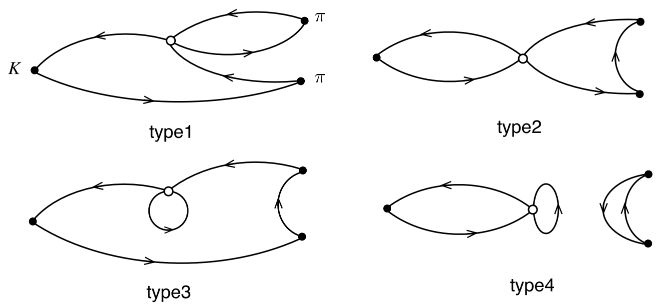

Figure 1: Quark contraction of $K \to \pi\pi$ decay.

where we introduce $U(1)$ noise $\xi_j(\mathbf{x},t)$ which satisfies $\delta^3(\mathbf{x}-\mathbf{y}) = 1/N_R \cdot \sum_{j=1}^{N_R} \xi_j^*(\mathbf{x},t)\xi_j(\mathbf{y},\mathbf{t})$ for $N_R \to \infty$. The effect of HPE for the quark loop is removing the $\bar{D}$ and $\bar{D}^3$ terms in (3.7) explicitly which make only statistical noise. We find that HPE reduce the statistical error of the decay amplitudes to about 50% compared with the normal stochastic method.

We also implement the truncated solver method (TSM) for (3.6) by

$$Q(\mathbf{x},t;\mathbf{x},t) = \frac{1}{N_T}\sum_{j=1}^{N_T} \xi_j^*(\mathbf{x},t)\, S_j^T(\mathbf{x},t) \; + \; \frac{1}{N_R}\sum_{j=N_T+1}^{N_T+N_R} \xi_j^*(\mathbf{x},t)\, \left[S_j(\mathbf{x},t) - S_j^T(\mathbf{x},t) \right] , \tag{3.8}$$

where $S_j^T(\mathbf{x},t)$ is a value given with the quark propagator W^{-1} calculated with a loose stopping condition in (3.7) and $S_j(\mathbf{x},t)$ is that with a stringent condition. We set $N_T = 5$ and the stopping condition $R \equiv |WW^{-1} - \xi|/|\xi| < 1.2 \times 10^{-6}$ for $S_j^T(\mathbf{x},t)$, and $N_R = 1$ and $R < 10^{-14}$ for $S_j(\mathbf{x},t)$. We find that contributions of the second term of (3.8) to the decay amplitudes are negligible compared with the statistical error. Thus we neglect the second term in (3.8) and estimate the quark loop by only the first term by setting $N_T = 6$ for TSM, confierming that the contributions of the second term are negligible by additional calculations of $S_j(\mathbf{x},t)$ for all gauge configurations. The numerical cost of TSM (3.8) is about twice of that without TSM (3.6) with $N_R = 1$.

4. Results

There are four types of quark contractions for the $K \to \pi\pi$ decay as shown in Fig. 1, where the naming of the contractions follows that by RBC-UKQCD [2]. The results for the time correlation function (3.1) of Q_2 for the $\Delta I = 1/2$ process are plotted in Fig. 2. We adopt $K^0 = -\bar{d}\gamma_5 s$ as the neutral K meson operator, so our correlation function has an extra minus from the usual convention. We find a large cancellation in $\overline{Q}_2$ between the contributions from the operator Q_2 and $\alpha(Q_2) \cdot P$ for the *type3* contraction. This is not seen for the *type4* contraction. In (d) we compare the correlation functions calculated with TSM (3.8) and without TSM (3.6) with $N_R = 1$. TSM improves the statistics drastically. The numerical cost of TSM is about twice of that without TSM as already mentioned. Thus TSM is a very efficient method.

The results for Q_6 for $\Delta I = 1/2$ are shown in Fig. 3. Here we find a large cancellation in $\overline{Q}_6$ between the contributions of Q_6 and $\alpha(Q_6) \cdot P$ for both the *type3* and *type4* contractions. In (c) a large cancellation is also seen between the *type1* and *type2* contractions. An efficiency of TSM is also observed for Q_6 in (d).

