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Analytical relation between confinement and chiral symmetry breaking in terms of Polyakov loop and Dirac eigenmodes in odd-number lattice QCD

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In lattice QCD formalism, we derive an analytical gauge-invariant relation between the Polyakov loop $\langle L_P \rangle$ and the Dirac eigenvalues λ_n in QCD, i.e., $\langle L_P \rangle \propto \sum_n \lambda_n^{N_t-1} \langle n|\hat{U}_4|n\rangle$, by considering $\mathrm{Tr}(\hat{U}_4 \hat{\not{D}}^{N_t-1})$ on a temporally odd-number lattice, where the temporal lattice size N_t is odd. This formula is a Dirac spectral representation of the Polyakov loop in terms of Dirac eigenmodes $|n\rangle$. We here use an ordinary square lattice with the normal (nontwisted) periodic boundary condition for link-variables $U_\mu(s)$ in the temporal direction. From this relation, one can estimate each contribution of the Dirac eigenmode to the Polyakov loop. Because of the factor $\lambda_n^{N_t-1}$ in the Dirac spectral sum, this analytical relation generally indicates quite small contribution of low-lying Dirac modes to the Polyakov loop in both confined and deconfined phases, while the low-lying Dirac modes are essential for chiral symmetry breaking. Also in lattice QCD calculations in confined and deconfined phases, we numerically confirm the analytical relation, non-zero finiteness of $\langle n|\hat{U}_4|n\rangle$ for each Dirac mode, and negligibly small contribution of low-lying Dirac modes to the Polyakov loop, i.e., the Polyakov loop is almost unchanged even by removing low-lying Dirac-mode contribution from the QCD vacuum generated by lattice QCD simulations. Thus, we conclude that low-lying Dirac modes are not essential modes for confinement, which indicates no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD.

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1. Introduction: Are color confinement and CSB one-to-one in QCD?

Quantum chromodynamics (QCD) has two outstanding nonperturbative phenomena of color confinement and spontaneous chiral-symmetry breaking [1] in the low-energy region, and their derivation is one of the most important problems in theoretical physics. For quark confinement, the Polyakov loop $\langle L_P \rangle$ is a typical order parameter, and relates to the single-quark free energy E_q as $\langle L_P \rangle \propto e^{-E_q/T}$ at temperature T. $\langle L_P \rangle$ is also an order parameter of Z_{N_c} center symmetry in QCD [2]. For chiral symmetry breaking, the standard order parameter is the chiral condensate $\langle \bar{q}q \rangle$, and low-lying Dirac modes play the essential role, as the Banks-Casher relation [3] indicates.

The relation between confinement and chiral symmetry breaking is also one of the important physical issues [4, 5, 6, 7, 8, 9, 10, 11], and there are several circumstantial evidence on their correlation. For example, lattice QCD simulations have shown almost coincidence between deconfinement and chiral-restoration temperatures [2, 12], although slight difference of about 25MeV between them is pointed out in some recent lattice QCD studies [13]. Their correlation is also suggested in terms of QCD-monopoles [4, 5], which topologically appear in QCD in the maximally Abelian gauge [14, 15, 16, 17, 18], leading to the dual-superconductor picture [19]. Actually, by removing the monopoles from the QCD vacuum generated in lattice QCD, confinement and chiral symmetry breaking are simultaneously lost [5], as schematically shown in Fig.1. This fact indicates an important role of monopoles to both confinement and chiral symmetry breaking, and thus these two phenomena seem to be related via the monopole. However, in spite of the essential role of monopoles, the direct relation of confinement and chiral symmetry breaking is still unclear.

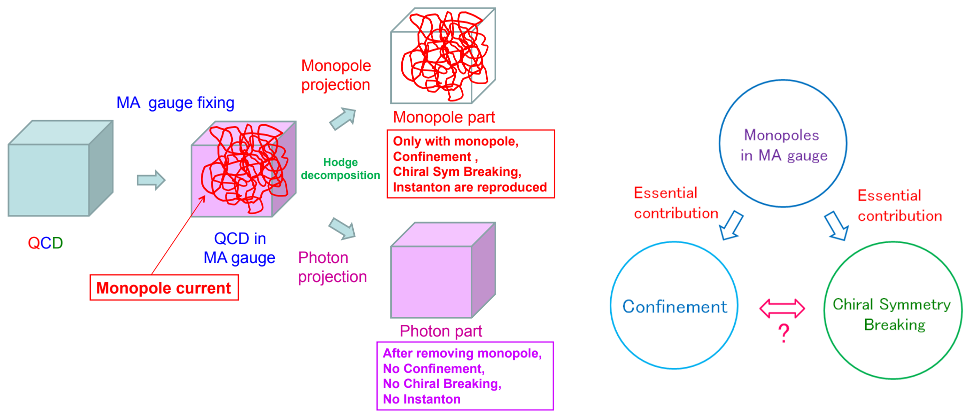

Figure 1: The role of monopoles to nonperturbative QCD. In the MA gauge, QCD becomes Abelian-like due to the large off-diagonal gluon mass of about 1GeV [17], and monopole current topologically appears [14, 15]. By the Hodge decomposition, the QCD system can be divided into the monopole part and the photon part. The monopole part has confinement [15], chiral symmetry breaking [5] and instantons [16], while the photon part does not have all of them, as shown in lattice QCD. In spite of the essential role of monopoles, the direct relation of confinement and chiral symmetry breaking is still unclear.

