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Vacuum polarization function in $N_f = 2+1$ domain-wall fermion

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We report lattice calculation of vacuum polarizatoion function (VPF) with $N_f = 2+1$ domain-wall fermions. In this calculation, by using the all-mode-averaging (AMA) techniques, the precise lattice calculation of VPF is perfomed. By comparison with two cut-off scales, we estimate the lattice discretization effect. Fitting VPF in high momentum region, using the operator product expansion (OPE) and perturbative QCD, we obtain the more reliable estimate of strong coupling constant, and furthermore quark condensate is also evaluated. On the other hand, in low momentum region, we show that the lowest Pade approximation is in good agreement with precise VPFs even at the lowest momentum point. This result encouranges for providing robust value of leading order of hadronic contribution to muon g-2.

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1. Introduction

In this proceedings, we report on the progress of calculation of strong coupling constant and the leading order of hadronic contribution (HLO) to muon g-2 from vacuum polarization function (VPF). VPF as a function of Euclidean momentum squared, $-q^2 = Q^2$, can be extracted from the two-point correlation function of vector current. VPF provides rich information for hadronic contribution to low-energy physics and perturbative QCD.

The determination of Q^2 dependence of VPFs is important to evaluate the non-perturbative theoretical value of muon g-2 and the precise test of the Standard model (SM). On the other hand, in high momentum regime, the operator product expansion (OPE) represents VPF as a pertubative diagram of quark-gluon and the multi-dimensional operator condensate with inverse of power of Q^2. Non-perturbative comparison with lattice calculation is important task for the check of valid regime of OPE formula and estimate of QCD scale $\Lambda_{\rm QCD}$.

Although there have been several attempts of lattice calculation of HLO muon g-2 [1, 2, 3, 4] ([5] and see references therein), the total uncertainty is more than 10 times larger than currently phenomenological estimate from experimental data [6]. Especially the large statistical error in low-Q^2 is significant in this calculation, since the QED weight function around muon mass squared ($Q^2 \simeq 0.01$ GeV^2) is much dominant in total Q^2 integral. The reduction of statistical error of VPFs is the first step for precise measurement of HLO muon g-2 on the lattice.

Comparing the Adler function to lattice VPF, we directly evaluate the renormalized strong coupling constant α_s in the $\overline{\rm MS}$ scheme. As shown in Ref.[7, 8], the precise value of $\alpha_s^{\overline{\rm MS}}$ is obtained from the pure perturbative formula at 3-loop and the contribution of quark and gluon condensate in OPE.

Here we show the preliminary results of these observables using the all-mode-averaging (AMA) technique to drastically reduce the statistical error. In this study we use two cut-off ensembles, $24^3 \times 64$ at $a^{-1} = 1.73$ GeV and $32^3 \times 64$ at $a^{-1} = 2.23$ GeV, and estimate the systematic error due to lattice discretization. Table 1 shows the details of simulation parameter. We obtain the Lanczos method to compute the exact N_λ low-lying modes of even-odd preconditioned Hermitian Dirac operator. In AMA, we employ relaxed CG solver with fixed CG iteration as approximation [9, 10].

Table 1: Lattice parameters in the simulation.N_λ means the number of low-lying mode we use in deflation method of CG process, and $N_{\rm CG}^{\rm AMA}$ means the stopping iteration of relaxed CG in AMA. N_G is the number of source locations of approximation.

Lattice size	a^{-1} GeV	m_s	m_{ud}	m_π (GeV)	N_λ	N_G	$N_{\rm CG}^{\rm AMA}$	$N_{\rm conf}$
$24^3 \times 64$	1.73	0.04	0.005	0.33	140	32	180	192
			0.01	0.42	140	32	150	172
			0.02	0.54	140	32	150	105
$32^3 \times 64$	2.23	0.03	0.004	0.28	140	32	180	100
			0.006	0.33	140	32	180	113
			0.008	0.38	140	32	180	122

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2. Extraction of vacuum polarization function

In the continuum theory, the VPF is given by the expansion of vector two-point correlation function in four-dimensional momentum representation,

$$i\int d^4x e^{iqx}\langle T\{V^a_\mu(x)V^{b\dagger}_\nu(0)\}\rangle = -\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\delta^{ab}\Pi^{\rm cont}_V(q^2) \equiv \delta^{ab}\Pi^{\rm cont}_{\mu\nu}(q). \qquad (2.1)$$