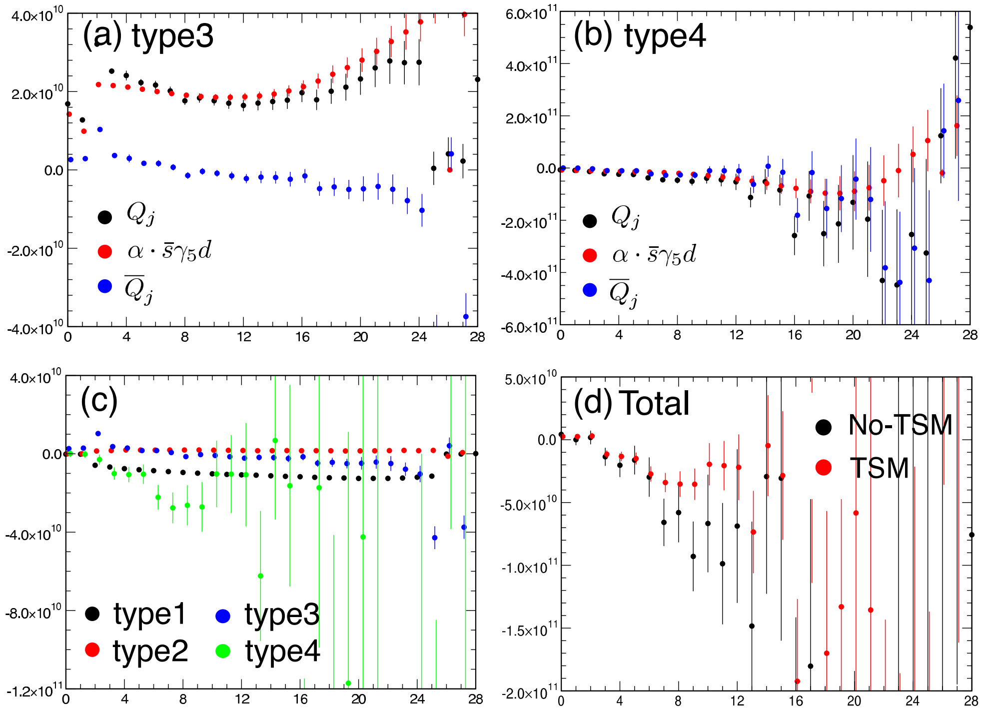

Figure 2: Time correlation function of Q_2 for the $\Delta I = 1/2$ decay. (a) *type3* contribution for Q_2, $\alpha(Q_2) \cdot P$ and $\overline{Q}_2 = Q_2 - \alpha(Q_2) \cdot P$, (b) *type4* contribution, (c) results for each type of contractions for $\overline{Q}_2$, (d) results of the total correlation functions calculated with TSM and without TSM.