Then, we have a question. *If only the relevant ingredient of chiral symmetry breaking is carefully removed from the QCD vacuum, how will be quark confinement?*

To obtain the answer, in this paper, we derive an analytical relation between the Polyakov loop and the Dirac modes in temporally odd-number lattice QCD, where the temporal lattice size is odd, and discuss the relation between confinement and chiral symmetry breaking.

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2. Lattice QCD formalism for Dirac operator, Dirac eigenvalues and Dirac modes

First, we clarify the mathematical condition of lattice QCD formalism adopted in this study [10, 11]. We use an ordinary Euclidean square lattice with spacing a and size $V \equiv N_s^3 \times N_t$. The normal (nontwisted) periodic boundary condition is used for the link-variable $U_\mu(s) = \mathrm{e}^{iagA_\mu(s)}$ in the temporal direction, which is physically required at finite temperatures. We take SU(N_c) as the gauge group, although any gauge group G can be taken for most arguments in this paper.

Note that, in our studies, we just consider the mathematical expansion by eigenmodes $|n\rangle$ of the Dirac operator $\not{D} = \gamma_\mu D_\mu$, using the completeness of $\sum_n |n\rangle\langle n| = 1$. In general, instead of $\not{D}$, one can consider any (anti)hermitian operator, e.g., $D^2 = D_\mu D_\mu$, and the expansion in terms of its eigenmodes [7]. In this paper, to consider chiral symmetry breaking, we adopt $\not{D}$ and the expansion by its eigenmodes.

In lattice QCD, the Dirac operator $\not{D} = \gamma_\mu D_\mu$ is expressed with $U_\mu(s) = \mathrm{e}^{iagA_\mu(s)}$ as

$$\not{D}_{s,s'} \equiv \frac{1}{2a} \sum_{\mu=1}^{4} \gamma_\mu \left[U_\mu(s)\delta_{s+\hat{\mu},s'} - U_{-\mu}(s)\delta_{s-\hat{\mu},s'} \right], \tag{2.1}$$

with $U_{-\mu}(s) \equiv U_\mu^\dagger(s-\hat{\mu})$ and lattice unit vector $\hat{\mu}$. Taking hermitian $\gamma_\mu = \gamma_\mu^\dagger$, the Dirac operator $\not{D}$ is anti-hermitian and satisfies $\not{D}^\dagger_{s',s} = -\not{D}_{s,s'}$. We introduce the normalized Dirac eigen-state $|n\rangle$ as

$$\not{D}|n\rangle = i\lambda_n|n\rangle, \qquad \langle m|n\rangle = \delta_{mn}, \tag{2.2}$$

with the Dirac eigenvalue $i\lambda_n$ ($\lambda_n \in \mathbf{R}$). Because of $\{\gamma_5, \not{D}\} = 0$, the state $\gamma_5|n\rangle$ is also an eigen-state of $\not{D}$ with the eigenvalue $-i\lambda_n$. Here, the Dirac eigen-state $|n\rangle$ satisfies the completeness of

$$\sum_n |n\rangle\langle n| = 1. \tag{2.3}$$

The Dirac eigenfunction $\psi_n(s) \equiv \langle s|n\rangle$ satisfies $\not{D}\psi_n(s) = i\lambda_n\psi_n(s)$, i.e.,

$$\frac{1}{2a} \sum_{\mu=1}^{4} \gamma_\mu [U_\mu(s)\psi_n(s+\hat{\mu}) - U_{-\mu}(s)\psi_n(s-\hat{\mu})] = i\lambda_n\psi_n(s). \tag{2.4}$$

By the gauge transformation of $U_\mu(s) \to V(s)U_\mu(s)V^\dagger(s+\hat{\mu})$, $\psi_n(s)$ is gauge-transformed as

$$\psi_n(s) \to V(s)\psi_n(s), \tag{2.5}$$

which is the same as that of the quark field. (To be strict, there can appear an irrelevant n-dependent global phase factor $e^{i\varphi_n[V]}$, according to arbitrariness of the phase in the basis $|n\rangle$ [8].)

The spectral density $\rho(\lambda)$ of the Dirac operator $\not{D}$ relates to chiral symmetry breaking, e.g., the zero-eigenvalue density $\rho(0)$ leads to $\langle\bar{q}q\rangle$ (Banks-Casher's relation) [3]. In fact, the low-lying Dirac modes are regarded as the essential modes responsible to chiral symmetry breaking in QCD.

Here, we take the operator formalism in lattice QCD [7, 8, 9] by introducing the link-variable operator $\hat{U}_{\pm\mu}$ defined by the matrix element of

$$\langle s|\hat{U}_{\pm\mu}|s'\rangle = U_{\pm\mu}(s)\delta_{s\pm\hat{\mu},s'}. \tag{2.6}$$

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With the link-variable operator, the Dirac operator and covariant derivative are simply expressed as

$$\hat{\not{D}} = \frac{1}{2a}\sum_{\mu=1}^{4}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}), \qquad \hat{D}_\mu = \frac{1}{2a}(\hat{U}_\mu - \hat{U}_{-\mu}). \tag{2.7}$$

The Polyakov loop is also simply written as the functional trace of $\hat{U}_4^{N_t}$, i.e., $\langle L_P\rangle = \frac{1}{N_c V}\langle \mathrm{Tr}_c\{\hat{U}_4^{N_t}\}\rangle$, where "$\mathrm{Tr}_c$" denotes the functional trace of $\mathrm{Tr}_c \equiv \sum_s \mathrm{tr}_c$ including the trace tr_c over color index. For large volume V, one can expect $\langle O\rangle \simeq \mathrm{Tr}\ O/\mathrm{Tr}\ 1$ for any operator O at each gauge configuration. The Dirac-mode matrix element of the link-variable operator $\hat{U}_\mu$ can be expressed with $\psi_n(s)$ as

$$\langle m|\hat{U}_\mu|n\rangle = \sum_s \langle m|s\rangle\langle s|\hat{U}_\mu|s+\hat{\mu}\rangle\langle s+\hat{\mu}|n\rangle = \sum_s \psi_m^\dagger(s)U_\mu(s)\psi_n(s+\hat{\mu}), \tag{2.8}$$

which is gauge invariant, because of (2.5), apart from an irrelevant global phase factor [8].