The above equation is satisfied with charge conservation, $q^\mu\Pi^{\rm cont}_{\mu\nu}(q) = q^\nu\Pi^{\rm cont}_{\mu\nu}(q) = 0$ for vector current $V^a_\mu = \bar{q}\gamma_\mu\tau^a q$. τ is SU(3) isospin generator matrix. For the computation of VPF in HLO muon g-2, we adopt the electromagnetic current, $V^{\rm EM}_\mu(x) = \bar{q}\gamma_\mu Q_e q$, in which charge matrix for light and strange quark is $Q_e = {\rm diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$. This is also described by the light and strange quark propagators D^{-1}_{ud}, D^{-1}_s as

$$\langle V^{\rm EM}_\mu V^{\rm EM}_\nu\rangle(x) \simeq (e_u^2 + e_d^2)\langle V^{ud}_\mu V^{ud}_\nu\rangle_c(x) + e_s^2\langle V^s_\mu V^s_\nu\rangle_c(x), \qquad (2.2)$$

where the above equation is used in the assumption of SU(3) isospin limit, in which there is no contribution of disconnected diagram due to the cancellation $e_u + e_d + e_s = 0$.

In lattice QCD, we use the combination of conserved and local current of two-point correlation function as well as used in [8, 3]. The local current $V^{\rm loc}_\mu$ is same form as continuum theory, however, it is not satisfied with charge conservation. We multiply the renormalization constant $Z_V = 0.7178$ which is computed non-perturbatively in Ref.[11]. The conserved current has the charge conservation $\sum_\mu \hat{Q}_\mu \Pi_{\mu\nu} = 0$, where we define $\hat{Q}_\mu = e^{iQ_\mu} - 1 = e^{iQ_\mu/2}2i\sin(Q_\mu/2)$. To extract the VPF from lattice vector two-point correlation function, we modify the expansion of Eq.(2.1) as

$$\begin{aligned}\Pi^q_{\mu\nu}(Q) &= \Pi^q_V(Q)(\hat{Q}_\mu\hat{Q}_\nu - \delta_{\mu\nu}\hat{Q}^2) + \mathscr{O}((aQ)^6)\\ &= \Pi^q_V(Q)\left[Q_\mu Q_\nu - \frac{Q_\mu^2 Q_\nu^2}{16} - \delta_{\mu\nu}\left(Q^2 - \frac{Q^4}{16}\right)\right] + \mathscr{O}((aQ)^6),\end{aligned} \qquad (2.3)$$

where, in the last equation, we also expand $\hat{Q}_\mu$ with $aQ_\mu = 2\pi n_\mu/L$. Below $(aQ)^2 = 0.7$ region, we ignore higher order contribution than $O((aQ)^6)$, which contains the lattice artifact due to using non-conserved current in this formula. As shown in [8, 3], this effect is negligibly small in this region. In practice, $\Pi_V(Q)$ is obtained by fitting lattice data for different combination of Q_μ at each Q^2 with function of Eq.(2.3).

Furthermore we exclude the momentum which has only finite value at one direction above $Q^2 \simeq 0.2$ GeV^2 region, such as $n^{\rm exclude}_\mu = \big\{(n_i = 1, n_t = 0), (n_i = 0, n_t = 3), (n_i = 0, n_t = 4), (n_i = 2, n_t = 0)\big\}$, except for $(n_i = 0, n_t = 1)$ and $(n_i = 0, n_t = 2)$ in the case of $24^3 \times 64$ at $a^{-1} = 1.73$ GeV, and $(n_i = 0, n_t = 1)$ at $a^{-1} = 2.23$ GeV. As a consequence of the above subtraction, we can conservatively avoid the lattice artifacts effect.

3. Estimate of strong coupling constant

The formula of OPE of vector VPF is expressed as

$$\begin{aligned}\Pi_V(Q^2) = &\frac{c}{\mu^2 a^2} + C_0(l_\mu(Q^2),\alpha_s) + C^V_m(l_\mu(Q^2),\alpha_s)\frac{m_q^2}{Q^2} + C^V_{\bar{q}q}(l_\mu(Q^2),\alpha_s)\frac{\langle m\bar{q}q\rangle}{Q^4}\\ &+ C^{\rm loop}_{\bar{q}q}(l_\mu(Q^2),\alpha_s)\frac{\sum_f\langle m_f\bar{q}q\rangle}{Q^4} + C_{GG}(l_\mu(Q^2),\alpha_s)\frac{\left\langle\frac{\alpha_s}{\pi}GG\right\rangle}{Q^4},\end{aligned} \qquad (3.1)$$

Table 2: Result of fitting lattice VPF with OPE formula.