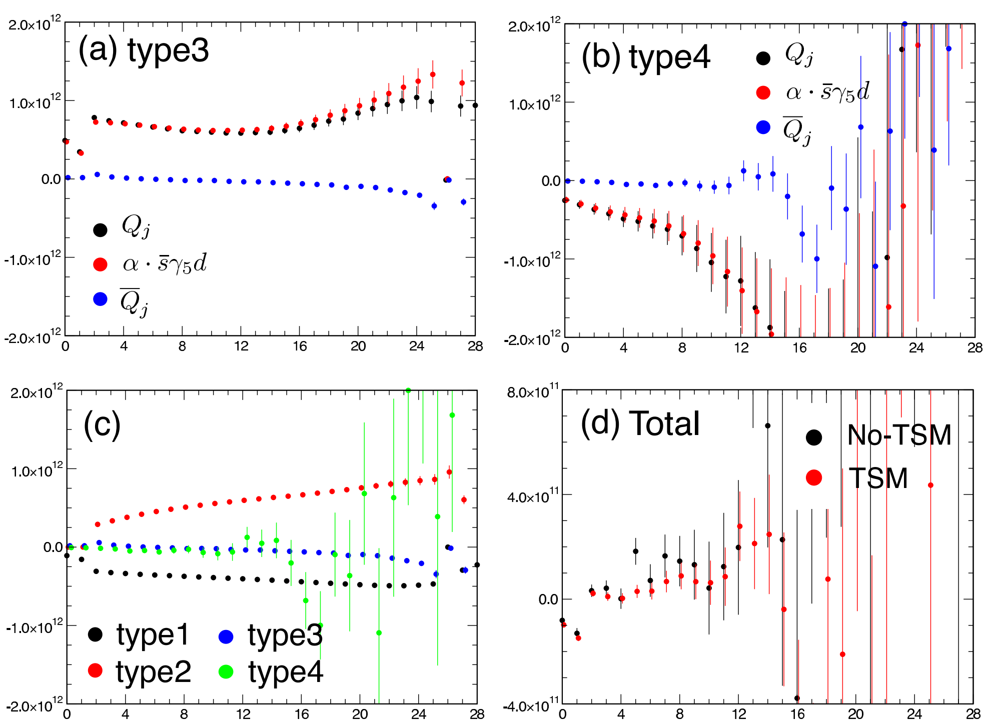

Figure 3: Time correlation function of Q_6 for the $\Delta I = 1/2$ decay following the same convention as in Fig. 2.

We extract the decay amplitude $M(Q^I) = \langle K|\overline{Q}|\pi\pi;I\rangle$ by fitting the time correlation function (3.1) with a fitting function,

$$G(Q^I)(t) = M(Q^I)\cdot(1/F_{LL})\cdot N_K N^I_{\pi\pi}\cdot \mathrm{e}^{-m_K(t_K-t)-E_{\pi\pi}(t-t_\pi)}\times(-1)\ , \qquad (4.1)$$

with the energy of the two-pion state $E_{\pi\pi}$ which is fixed at a value obtained from the $\pi\pi \to \pi\pi$ correlation function. The factor (-1) comes from the convention of the K^0 operator. The factor $N_K = \langle 0|W_K|K\rangle$ and $N^I_{\pi\pi} = \langle 0|W^I_{\pi\pi}|\pi\pi;I\rangle$ are estimated from the wall to wall propagator of the K meson and the two-pion. F_{LL} is the Lellouch-Lüscher factor [8]. In the present work, precise results for the scattering phase shift for the two-pion system are not yet available, so we adopt the factor for the noninteracting case, $F_{LL} = \sqrt{2m_K L^3}(\sqrt{2m_\pi L^3})^2$. Our results are given by

$$a\cdot M(Q_2^{I=0}) = (+4.43\pm1.62)\times10^{-2}\ , \quad a\cdot M(Q_6^{I=0}) = (-1.34\pm0.85)\times10^{-1}\ , \qquad (4.2)$$

for the fitting range $t = [8,12]$. The signal to noise ratios are comparable to those of RBC-UKQCD [2]. In the next step, we will correct these bare values by the renormalization factors and multiply with the coefficient functions to obtain the physical results for A_0 and A_2.

5. Summary

In the present work we have reported on our attempt at a direct calculation of the $K \to \pi\pi$ decay amplitudes for both the $\Delta I = 1/2$ and $3/2$ channels with the Wilson fermion action. We have shown that the unwanted mixings with wrong chirality operators are absent for the $K \to \pi\pi$ decay even for the Wilson fermion action. We have calculated the decay amplitudes by using the stochastic method with the hopping parameter expansion technique and the truncated solver method. We have shown that these two methods are efficient.

Acknowledgments

This work is supported in part by Grants-in-Aid of the Ministry of Education No. 23340054. The numerical calculations have been carried out on T2K-TSUKUBA at University of Tsukuba, SR16000 at University of Tokyo, and K-computer at RIKEN AICS.

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