3. Previous numerical study: Dirac-mode expansion and Dirac-mode projection

We here review our previous studies [7, 8, 9] on "Dirac-mode expansion", "Dirac-mode projection" where the Dirac-mode space is restricted, and the role of low-lying Dirac modes to confinement in SU(3) lattice QCD.

From the completeness of the Dirac-mode basis, $\sum_n |n\rangle\langle n| = 1$, arbitrary operator $\hat{O}$ can be expanded in terms of the Dirac-mode basis $|n\rangle$ as $\hat{O} = \sum_n\sum_m |n\rangle\langle n|\hat{O}|m\rangle\langle m|$. With this relation, we consider the Dirac-mode expansion and Dirac-mode projection. We define the projection operator $\hat{P} \equiv \sum_{n\in A}|n\rangle\langle n|$, which restricts the Dirac-mode space to its arbitrary subset A. Using the projection operator $\hat{P}$, we define the Dirac-mode projected link-variable operator $\hat{U}_\mu^P \equiv \hat{P}\hat{U}_\mu\hat{P} = \sum_{m\in A}\sum_{n\in A}|m\rangle\langle m|\hat{U}_\mu|n\rangle\langle n|$, the Dirac-mode projected Wilson-loop operator $\hat{W}^P \equiv \prod_{k=1}^{N}\hat{U}_{\mu_k}^P$, the Dirac-mode projected inter-quark potential $V^P(R) \equiv -\lim_{T\to\infty}\frac{1}{T}\ln\langle \mathrm{Tr}_c\ \hat{W}^P(R,T)\rangle$ from the $R\times T$ rectangular Wilson loop, and the Dirac-mode projected Polyakov loop $\langle L_P\rangle_{\mathrm{proj.}} \equiv \frac{1}{N_c V}\langle \mathrm{Tr}_c\ (\hat{U}_4^P)^{N_t}\rangle$.

In Refs.[7, 8], we use SU(3) quenched lattice QCD at $\beta = 5.6$ (i.e., $a \simeq 0.25$fm) on 6^4, and take the IR-cutoff $\Lambda_{\mathrm{IR}} = 0.5a^{-1} \simeq 0.4$GeV for the Dirac modes, which leads an extreme reduction of the chiral condensate as $\langle\bar{q}q\rangle_{\Lambda_{\mathrm{IR}}}/\langle\bar{q}q\rangle \simeq 0.02$ for the current quark mass $m \simeq 5$MeV. Figure 2 shows the IR-Dirac-mode-cut Wilson loop $\langle \mathrm{Tr}_c\hat{W}^P(R,T)\rangle$, the IR-cut inter-quark potential $V^P(R)$, and the IR-Dirac-mode-cut Polyakov loop $\langle L_P\rangle_{\mathrm{IR\text{-}cut}}$, after the removal of the low-lying Dirac modes.

Remarkably, even after removing the coupling to the low-lying Dirac modes, the IR-Dirac-mode-cut Wilson loop obeys the area law as $\langle W^P(R,T)\rangle \propto e^{-\sigma^P RT}$, and the slope σ^P, i.e., the string tension, is almost unchanged as $\sigma^P \simeq \sigma$. As shown in Fig.2(b), the IR-cut inter-quark potential $V^P(R)$ is almost unchanged from the original one, apart from an irrelevant constant. Also from Fig.2(c), we find that the IR-Dirac-mode-cut Polyakov loop is almost zero, $\langle L_P\rangle_{\mathrm{IR\text{-}cut}} \simeq 0$, which means Z_3-unbroken confinement phase. In this way, confinement is kept even in the absence of low-lying Dirac modes or the essence of chiral symmetry breaking [7, 8, 9].

We also investigate the UV-cut of Dirac modes in lattice QCD, and find that the confining force is almost unchanged by the UV-cut [7, 8, 9]. Furthermore, we examine "intermediate(IM)-cut" of Dirac modes, and obtain almost the same confining force [7, 8].

From these lattice QCD results, there is no specific region of the Dirac modes responsible to confinement. In other words, we conjecture that the "seed" of confinement is distributed not only in low-lying Dirac modes but also in a wider region of the Dirac-mode space.