Lattice	a^{-1} GeV	α_s/π	$\Lambda^{(3)}_{\overline{MS}}$ (GeV)	$-\langle q\bar{q}\rangle^{1/3}_{\overline{MS}}$ (GeV)	$\langle \frac{\alpha_s}{\pi} GG\rangle$ (GeV^4)	χ^2/dof
$24^3 \times 64$	1.73	0.08192(39)	0.2486(27)	0.276(11)	0.237(18)	1.8
		0.08193(34)	0.2486(24)	0.256[fix]	0.205(3)	2.5
$32^3 \times 64$	2.23	0.08317(85)	0.2572(58)	0.325(29)	0.741(138)	1.3
		0.08196(28)	0.2489(19)	0.256[fix]	0.475(10)	2.7
$a \to 0$		0.0844(20)	0.265(14)	0.390(68)		
JLQCD[8]	1.83	0.0817(6)	0.247(5)	0.242[fix]	-0.020(2)	2.8

where $\alpha_s = \alpha_s(\mu^2)$ and $m_q = m_q(\mu^2)$ at renormalization scale $\mu = 2$ GeV. Note that we use the α_s and mass renormalization in $\overline{\rm MS}$ scheme. The mass renormalization constant is determined non-perturbatively as $Z_m^{\overline{MS}}(\mu = 2\text{GeV}) = 1.498$ in $24^3 \times 64$ and $Z_m^{\overline{MS}}(\mu = 2\text{GeV}) = 1.527$ in $32^3 \times 64$ in Ref.[11]. We also define $l_\mu(Q^2) = \ln(Q^2/\mu^2)$. The first term and second term are scheme dependence and pure QCD perturbation up to N^{3}LO $\mathcal{O}(\alpha_s^2)$ referring to [12, 13], The third term is proportional to quark mass, which is derived from expansion with respect to m_q^2, and here we use 3-loop formula referring to [14]. From the forth term, since these are sub-leading contribution to VPF, we use the leading order of Wilson coefficient, $C^V_{\bar{q}q} = 2$, $C^{\rm loop}_{\bar{q}q} = 0$ and $C_{GG} = 1/12$ [15, 13].

In this calculation, we use four free parameters in the fitting of VPF, which are c, α_s, $\langle q\bar{q}\rangle_{\overline{MS}}$ and $\langle \frac{\alpha_s}{\pi} GG\rangle$. We perform the simultaneous fitting for all quark mass in each cut-off scale. Figure 1 shows the VPF in our ensemble in the fitting Q^2 region, 0.93 GeV^2<Q^2<1.8 GeV^2. The accuracy of VPF at each momentum is almost subpercent in this fitting region, which is improved by factor 10 and more compared to [8]. One sees that fitting function is in good agreement with our accurate lattice data within 1σ statistical error, and thus it turns out that OPE formula up to N^{3}LO in Eq.(3.1) is able to precisely describe the q^2 dependence of lattice result above $Q^2 = 0.9$ GeV^2.

In Table 2, we compare the lattice results obtained in this calculation and previous work on JLQCD collaboration [8]. In Ref.[8], they have used the overlap fermion at relatively small volume and one lattice spacing, and also the condensate is fixed in computed value in the others. The strong coupling constant at $\mu = 2$ GeV is consistent between our result and JLQCD, and our results are around factor 1.5 improved for 24^3 lattice. The fitting result with free chiral condensate also makes improvement of χ^2/dof, and furthermore which is compatible with 0.256(6) [11] given from hadron spectroscopy. It is also compatible between different cut-off results, and taking continuum limit for α_s and $\langle q\bar{q}\rangle$ increases these values less than 5% for α_s and 20–30% for quark condensate. We notice that chiral condensate obtained by analysis in OPE formula is large over 1σ error, although statistical and systematic errors are still large. It indicates that there is necessary to do more detailed analysis of systematic uncertainties for $\langle q\bar{q}\rangle$.