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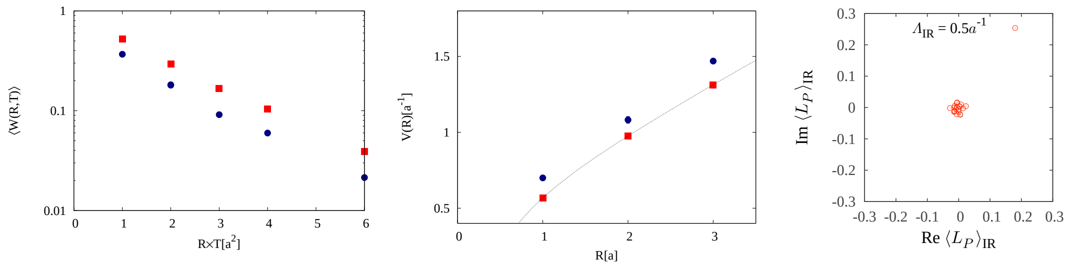

Figure 2: Lattice QCD results [7, 8] after the removal of low-lying Dirac modes below the IR-cutoff $\Lambda_{\rm IR} = 0.5a^{-1} \simeq 0.4$GeV. (a) The IR-cut Wilson loop $\langle {\rm Tr}_c\, W^P(R,T)\rangle$ (circle) plotted against $R \times T$. The slope σ^P is almost the same as that of the original Wilson loop (square). (b) The IR-cut inter-quark potential $V^P(R)$ (circle). $V^P(R)$ is almost unchanged from the original one (square), apart from an irrelevant constant. (c) The scatter plot of the IR-cut Polyakov loop $\langle L_P\rangle_{\rm IR\text{-}cut}$. $\langle L_P\rangle_{\rm IR\text{-}cut} \simeq 0$ means Z_3-unbroken confinement phase.

4. Analytical relation of the Polyakov loop and Dirac modes in odd-N_t lattice QCD

Now, we consider temporally odd-number lattice QCD, where the temporal lattice size N_t is odd [10, 11]. Here, we use an ordinary square lattice with the normal (nontwisted) periodic boundary condition for link-variables $U_\mu(s)$ in the temporal direction. The spatial lattice size N_s is taken to be larger than N_t, i.e., $N_s > N_t$. Note that, in the continuum limit of $a \to 0$ and $N_t \to \infty$, any large number N_t gives the same physical result. Then, to use the odd-number lattice is no problem.

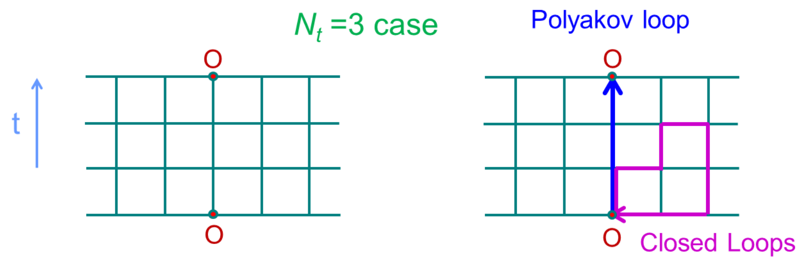

Figure 3: An example of the temporally odd-number lattice ($N_t = 3$ case). Only gauge-invariant quantities such as closed loops and the Polyakov loop survive in QCD, after taking the expectation value, i.e., the gauge-configuration average. Geometrically, closed loops have even-number links on the square lattice.

As a general mathematical argument of the Elitzur theorem [2], only gauge-invariant quantities such as closed loops and the Polyakov loop survive in QCD. In fact, all the non-closed lines are gauge-variant and their expectation values are zero. Note here that any closed loop needs even-number link-variables on the square lattice, except for the Polyakov loop [10]. (See Fig.3.)

In temporally odd-number lattice QCD [10, 11], we consider the functional trace of

$$I \equiv {\rm Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_t-1}), \tag{4.1}$$

where ${\rm Tr}_{c,\gamma} \equiv \sum_s {\rm tr}_c {\rm tr}_\gamma$ includes ${\rm tr}_c$ and the trace ${\rm tr}_\gamma$ over spinor index. Its expectation value

$$\langle I\rangle = \langle {\rm Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_t-1})\rangle \tag{4.2}$$

is obtained as the gauge-configuration average in lattice QCD. When the volume V is enough large, one can expect $\langle O\rangle \simeq {\rm Tr}\, O/{\rm Tr}\, 1$ for any operator O even in each gauge configuration.

From Eq.(2.7), $\hat{U}_4\, \hat{\not{D}}^{N_t-1}$ can be expressed as a sum of products of N_t link-variable operators, since the Dirac operator $\hat{\not{D}}$ includes one link-variable operator in each direction of $\pm\mu$. In fact, $\hat{U}_4\, \hat{\not{D}}^{N_t-1}$ includes "many trajectories" with the total length N_t (in lattice unit) on the square lattice, as shown in Fig.4. Note that all the trajectories with the odd-number length N_t cannot form a closed loop on the square lattice, and give gauge-variant contribution, except for the Polyakov loop [10].

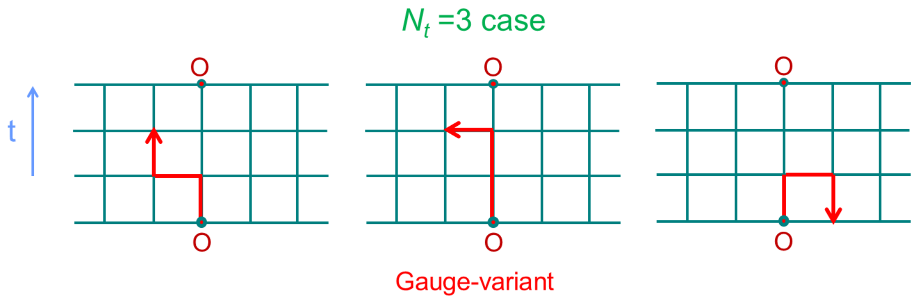

Figure 4: Partial examples of the trajectories stemming from $\langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$. For each trajectory, the total length is N_t, and the "first step" is positive temporal direction corresponding to $\hat{U}_4$. All the trajectories with the odd-number length N_t cannot form a closed loop on the square lattice, so that they are gauge-variant and give no contribution in $\langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$, except for the Polyakov loop.