4. Fitting VPFs in the low-Q^2 regime

In this section, we perform the analysis of VPF for estimate of HLO muon g-2, in which the current correlator has quark charge matrix instead of SU(2) isospin generator. In this section,

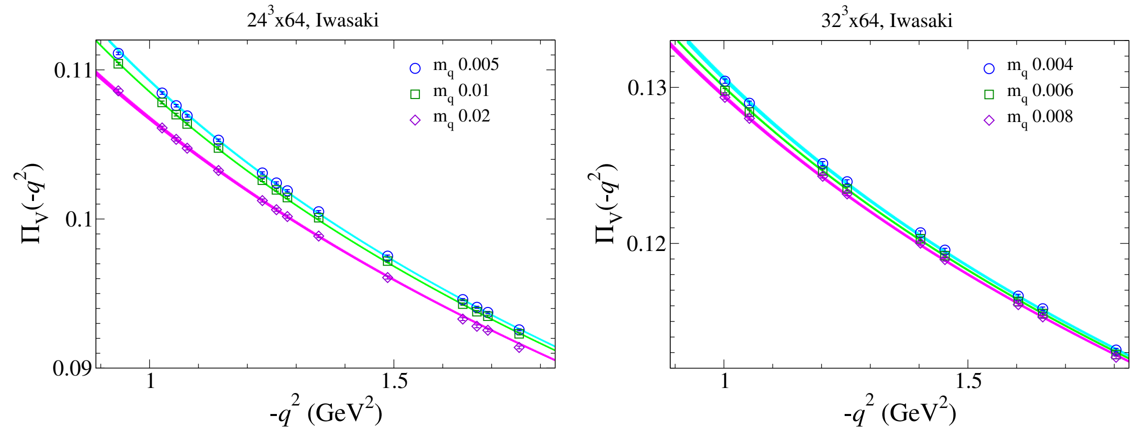

Figure 1: VPF in high-Q^2 in $24^3 \times 64$ lattice (left) and $32^3 \times 64$ lattice (right). The different symbols denote the lattice data at different masses. The solid lines denote the fitting function with OPE formula.

we use $\Pi_{\rm EM}$ as VPF from EM current correlator. First in Figure 2 we plot VPFs for light and strange sector weighted by the quark charge, and these summantion. The amount of strange VPF contribution is about 18% for total VPF, and this is due to the different factor of squared of quark charge, $e_s^2/(e_u^2+e_d^2) = 20\%$.

Here, in order to fit lattice VPF with continuous function of $-q^2 = Q^2$, we employ the following function,

$$f(-q^2) = a_0 - Q^2\Big(\frac{a_1}{b_1+Q^2} + a_2\Big), \tag{4.1}$$

which has four sorts of unknown parameters, a_0, a_1, a_2 and b_1. a_0 gives $\Pi_{\rm EM}(0)$ extrapolated into $-q^2 = 0$. This is similar construction as Pade approximation called "PA_{11}" [4], however in our fitting function, b_1 is not any constrained. We set the fitting range as $-q^2 < 0.74$ GeV^2, where is close to the minimum point for fitting with OPE as shown in previous section. In Figure 3 one sees that the function in Eq.(4.1) is in agreement with lattice data even in the lowest $-q^2$ point. In table 3, we show the value of each fitting parameter. χ^2/dof is less than 0.5, and accuracy of $\Pi_{\rm EM}(0)$ is less than 5% statistical error. Note that, at $m = 0.005$ in 24^3 lattice, the fitting parameter b_1 is smaller than $4m_\pi^2$, which regards a cut of integral for spectral function in $q^2 > 0$ axis. In Pade approximation proposed in [4], they have taken such constraint. If we constraint b_1 as $b_1 > 4m_\pi^2$, the chi-squared fitting does not convergence reasonably. This result suggests that it is necessary to take into account different fitting function from vector-dominance-like model which is assuming the contribution of tail of vector meason pole in time-like q^2 region. Currently we study the systematic uncertainties of finite size effect and lattice artifacts in low q^2 region under way.

For muon g-2 calculation of leading order of hadronic contribution, it is necessary to perform the q^2 integral with QED kernel [1, 16] for subtracted VPF

$$\hat{\Pi}_{\rm EM}(-q^2) = \Pi_{\rm EM}(0) - \Pi_{\rm EM}(-q^2). \tag{4.2}$$

We propose to divide the q^2 integral into two parts as low q^2 using Eq.(4.1) and high q^2 using OPE formula Eq.(3.1), and perform the numerical integral for $\Pi_{\rm EM}$ from $[0,\infty]$. In Eq.(4.2), $\Pi_{\rm EM}(0)$ is given from extrapolation to $-q^2 = 0$ in low q^2 fitting. Our preliminary result shows that its value