Hence, among the trajectories stemming from $\langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$, all the non-loop trajectories are gauge-variant and give no contribution, according to the Elitzur theorem [2]. Only the exception is the Polyakov loop. (Compare Figs.4 and 5.) Note that $\langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$ does not include the anti-Polyakov loop $\langle L_P^\dagger \rangle$, since the "first step" is positive temporal direction corresponding to $\hat{U}_4$.

Thus, in the functional trace $\langle I \rangle = \langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$, only the Polyakov-loop ingredient can survive as the gauge-invariant quantity, and $\langle I \rangle$ is proportional to the Polyakov loop $\langle L_P \rangle$.

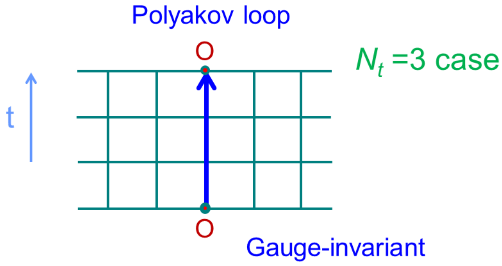

Figure 5: Among the trajectories stemming from $\langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$, only the Polyakov-loop ingredient can survive as the gauge-invariant quantity. Owing to the first factor $\hat{U}_4$, $\langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_t-1})\rangle$ does not include $\langle L_P^\dagger \rangle$.

Actually, we can mathematically derive the following relation [10]:

$$
\begin{aligned}
\langle I \rangle &= \langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_t-1})\rangle \\
&= \langle \mathrm{Tr}_{c,\gamma}\{\hat{U}_4(\gamma_4 \hat{D}_4)^{N_t-1}\}\rangle && (\because \text{ only gauge-invariant terms survive}) \\
&= 4\langle \mathrm{Tr}_c(\hat{U}_4 \hat{D}_4^{N_t-1})\rangle && (\because\ \gamma_4^{N_t-1} = 1,\ \mathrm{tr}_\gamma 1 = 4) \\
&= \frac{4}{(2a)^{N_t-1}}\langle \mathrm{Tr}_c\{\hat{U}_4(\hat{U}_4 - \hat{U}_{-4})^{N_t-1}\}\rangle && (\because\ \hat{D}_4 = \frac{1}{2a}(\hat{U}_4 - \hat{U}_{-4})) \\
&= \frac{4}{(2a)^{N_t-1}}\langle \mathrm{Tr}_c\{\hat{U}_4^{N_t}\}\rangle = \frac{12V}{(2a)^{N_t-1}}\langle L_P \rangle. && (\because \text{ only gauge-invariant terms survive}) \qquad (4.3)
\end{aligned}
$$

We thus obtain the relation between $\langle I \rangle = \langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_t-1}) \rangle$ and the Polyakov loop $\langle L_P \rangle$,

$$\langle I \rangle = \langle \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_t-1}) \rangle = \frac{12V}{(2a)^{N_t-1}} \langle L_P \rangle. \tag{4.4}$$

On the other hand, we calculate the functional trace in Eq.(4.2) using the complete set of the Dirac-mode basis $|n\rangle$ satisfying $\sum_n |n\rangle\langle n| = 1$, and find the Dirac-mode representation of

$$\langle I \rangle = \sum_n \langle n|\hat{U}_4\, \hat{\not{D}}^{N_t-1}|n\rangle = i^{N_t-1} \sum_n \lambda_n^{N_t-1} \langle n|\hat{U}_4|n\rangle. \tag{4.5}$$

Combing Eqs.(4.4) and (4.5), we obtain the analytical relation between the Polyakov loop $\langle L_P \rangle$ and the Dirac eigenvalues $i\lambda_n$ [10] in QCD:

$$\langle L_P \rangle = \frac{(2ai)^{N_t-1}}{12V} \sum_n \lambda_n^{N_t-1} \langle n|\hat{U}_4|n\rangle. \tag{4.6}$$

This is a direct relation between the Polyakov loop $\langle L_P \rangle$ and the Dirac modes in QCD, i.e., a "Dirac spectral representation of the Polyakov loop", and is mathematically valid on the temporally odd-number lattice in both confined and deconfined phases. Based on Eq.(4.6), we can investigate each Dirac-mode contribution to the Polyakov loop individually, e.g., by evaluating each contribution specified by n numerically in lattice QCD. In particular, by paying attention to low-lying Dirac modes in Eq.(4.6), the relation between confinement and chiral symmetry breaking can be discussed in QCD.

As a remarkable fact, because of the factor $\lambda_n^{N_t-1}$, the contribution from low-lying Dirac-modes with $|\lambda_n| \simeq 0$ is negligibly small in the Dirac spectral sum of RHS in Eq.(4.6), compared to the other Dirac-mode contribution. In fact, the low-lying Dirac modes have little contribution to the Polyakov loop, regardless of confined or deconfined phase [10, 11]. (This result agrees with the previous numerical lattice results that confinement properties are almost unchanged by removing low-lying Dirac modes from the QCD vacuum [7, 8, 9].) Thus, we conclude from the relation (4.6) that low-lying Dirac modes are not essential modes for confinement, which indicates no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD.

Here, we give several comments on the relation (4.6) in order.

1. Equation (4.6) is a manifestly gauge-invariant relation. Actually, the matrix element $\langle n|\hat{U}_4|n\rangle$ can be expressed with the Dirac eigenfunction $\psi_n(s)$ and the temporal link-variable $U_4(s)$ as

$$\langle n|\hat{U}_4|n\rangle = \sum_s \langle n|s\rangle\langle s|\hat{U}_4|s+\hat{t}\rangle\langle s+\hat{t}|n\rangle = \sum_s \psi_n^\dagger(s) U_4(s) \psi_n(s+\hat{t}), \tag{4.7}$$

   and each term $\psi_n^\dagger(s) U_4(s) \psi_n(s+\hat{t})$ is manifestly gauge invariant, because of the gauge transformation property (2.5). [Global phase factors also cancel exactly between $\langle n|$ and $|n\rangle$.]

2. In RHS of Eq.(4.6), there is no cancellation between chiral-pair Dirac eigen-states, $|n\rangle$ and $\gamma_5|n\rangle$, because (N_t-1) is even, i.e., $(-\lambda_n)^{N_t-1} = \lambda_n^{N_t-1}$, and $\langle n|\gamma_5\hat{U}_4\gamma_5|n\rangle = \langle n|\hat{U}_4|n\rangle$.

3. Even in the presence of a possible multiplicative renormalization factor for the Polyakov loop, the contribution from the low-lying Dirac modes (or the small $|\lambda_n|$ region) is relatively negligible, compared to other Dirac-mode contribution in the sum of RHS in Eq.(4.6).

4. For the arbitrary color number N_c, Eq.(4.6) is true and applicable in the SU(N_c) gauge theory.

5. If RHS in Eq.(4.6) *were* not a sum but a product, low-lying Dirac modes should have given an important contribution to the Polyakov loop as a crucial reduction factor of $\lambda_n^{N_t-1}$. In the sum, however, the contribution ($\propto \lambda_n^{N_t-1}$) from the small $|\lambda_n|$ region is negligible.

6. Even if $\langle n|\hat{U}_4|n\rangle$ behaves as the δ-function $\delta(\lambda)$, the factor $\lambda_n^{N_t-1}$ is still crucial in RHS of Eq.(4.6), because of $\lambda\delta(\lambda)=0$.

7. The relation (4.6) is correct regardless of presence or absence of dynamical quarks, although dynamical quark effects appear in $\langle L_P\rangle$, the Dirac eigenvalue distribution $\rho(\lambda)$ and $\langle n|\hat{U}_4|n\rangle$.

8. The relation (4.6) is correct also at finite baryon density and finite temperature.

9. Equation (4.6) obtained on the odd-number lattice is correct in the continuum limit of $a\to 0$ and $N_t\to\infty$, since any number of large N_t gives the same physical result.

Note that most of the above arguments can be numerically confirmed by lattice QCD calculations. Using actual lattice QCD calculations at the quenched level, we numerically confirm the analytical relation (4.6), non-zero finiteness of $\langle n|\hat{U}_4|n\rangle$ for each Dirac mode, and the negligibly small contribution of low-lying Dirac modes to the Polyakov loop, in both confined and deconfined phases [10, 11], as will be shown in Sec.5. Although we numerically find an interesting drastic change of the behavior of $\langle n|\hat{U}_4|n\rangle$ between confined and deconfined phases, we find also quite small contribution of low-lying Dirac modes to the Polyakov loop.

5. Modified KS formalism for temporally odd-number lattice

The Dirac operator $\not{D}$ has a large dimension of $(4\times N_\mathrm{c}\times V)^2$, and hence the numerical cost for solving the Dirac eigenvalue equation is quite huge. This numerical cost can be partially reduced using the Kogut-Susskind (KS) formalism [2, 8, 11, 20]. However, the original KS formalism can be applied only to the "even lattice" where all the lattice sizes N_μ are even number. In this section, we modify the KS formalism to be applicable to the odd-number lattice [11]. Using the "modified KS formalism", we can reduce the numerical cost in the case of the temporally odd-number lattice.

In the original KS formalism for even lattices, using the matrix $T(s)\equiv\gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_4}$, all the γ-matrices can be diagonalized as $T^\dagger(s)\gamma_\mu T(s\pm\hat{\mu})=\eta_\mu(s)\mathbf{1}$, where $\eta_\mu(s)$ is the staggered phase, $\eta_1(s)\equiv 1,\ \ \eta_\mu(s)\equiv(-1)^{s_1+\cdots+s_{\mu-1}}$ $(\mu\geq 2)$. Then, the Dirac operator $\not{D}$ is spin-diagonalized as

$$\sum_\mu T^\dagger(s)\gamma_\mu D_\mu T(s+\hat{\mu})=\mathrm{diag}(\eta_\mu D_\mu,\eta_\mu D_\mu,\eta_\mu D_\mu,\eta_\mu D_\mu), \tag{5.1}$$

where $\eta_\mu D_\mu$ is the KS Dirac operator given by

$$(\eta_\mu D_\mu)_{ss'}=\frac{1}{2a}\sum_{\mu=1}^{4}\eta_\mu(s)\left[U_\mu(s)\delta_{s+\hat{\mu},s'}-U_{-\mu}(s)\delta_{s-\hat{\mu},s'}\right]. \tag{5.2}$$