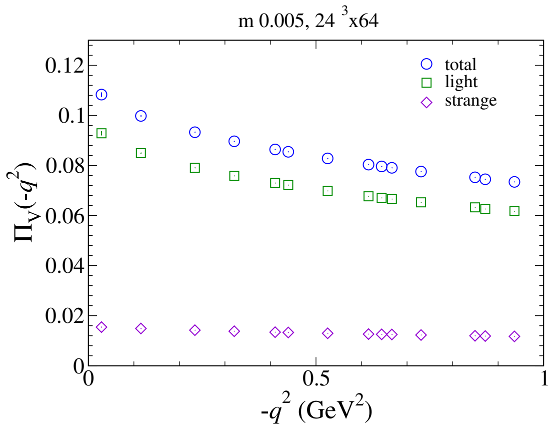

Figure 2: VPF at $m = 0.005$ on 24^3 lattice. Different symbols are VPF at light quark (squared), strange quark (diamond) and total (circle) respectively.

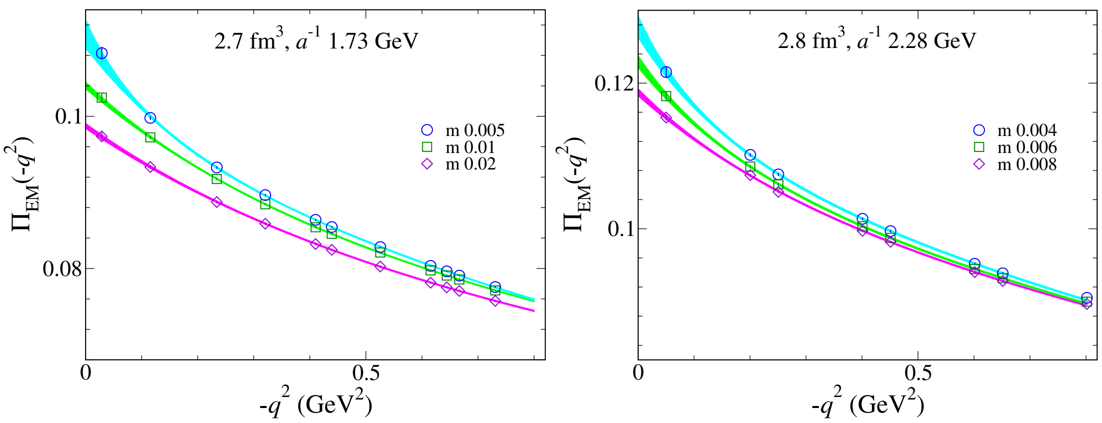

Figure 3: VPF of EM current on 24^3 (left) and 32^3 (right) lattice as a function of $-q^2$. The different symbols are VPF in different light quark mass ensembles, and solid lines are fit function.

is roughly 1.6 times larger than previous analysis with same gauge ensemble in [3], although the statistical error is 20–30% for $m = 0.005$ on 24 cube, and 10–15% for $m = 0.004$ on 32 cube. We need to carefully compare our results with [3].

5. Summary

In this proceedings, we present the recent analysis of vector vacuum polarization function (VPF) calculated in lattice QCD using domain-wall fermion. In this work, we adopt the new error reduction technique, so called as *All-mode-averaging (AMA)*, and thus the precise VPF is obtained. For the calculation of strong coupling constant, $\alpha_s^{\overline{\rm MS}}$ is given from OPE formula of vector VPF in two different cut-offs, and so that the lattice artifact effect is also taken into account. On the other hand, in low q^2 regime, using the fitting function analogous to Pade approximation, this function is

Table 3: Table of fitting parameters using the range of $-q^2 < 0.74$ GeV^2.

Lattice	m	a_0	a_1	b_1 (GeV^2)	a_2 (GeV^{-2})	χ^2/dof	$4m_\pi^2$ (GeV^2)
$24^3 \times 64$	0.005	0.1108(19)	0.0288(42)	0.258(111)	0.0164(35)	0.5	0.44
	0.01	0.1040(4)	0.0466(30)	0.763(81)	0.0056(10)	0.1	0.71
	0.02	0.0987(3)	0.0494(5)	1.023(53)	0.0033(2)	0.03	1.17
$32^3 \times 64$	0.004	0.1278(15)	0.0323(11)	0.249(50)	0.0161(13)	0.04	0.31
	0.006	0.1229(8)	0.0319(17)	0.347(53)	0.0137(12)	0.02	0.44
	0.008	0.1187(5)	0.0366(38)	0.576(86)	0.0100(16)	0.2	0.58

in good agreement with the precise lattice VPF below $-q^2 = 0.74$ GeV^2. The systematic analysis of heading order of hadronic contribution to muon g-2 is under way.

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