Equation (5.1) shows fourfold degeneracy of the Dirac eigenvalue relating to the spiror structure, and then all the eigenvalues $i\lambda_n$ are obtained by solving the reduced Dirac eigenvalue equation

$$\eta_\mu D_\mu|n)=i\lambda_n|n). \tag{5.3}$$

Using the eigenfunction $\chi_n(s) \equiv \langle s|n\rangle$ of the KS Dirac operator, the explicit form of Eq.(5.3) reads

$$\frac{1}{2a}\sum_{\mu=1}^{4}\eta_\mu(x)[U_\mu(x)\chi_n(x+\hat{\mu})-U_{-\mu}(x)\chi_n(x-\hat{\mu})]=i\lambda_n\chi_n(x), \tag{5.4}$$

where the relation between the Dirac eigenfunction $\psi_n(s)$ and the spinless eigenfunction $\chi_n(s)$ is

$$\psi_n(s)=T(s)\chi_n(s). \tag{5.5}$$

Note here that the original KS formalism is applicable only to even lattices in the presence of the periodic boundary condition [11]. In fact, the periodic boundary condition requires

$$T(s+N_\mu\hat{\mu})=T(s)\ (\mu=1,2,3,4), \tag{5.6}$$

however, it is satisfied only on even lattices. Note also that, while the spatial boundary condition can be changed arbitrary, the temporal periodic boundary condition physically appears and cannot be changed at finite temperatures. Thus, the original KS formalism cannot be applied on the temporally odd-number lattice.

Now, we consider the temporally odd-number lattice, with all the spatial lattice size being even. Instead of the matrix $T(s)$, we introduce a new matrix [11]

$$M(s)\equiv\gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_1+s_2+s_3}, \tag{5.7}$$

where the exponent of γ_4 differs from $T(s)$. As a remarkable feature, the requirement from the periodic boundary condition is satisfied on the temporally odd-number lattice [11]:

$$M(s+N_\mu\hat{\mu})=M(s)\ (\mu=1,2,3,4). \tag{5.8}$$

Using the matrix $M(s)$, all the γ-matrices are transformed to be proportional to γ_4:

$$M^\dagger(s)\gamma_\mu M(s\pm\hat{\mu})=\eta_\mu(s)\gamma_4, \tag{5.9}$$

where $\eta_\mu(x)$ is the staggered phase. In the Dirac representation, γ_4 is diagonal as

$$\gamma_4=\mathrm{diag}(1,1,-1,-1)\quad\text{(Dirac representation)}, \tag{5.10}$$

and we take the Dirac representation. Thus, we can spin-diagonalize the Dirac operator $\not{D}$ in the case of the temporally odd-number lattice [11]:

$$\sum_\mu M^\dagger(s)\gamma_\mu D_\mu M(s+\hat{\mu})=\mathrm{diag}(\eta_\mu D_\mu,\eta_\mu D_\mu,-\eta_\mu D_\mu,-\eta_\mu D_\mu), \tag{5.11}$$

where $\eta_\mu D_\mu$ is the KS Dirac operator given by Eq.(5.2). Then, for each λ_n, two positive modes and two negative modes appear relating to the spinor structure on the temporally odd-number lattice. (Note also that the chiral partner $\gamma_5|n\rangle$ gives an eigenmode with the eigenvalue $-i\lambda_n$.) In any case, all the eigenvalues $i\lambda_n$ can be obtained by solving the reduced Dirac eigenvalue equation

$$\eta_\mu D_\mu|n)=\pm i\lambda_n|n) \tag{5.12}$$

just like the case of even lattices. The relation between the Dirac eigenfunction $\psi_n(s)$ and the sponless eigenfunction $\chi_n(s)\equiv\langle s|n\rangle$ is given by

$$\psi_n(s)=M(s)\chi_n(s) \tag{5.13}$$

on the temporally odd-number lattice.

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6. Numerical confirmation for the relation between Polyakov loop and Dirac modes

Using the modified KS formalism, Eq.(4.6) is rewritten as

$$\langle L_P \rangle = \frac{(2ai)^{N_t-1}}{3V} \sum_n \lambda_n^{N_t-1} (n|\hat{U}_4|n). \tag{6.1}$$

Note that the (modified) KS formalism is an exact method for diagonalizing the Dirac operator and is not an approximation, so that Eqs.(4.6) and (6.1) are completely equivalent. In fact, the relation (4.6) can be confirmed by the numerical test of the relation (6.1).

We numerically calculate LHS and RHS of the relation (6.1), respectively, and compare them [11]. We perform SU(3) lattice QCD Monte Carlo simulations with the standard plaquette action at the quenched level in both cases of confined and deconfined phases. For the confined phase, we use $10^3 \times 5$ lattice with $\beta \equiv 2N_c/g^2 = 5.6$ (i.e., $a \simeq 0.25$ fm), corresponding to $T \equiv 1/(N_t a) \simeq$ 160 MeV. For the deconfined phase, we use $10^3 \times 3$ lattice with $\beta = 5.7$ (i.e., $a \simeq 0.20$ fm), corresponding to $T \equiv 1/(N_t a) \simeq 330$ MeV.

As the numerical result, comparing LHS and RHS of the relation (6.1), we find that the relation (6.1) is almost exact even for each gauge configuration in both confined and deconfined phases [11]. Therefore, the relation (6.1) is satisfied also for the gauge-configuration average.

Next, we numerically confirm that the low-lying Dirac modes have negligible contribution to the Polyakov loop using Eq.(6.1). By checking all the Dirac modes, we find that the matrix element $(n|\hat{U}_4|n)$ is generally nonzero for each Dirac mode [10, 11]. In fact, for low-lying Dirac modes, the factor $\lambda_n^{N_t-1}$ plays a crucial role in RHS of Eq.(6.1). Since RHS of Eq.(6.1) is expressed as a sum of the Dirac-mode contribution, we calculate the Polyakov loop without low-lying Dirac-mode contribution as

$$\langle L_P \rangle_{\text{IR-cut}} \equiv \frac{(2ai)^{N_t-1}}{3V} \sum_{|\lambda_n|>\Lambda_{\text{IR}}} \lambda_n^{N_t-1} (n|\hat{U}_4|n), \tag{6.2}$$

with the IR cut Λ_{IR} for the Dirac eigenvalue. The chiral condensate without the contribution from the low-lying Dirac-mode below IR cut Λ_{IR} is given by [8, 11]

$$\langle \bar{q}q \rangle_{\Lambda_{\text{IR}}} = -\frac{1}{V} \sum_{\lambda_n>\Lambda_{\text{IR}}} \frac{2m}{\lambda_n^2 + m^2}, \tag{6.3}$$

where m is the current quark mass. Here, we take the IR cut of $\Lambda_{\text{IR}} \simeq 0.4$GeV. In the confined phase, this IR Dirac-mode cut leads to $\langle \bar{q}q \rangle_{\Lambda_{\text{IR}}} / \langle \bar{q}q \rangle \simeq 0.02$ and almost chiral-symmetry restoration for the current quark mass $m \simeq 5$MeV.

We find that $\langle L_P \rangle \simeq \langle L_P \rangle_{\text{IR-cut}}$ is numerically satisfied even for each gauge configuration in both confined and deconfined phases. Table 1 and 2 show the numerical result of $\langle L_P \rangle$ and $\langle L_P \rangle_{\text{IR-cut}}$ in each gauge configuration for confined and deconfined phases, respectively. Thus, the Polyakov loop is almost unchanged by removing the contribution from the low-lying Dirac modes [11], which are essential for chiral symmetry breaking. From both analytical and numerical results, we conclude that low-lying Dirac modes are not essential modes for confinement, which indicates no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD.

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Table 1: Numerical results for $\langle L_P \rangle$ and $\langle L_P \rangle_{\text{IR-cut}}$ in lattice QCD with $10^3 \times 5$ and $\beta = 5.6$ for each gauge configuration, where the system is in confined phase.

configuration No.	1	2	3	4	5	6	7
Re$\langle L_P \rangle$	0.00961	-0.00161	0.0139	-0.00324	0.000689	0.00423	-0.00807
Im$\langle L_P \rangle$	-0.00322	-0.00125	-0.00438	-0.00519	-0.0101	-0.0168	-0.00265
Re$\langle L_P \rangle_{\text{IR-cut}}$	0.00961	-0.00160	0.0139	-0.00325	0.000706	0.00422	-0.00807
Im$\langle L_P \rangle_{\text{IR-cut}}$	-0.00321	-0.00125	-0.00437	-0.00520	-0.0101	-0.0168	-0.00264

Table 2: Numerical results for $\langle L_P \rangle$ and $\langle L_P \rangle_{\text{IR-cut}}$ in lattice QCD with $10^3 \times 3$ and $\beta = 5.7$ for each gauge configuration, where the system is in deconfined phase.

configuration No.	1	2	3	4	5	6	7
Re$\langle L_P \rangle$	0.316	0.337	0.331	0.305	0.314	0.316	0.337
Im$\langle L_P \rangle$	-0.00104	-0.00597	0.00723	-0.00334	0.00167	0.000120	0.0000482
Re$\langle L_P \rangle_{\text{IR-cut}}$	0.319	0.340	0.334	0.307	0.317	0.319	0.340
Im$\langle L_P \rangle_{\text{IR-cut}}$	-0.00103	-0.00597	0.00724	-0.00333	0.00167	0.000121	0.0000475

7. Summary and concluding remarks

In this study, we have analytically derived a direct relation between the Polyakov loop and the Dirac modes in temporally odd-number lattice QCD [10, 11], on ordinary square lattices with the normal (nontwisted) periodic boundary condition for link-variables. We have shown that the low-lying Dirac modes have quite small contribution to the Polyakov loop [10, 11].

As a new method, we have modified the KS formalism to perform the spin-diagonalization of the Dirac operator on the temporally odd-number lattice [11]. In lattice QCD calculations, using the “modified KS formalism”, we have numerically shown that the contribution of low-lying Dirac modes to the Polyakov loop is negligibly small in both confined and deconfined phases [11].

From the analytical relation (4.6) and the numerical confirmation, we conclude that low-lying Dirac-modes have little contribution to the Polyakov loop, and are not essential for confinement, while these modes are essential for chiral symmetry breaking. This conclusion indicates no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD.

Since the relation (4.6) is correct in the presence of dynamical quarks and also at finite density, it is interesting to investigate Eq.(4.6) in full QCD simulations and at finite baryon density.

Our results suggest some independence between chiral symmetry breaking and color confinement, which may lead to richer phase structure in QCD. For example, the phase transition point can be different between deconfinement and chiral restoration in the presence of strong electromagnetic fields, because of their nontrivial effect on chiral symmetry [21].

As a future work, it is interesting to investigate the Polyakov-loop fluctuation, which is recently found to be important in the QCD phase transition [22]. It is also meaningful to compare with other lattice QCD result on importance of infrared gluons to confinement, i.e., confinement originates from the low-momentum gluons below 1.5GeV in Landau gauge [23].

In any case, the research for the direct relation between confinement and chiral symmetry breaking would give a new direction in the theoretical study of nonperturbative QCD.

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